AN EXTENSION OF THE ALMOST ISOLATED SINGULARITY OF FINITE EXPONENTIAL ORDER

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1. The almost isolated singularity

Let $f(z)$ be represented on its circle of convergence $|z| = 1$ by the Taylor series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots \qquad \text{.....................(1)}$$

and suppose that its sole singularity on $|z| = 1$ is an almost isolated singularity at $z = 1$. In the neighbourhood of such a singularity $f(z)$ is regular on a sufficiently small disk, centre $z = 1$, with the outward drawn radius along the positive real axis excised.† If also in this neighbourhood $|f(z)|\, e^{-(1/\delta)^\rho}$ remains bounded for some finite ρ, where δ is the distance from the excised radius, then the singularity is said to be of *finite exponential order*.‡

Pólya has created an appropriate process of dissection whereby the singularity can be separated off from the rest of $f(z)$.§ In this, the singular stretch of the positive real axis is cut so as to isolate the singular segment $(1, k)$ for some $k > 1$. The point $z = 1$ is singular while the point $z = k$ and any interior point of the segment may or may not be singular.‖ The associated decomposition gives

$$f(z) = f_1(z) + f^*(z), \qquad \text{.............................(2)}$$

where $f^*(z)$ is regular inside the circle $|z| = k$ and

$$f_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)dw}{w-z}. \qquad \text{.............................(3)}$$

Here Γ is a simple closed curve containing the segment $(1, k)$ but passing through the point $w = k$ and excluding the origin. It is described in a negative direction.

The coefficients c_n of (1) may be interpolated by a function $G(z)$ as follows:¶

$$c_n = G(n) + c_n^* \qquad \text{..................................(4)}$$

† (5), 735.

‡ (5), 777.

§ (5), 738-41.

‖ The isolated critical point and the essential point which is a limit point of poles along a line are particular cases. So is the isolated essential point and for this the point $z = k$ is not singular.

¶ (5), 741-4.

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where

$$G(z) = \frac{1}{2\pi i}\int_C f_1(w)w^{-z-1}dw, \quad \text{.........................(5)}$$

the integral is taken round a circle $|w| = h(<1)$, $f_1(z)$ is defined by (3) and

$$\lim_{n\to\infty} |c_n^*| < 1,$$

as follows from (2).

The substitution $w = e^{-\zeta}$ in (5) shows that

$$G(z) = \frac{1}{2\pi i}\int f_1(e^{-\zeta})e^{z\zeta}d\zeta,$$

where the path of integration is from $-\log h - i\pi$ to $-\log h + i\pi$ in the ζ-plane.

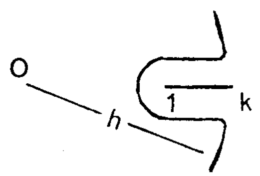
The function $G(z)$ is of exponential type and its indicator diagram is the segment $(-\log k, 0)$ of the negative real axis. It is at most of order 1, minimum type in the right half-plane, including the boundary.†

2. Finite exponential order

When the singularity of $f(z)$ at $z = 1$ is of finite exponential order these results can be sharpened. To this end we first prove the lemma below.

Lemma 1. *When the almost isolated singularity of $f(z)$ at $z = 1$ is of finite exponential order ρ then $G(z)$ is at most of order $\frac{\rho}{\rho+1}$ in the right half-plane and on its boundary.*‡

The path of integration in (5) can be deformed so that $h > 1$, provided that the contour deviates inside the circle $|z| = 1$ round the point $z = 1$. In fact, we can assume it to enclose the point 1 in a semi-circle of radius δ and to be



distant δ from the segment $(1, k)$ until the straight pieces of contour meet the circle $|z| = h > 1$. Now, on this contour C_1,

$$|f(z)| < K(\delta)e^{(1/\delta)^\rho}, \quad \text{...............................(6)}$$

and so, from (2),

$$|f_1(z)| < K_1(\delta)e^{(1/\delta)^\rho}, \quad \text{............................(7)}$$

since $f^*(z)$ is bounded on the contour C_1. On the part near $w = 1$ we have $|\arg w| \leq \delta$ and so, from (5) and (7),

$$|G(z)| < K_2(\delta)\exp(\delta^{-\rho} + |z|\delta). \quad \text{....................(8)}$$

† (5), 744, Satz III.

‡ I am indebted to Dr M. L. Cartwright for the use of notes from which the proof of Lemma 1 has been adapted.

Hence, if we take $|z| = (1/\delta)^{\rho+1}$ then

$$|G(z)| < K_2(\delta) \exp |z|^{\frac{\rho}{\rho+1}} \quad \text{.............................(9)}$$

in the right half-plane and, in particular,

$$|G(\pm iy)| < K_2(\delta) \exp |y|^{\frac{\rho}{\rho+1}} \quad \text{........................(10)}$$

on the imaginary axis.

The property of finite exponential order is critical in regard to the coefficient theory of the almost isolated singularity because, in general,

$$\log |G(n)| > -\gamma n \quad \text{.................................(11)}$$

for prescribed positive γ, for a sequence of n of upper density 1, at least, while if the singularity is of finite exponential order then the inequality (11) holds for a sequence of n of density 1.† While the upper density property is, in the general case, minimal for the class of almost isolated singularity and may be exceeded (as happens in the case of finite exponential order), Pólya has given examples ‡ for which this property is the best possible.

The condition on $G(z)$ which makes it possible for (11) to hold for a sequence of n of density 1, rather than upper density 1, for functions of at most order 1, minimum type, in the right half-plane, is that of being of convergent minimum type on the boundary, so as to satisfy

$$\int_0^\infty \frac{\log^+ |G(\pm it)|}{1+t^2}\, dt < +\infty, \text{ or } \int_1^\infty \log^+ |G(\pm iy)| \frac{dy}{y^2} < +\infty \quad \text{...(12)}$$

on the arms of the angle in which it is of order 1, minimum type.§

This condition is clearly satisfied when the almost isolated singularity of $f(z)$ at $z = 1$ is of finite exponential order for then, in accordance with Lemma 1, $G(z)$ is of order less than 1 in the right half-plane, as is seen from (9) and (10).

However, it is possible to satisfy (12) even if $G(z)$ is of order 1, provided that it is of convergent minimum type. In this case $f(z)$ would satisfy (6) with $\rho = \infty$, subject to certain restrictions. The determination of these restrictions is the object of the present paper.

3. Restricted order

In place of (6) we now suppose that, on the contour C_1,

$$|f(z)| < e^{\psi(R)/R} \quad \text{..................................(13)}$$

where $R = 1/\delta$ and $\psi(R)$ is a positive increasing differentiable real function for increasing positive R such that $\log \psi(R)$ is convex. ‖ We now restrict the rate of growth of $\psi(R)$ by the condition

$$\int_1^\infty \frac{d \log \psi(R)}{R} < +\infty. \quad \text{............................(14)}$$

† (5), 777, Satz X; (4), 222.
‡ (5), 763-5.
§ (3), 430-31; (4), 204.
‖ In the sense that if $0 < R_1 < R_2$ then the points on the straight segment joining $\log \psi(R_1)$ to $\log \psi(R_2)$ lie below the curve $y = \log \psi(x)$ unless it is a straight line.

Thus $\psi(R)$ cannot be as large as e^R but it can be larger than $e^{R^{1-\varepsilon}}$ with $\varepsilon > 0$. Hence, from (13), $f(z)$ can be of infinite order. From (2) and (13) it follows that

$$|f_1(z)| < e^{\psi(R)/R} \quad \text{.................................(15)}$$

since $f^*(z)$ is bounded on C_1.

Now consider the integral for $G(z)$ in (5) but with the modified contour C_1. The value of the integral is largely determined by the behaviour of $f_1(w)$ on that part of the contour C_1 close to the segment $(1, k)$. Here (15) holds and $w = e^{-z\log(1-\delta e^{i\chi})}$ with $-\frac{1}{2}\pi \leqq \chi \leqq \frac{1}{2}\pi$. Thus, as in (8) with $|z| = r$,

$$|G(z)| < K(\delta) \exp\left(\frac{\psi(R)}{R} + \frac{r}{R}\right). \quad \text{.......................(16)}$$

To maximise the integral we now take $r = \psi(R)$. Let $R = \phi(r)$ be the relation inverse to this. Then condition (14) becomes

$$\int_1^\infty \frac{d\log r}{R} = \int_1^\infty \frac{dr}{Rr} = \int_1^\infty \frac{dr}{r\phi(r)} < +\infty. \quad \text{.................(17)}$$

Now with $r = \psi(R)$, $R = \phi(r)$, relation (16) becomes

$$|G(z)| < K(\delta) e^{r/\phi(r)} \quad \text{.....................................(18)}$$

and so

$$\int_1^\infty \frac{\log^+ |G(z)|}{r^2}\, dr < \int_1^\infty \frac{K dr}{r\phi(r)} < +\infty$$

from (17) and (18), where K is the finite upper bound of $K(\delta)$, $K_1(\delta)$, $K_2(\delta)$.

Putting $z = \pm iy$, we now obtain the required convergence condition on $G(z)$ when it is of order 1, minimum type, and $f(z)$ is of infinite order, namely

$$\int_1^\infty \frac{\log^+ |G(\pm iy)|}{y^2}\, dy < +\infty. \quad \text{.......................(19)}$$

We note that $\phi(r)$ is also an increasing function for positive increasing r and that, from (18),

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \log |G(r)| \leqq \lim_{r\to\infty} \frac{1}{\phi(r)} = 0.$$

Since $G(n) = c_n - c_n^*$, from (4), the equality sign must hold and so

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \log |G(r)| = 0. \quad \text{.............................(20)}$$

Conditions (19) and (20), together with the fact that $G(z)$ is of order 1, minimum type, in the right half-plane enable us to make use of well-known theorems of M. L. Cartwright † set out in the lemma below.

Lemma 2. *If* $G(z)$ *is regular and of order* 1, *minimum type, in the right*

† (1), Th. III using the method of (2), Th. II.

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half-plane and if

$$\text{(i)} \int_1^{\infty} \log^+ |G(\pm iy)| \frac{dy}{y^2} < +\infty, \quad \text{(ii)} \lim_{r\to\infty} \frac{1}{r} \log |G(r)| = 0,$$

then along every radius $\arg z = \theta$ *inside the half-plane and for every* $\gamma > 0$, $\log |G(re^{i\phi})| > -\gamma r$ *for a set of r of linear density* 1. *In particular*, (11) *is satisfied for a sequence of n of density* 1.

If $f(z)$ has at $z = 1$ an almost isolated singularity and satisfies in the neighbourhood of this singularity conditions (13) and (14) then the almost isolated singularity will be said to be of *restricted order*. Clearly an almost isolated singularity of finite exponential order is also of restricted order. The results concerning this singularity are embodied in the following theorem:

Theorem. *If* $f(z) = \sum_0^{\infty} c_n z^n$ *has at* $z = 1$ *an almost isolated singularity of restricted order, and if this is the only singularity of* $f(z)$ *on its circle of convergence then, for every* $\gamma > 0$, $|c_n| > e^{-\gamma n}$ *for a sequence of n of density* 1.

The approximate interpolation in (4), together with the results of Lemma 2, lead directly to the form of result given in the theorem.

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