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ESSENTIAL AMENABILITY OF ABSTRACT SEGAL ALGEBRAS

H. SAMEA

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Abstract

A number of well-known results of Ghahramani and Loy on the essential amenability of Banach algebras are generalized. It is proved that a symmetric abstract Segal algebra with respect to an amenable Banach algebra is essentially amenable. Applications to locally compact groups are given.

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1. Introduction

Ghahramani and Loy [5] introduced the notion of an essential amenable Banach algebra and investigated the essential amenability of certain Banach algebras such as symmetric Segal algebras. In this paper the notion of essential amenability is studied for a larger class of Banach algebras such as symmetric abstract Segal algebras.

The organization of this paper is as follows. Section 2 is devoted to preliminaries and notation needed throughout the rest of the paper. In Section 3 the structure of abstract Segal algebras is studied. In Section 4, by applying the results of Section 3, [5, Theorem 7.1] is generalized. Indeed, it is proved that any symmetric abstract Segal algebra with respect to an amenable Banach algebra is essentially amenable. As a consequence, it is shown that any symmetric abstract Segal algebra with respect to the group algebra $L^1(G)$ of an amenable locally compact group G is essentially amenable. This result generalizes [5, Corollary 7.1]. As a consequence it is shown that the convolution Banach algebra $L^\infty(G)$, for any compact group G, is essentially amenable. Moreover, this Banach algebra is amenable if and only if G is finite.

2. Preliminaries

Let A be a Banach algebra, and let B and C be nonempty subsets of A. Define $B.C = \{bc \mid b \in B, c \in C\}$, and BC to be the linear span of $B.C$. Write B^2 for BB.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Let X be a Banach A-bimodule. A bounded linear map $D : A \to X$ is an X-*derivation* if

$$D(ab) = D(a).b + a.D(b) \quad (a, b \in A).$$

For example, take $x \in X$, and set $ad_x^A(a) = a.x - x.a$ $(a \in A)$. Then ad_x^A is a derivation; such derivations are termed *inner* derivations. A Banach algebra A is *amenable* if every derivation from A into X^* is inner for all Banach A-bimodules X. An A-bimodule X is *neo-unital* if $X = A.X.A$, where $A.X.A = \{a.x.b \mid a, b \in A, x \in X\}$. Recall from [5] that a Banach algebra A is *essentially amenable* if for any neo-unital A-bimodule X, every derivation $D : A \to X^*$ is inner.

Let $(A, \|.\|_A)$ be a Banach algebra. Recall from [4] that a Banach algebra $(B, \|.\|_B)$ is an *abstract Segal algebra* with respect to A if:

(i) B is a dense left ideal in A;
(ii) there exists $M > 0$ such that $\|b\|_A \le M\|b\|_B$ for each $b \in B$;
(iii) there exists $C > 0$ such that $\|ab\|_B \le C\|a\|_A\|b\|_B$ for each $a, b \in B$.

A linear subspace $S(G)$ of the convolution group algebra $L^1(G)$ of a locally compact group G is said to be a *Segal algebra* if it satisfies the following conditions.

(i) $S(G)$ is dense in $L^1(G)$.
(ii) $S(G)$ is a Banach space under some norm $\|.\|_{S(G)}$ and there exists $M > 0$ such that $\|f\|_1 \le M\|f\|_{S(G)}$ for all $f \in S(G)$.
(iii) $S(G)$ is left translation invariant and the map $x \mapsto \delta_x * f$ from G into $S(G)$ is continuous for all $f \in S(G)$.
(iv) $\|\delta_x * f\|_{S(G)} = \|f\|_{S(G)}$, for all $f \in S(G)$ and $x \in G$.

That a Segal algebra on G is an abstract Segal algebra is well known; see [4, p. 359].

A Segal algebra $S(G)$ is *symmetric* if it is right translation invariant, and for all $f \in S(G)$, $\|f * \delta_x\|_{S(G)} = \|f\|_{S(G)}$ $(x \in G)$, and the map $x \mapsto f * \delta_x$ from G into $S(G)$ is continuous.

3. Abstract Segal algebras

We start this section with the following lemma.

LEMMA 3.1. *Let A be a Banach algebra such that $A.A = A$, B an abstract Segal algebra with respect to A. Then $\overline{B^2}^{\|\cdot\|_B}$ is an abstract Segal algebra with respect to A.*

PROOF. Since $A = A.A$, so $A = A.A = \overline{B}^{\|\cdot\|_A}\overline{B}^{\|\cdot\|_A} \subseteq \overline{B.B}^{\|\cdot\|_A}$. Therefore $B.B$ is dense in A, and so B^2 is dense in A. Let $a \in A$. On the one hand, $a(B.B) = (a.B).B \subseteq B.B$, and so $aB^2 \subseteq B^2$. On the other hand, since B is an abstract Segal algebra, there exists $C > 0$ such that $\|ab\|_B \le C\|a\|_A\|b\|_B$ $(b \in B)$. Hence, the mapping $\lambda_a : B \to B$; $b \mapsto ab$ is continuous. Therefore,

$$a\overline{B^2}^{\|\cdot\|_B} = \lambda_a(\overline{B^2}^{\|\cdot\|_B}) \subseteq \overline{\lambda_a(B^2)}^{\|\cdot\|_B} \subseteq \overline{B^2}^{\|\cdot\|_B}.$$

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This proves that $\overline{B^2}^{\|\cdot\|_B}$ is a left ideal in A. It follows that $\overline{B^2}^{\|\cdot\|_B}$ is an abstract Segal algebra with respect to A. □

LEMMA 3.2. *Let A be a Banach algebra with a bounded left approximate identity, B an abstract Segal algebra with respect to A. Then there exists a left approximate identity for $\overline{B^2}^{\|\cdot\|_B}$ which is also a bounded left approximate identity for A.*

PROOF. Let (e_α) be a bounded left approximate identity for A. By Cohen's factorization theorem [7, Theorem 32.23], $A.A = A$. By the proof of Lemma 3.1, $B.B$ is dense in A, so for each $\varepsilon > 0$ and α, there exists $e_{(\alpha,\varepsilon)} \in B.B$ such that $\|e_\alpha - e_{(\alpha,\varepsilon)}\|_A \le \min\{\varepsilon, 1\}$. Then $(e_{(\alpha,\varepsilon)})_{(\alpha,\varepsilon)}$ is a directed net in $B.B$ with $(\alpha_1, \varepsilon_1) \preceq (\alpha_2, \varepsilon_2)$ if and only if $\alpha_1 \preceq \alpha_2$, and $\varepsilon_1 \ge \varepsilon_2$. Moreover, $\sup_{(\alpha,\varepsilon)} \|e_{(\alpha,\varepsilon)}\|_A \le \sup_\alpha \|e_\alpha\|_A + 1$. For $a \in A$,

$$\|a - e_{(\alpha,\varepsilon)}a\|_A \le \|a - e_\alpha a\|_A + \|e_\alpha a - e_{(\alpha,\varepsilon)}a\|_A \le \|a - e_\alpha a\|_A + \varepsilon\|a\|_A,$$

and so

$$\limsup_{(\alpha,\varepsilon)} \|a - e_{(\alpha,\varepsilon)}a\|_A \le \lim_\alpha \|a - e_\alpha a\|_A + \lim_{\varepsilon\to 0} \varepsilon\|a\|_A = 0.$$

This shows that $(e_{(\alpha,\varepsilon)})$ is a bounded left approximate identity for A. Since B is an abstract Segal algebra, there exists $C > 0$ such that $\|ab\|_B \le C\|a\|_A\|b\|_B$ $(b \in B)$. Now, if $b_1, b_2 \in B$, then

$$\limsup_{(\alpha,\varepsilon)} \|e_{(\alpha,\varepsilon)}(b_1b_2) - (b_1b_2)\|_B \le C \limsup_{(\alpha,\varepsilon)} \|e_{(\alpha,\varepsilon)}b_1 - b_1\|_A\|b_2\|_B = 0,$$

and so for each $b \in B^2$, $\lim_{(\alpha,\varepsilon)} e_{(\alpha,\varepsilon)}b = b$ (note that B^2 is the linear span of $B.B$). Now if $b \in \overline{B^2}^{\|\cdot\|_B}$ and $\delta > 0$, then there exist $b_\delta \in B^2$ such that $\|b - b_\delta\|_B < \delta$. Therefore,

$$\begin{aligned}\|b - e_{(\alpha,\varepsilon)}b\|_B &\le \|b - b_\delta\|_B + \|b_\delta - e_{(\alpha,\varepsilon)}b_\delta\|_B + \|e_{(\alpha,\varepsilon)}b_\delta - e_{(\alpha,\varepsilon)}b\|_B \\ &\le \delta + \|b_\delta - e_{(\alpha,\epsilon)}b_\delta\|_B + C\|e_{(\alpha,\epsilon)}\|_A\delta,\end{aligned}$$

and hence $\limsup_{(\alpha,\varepsilon)} \|b - e_{(\alpha,\varepsilon)}b\|_B \le (1 + C \sup_{(\alpha,\varepsilon)} \|e_{(\alpha,\varepsilon)}\|_A)\delta$. Since $\delta > 0$ is arbitrary, so $(e_{(\alpha,\varepsilon)})$ is a left approximate identity for $\overline{B^2}^{\|\cdot\|_B}$. □

DEFINITION 3.3. Let $(A, \|.\|_A)$ be a Banach algebra. A Banach algebra $(B, \|.\|_B)$ is a symmetric abstract Segal algebra with respect to A if:

(i) B is a dense ideal in A;
(ii) there exists $M > 0$ such that $\|b\|_A \le M\|b\|_B$ for each $b \in B$;
(iii) there exists $C > 0$ such that $\|ab\|_B, \|ba\|_B \le C\|a\|_A\|b\|_B$ for each $a, b \in B$.

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REMARK 3.4. Let G be a locally compact group, and $S(G)$ be a symmetric Segal algebra. By a method similar to [8, Proposition 1, p. 19], one can prove that for each $f \in L^1(G)$ and $g \in S(G)$, $f * g$, $g * f \in S(G)$, and

$$\|f * g\|_{S(G)}, \|g * f\|_{S(G)} \leq \|f\|_1 \|g\|_{S(G)}.$$

This shows that $S(G)$ is a symmetric abstract Segal algebra with respect to $L^1(G)$.

The following theorem is the main result of this section.

THEOREM 3.5. *Let A be a Banach algebra with a bounded approximate identity, B a symmetric abstract Segal algebra with respect to A. Then $\overline{B^2}^{\|\cdot\|_B}$ is a symmetric Segal algebra with respect to A such that there exists an approximate identity for $\overline{B^2}^{\|\cdot\|_B}$ which is also a bounded approximate identity for A.*

PROOF. Let $(e_{(\alpha,\varepsilon)})$ be as in the proof of Lemma 3.2. By a method similar to the proof of Lemma 3.2, it can be proved that $(e_{(\alpha,\varepsilon)})$ is also a right approximate identity for the Banach algebras A and $\overline{B^2}^{\|\cdot\|_B}$. The proof of the rest of the thorem is similar. □

4. Amenability and essential amenability of symmetric abstract Segal algebras

PROPOSITION 4.1. *Let A be a Banach algebra, and I a closed subalgebra of A that contains $A.A$. If I is essentially amenable, then so is A.*

PROOF. Let X be a neo-unital Banach A-bimodule and D be a derivation from A into X^*. Since X is a neo-unital Banach A-bimodule,

$$X = A.X.A = A.(A.X.A).A = (A.A).X.(A.A) \subseteq I.X.I \subseteq X.$$

Hence X is also a neo-unital Banach I-bimodule. It is clear that $D_{|I}$ is a derivation from I into X^*. Since I is essentially amenable, there exists $\xi \in X^*$ such that $D_{|I} = ad_\xi^I$. Define $\overline{D} = D - ad_\xi^A$. Clearly $\overline{D}$ is a derivation from A into X^*, and $\overline{D}(I) = \{0\}$. Let $a \in A$ and $x \in X$. Since X is a neo-unital I-bimodule, there exist $b \in I$ and $y \in X$ such that $x = b.y$. Now b, $ab \in I$, so

$$\overline{D}(a)(x) = \overline{D}(a)(b.y) = (\overline{D}(a).b)(y) = \left(\overline{D}(ab) - a.\overline{D}(b)\right)(y) = 0.$$

Hence $\overline{D} = 0$, and so $D = ad_\xi^A$. Therefore A is essentially amenable. □

EXAMPLE 4.2.

(a) Let A be a nonzero Banach space. Define $a.b = 0$ for all $a, b \in A$. With this multiplication, A is a Banach algebra. Since $I = A.A = 0$ is amenable, A is essentially amenable.

(b) Let S be a null semigroup, that is S has an element 0 such that $st = 0$ for all $s, t \in S$. Then the convolution Banach algebra $\ell^1(S)$ is essentially amenable, since $I = \ell^1(S) * \ell^1(S) = \mathbb{C}\delta_0$ is amenable.

LEMMA 4.3. *Let A be a Banach algebra with a bounded left approximate identity, B an abstract Segal algebra with respect to A. Then the following statements are equivalent:*

(i) *B has a bounded left approximate identity.*
(ii) *$B = A$ (understood here as sets).*
(iii) *B is Banach algebra isomorphic to A.*

PROOF. (i) $\Rightarrow$ (ii). This is a result of Burnham [2] (see also [4, Lemma 1.1]).

(ii) $\Rightarrow$ (iii). Let (e_α) be a bounded left approximate identity for A. Since $A = B$, and B is an abstract Segal algebra, there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ $(a, b \in A)$. So for each $a \in A$,

$$\|a\|_A = \lim \|e_\alpha a\|_A \leq C \liminf_\alpha \|e_\alpha\|_A \|a\|_B \leq C\left(\sup_\alpha \|e_\alpha\|_A\right)\|a\|_B.$$

Now, by the inverse mapping theorem (see [3, Theorem 12.5 of Section III]), B is Banach algebra isomorphic to A.

(iii) $\Rightarrow$ (i). Since B is Banach algebra isomorphic to A, and A has a bounded left approximate identity, it follows that B also has a bounded left approximate identity. □

THEOREM 4.4. *Let B be a symmetric abstract Segal algebra with respect to an amenable Banach algebra A. Then B is essentially amenable. Moreover, B is amenable if and only if $B = A$ (understood here as sets).*

PROOF. Since A is amenable, by [9, Proposition 2.2.1] it has a bounded approximate identity. If $B = A$ then, by Lemma 4.3, B is Banach algebra isomorphic to A, and so is amenable. Conversely if B is amenable, then B has a bounded approximate identity, and so, by Lemma 4.3, $B = A$.

Now, suppose $B \neq A$. By Theorem 3.5 $\overline{B^2}^{\|\cdot\|_B}$ is a nonclosed ideal in A that has an approximate identity which is also an approximate identity for A. Since, by Theorem 3.5, $\overline{B^2}^{\|\cdot\|_B}$ is a symmetric abstract Segal algebra with respect to A, there exists $C > 0$ such that

$$\|ab\|_{\overline{B^2}^{\|\cdot\|_B}}, \|ba\|_{\overline{B^2}^{\|\cdot\|_B}} \leq C\|a\|_A\|b\|_{\overline{B^2}^{\|\cdot\|_B}} \quad (a \in A, b \in \overline{B^2}^{\|\cdot\|_B}).$$

Therefore, by [5, Theorem 7.1], $\overline{B^2}^{\|\cdot\|_B}$ is essentially amenable. Now, since $\overline{B^2}^{\|\cdot\|_B}$ is an ideal of B that contains $B.B$, B is essentially amenablenby Proposition 4.1. □

LEMMA 4.5. *Let A be a Banach algebra with a bounded approximate identity, J a dense ideal in A. Suppose that J is a Banach algebra under a norm $\|.\|_J$ such that*

$$\|ab\|_J, \|ba\|_J \leq \|a\|_A\|b\|_J \quad (a \in A, b \in J).$$

Then J is a symmetric abstract Segal algebra with respect to A.

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PROOF. Let (e_α) be a bounded approximate identity for A, and $M = \sup_\alpha \|e_\alpha\|_A$. If $b \in J$, then

$$\|b\|_A = \lim_\alpha \|e_\alpha b\|_A \leq \left(\sup_\alpha \|e_\alpha\|_A\right)\|b\|_J = M\|b\|_J$$

and this shows that J is a symmetric abstract Segal algebra with respect to A. □

The following theorem is a generalization of [5, Theorem 7.1].

THEOREM 4.6. *Let A be an amenable Banach algebra, J a dense ideal in A. Suppose that J is a Banach algebra under a norm $\|.\|_J$ such that*

$$\|ab\|_J, \|ba\|_J \leq \|a\|_A\|b\|_J \quad (a \in A, b \in J).$$

Then J is essentially amenable. Moreover, J is amenable if and only if $J = A$ (here means as sets).

PROOF. Since A is amenable, by [9, Proposition 2.2.1] it has a bounded approximate identity. By Lemma 4.5, J is a symmetric abstract Segal algebra with respect to the amenable Banach algebra A, and so by Theorem 4.4 is essentially amenable. Moreover, J is amenable if and only if $J = A$. □

The following theorem is a generalization of [5, Corollary 7.1(2)].

THEOREM 4.7. *Any symmetric abstract Segal algebra $S(G)$ with respect to $L^1(G)$, for any amenable locally compact group G, is essentially amenable. Moreover, $S(G)$ is amenable if and only if $S(G) = L^1(G)$.*

PROOF. By Johnson's theorem (see [9, Theorem 2.1.8]) $L^1(G)$ is amenable. The rest of the result is a corollary of Theorem 4.4. □

EXAMPLE 4.8. (a) Let G be a compact group with the normalized Haar measure. Then the convolution Banach algebra $L^\infty(G)$ is a symmetric abstract Segal algebra with respect to $L^1(G)$. Since $L^1(G)$ is amenable, by Theorem 4.7 the convolution Banach algebra $L^\infty(G)$ is essentially amenable.

(b) The convolution Banach algebra $L^\infty(G)$ is amenable if and only if $L^1(G) = L^\infty(G)$. But, in this case, by [6, Theorem 20.16],

$$L^\infty(G) = L^1(G) = L^1(G) * L^1(G) = L^1(G) * L^\infty(G) \subseteq C(G),$$

and so, by [7, Lemma 37.3], G is finite.

(c) The convolution Banach algebra $L^\infty(G)$ is a Segal algebra on G if and only if G is finite. To see this, suppose that $L^\infty(G)$ is a Segal algebra. On the one hand, by [8, Proposition 1(i), p. 34], $L^\infty(G)$ has a left approximate identity which is bounded in $\|.\|_{L^1(G)}$-norm. On the other hand $L^\infty(G)$ is a Banach left $L^1(G)$-module with the convolution giving the left module multiplication. Hence, by Cohen's factorization theorem (see [1], Theorem 10, p. 61), $L^\infty(G) = L^1(G) * L^\infty(G)$. By applying the same method as (b), one can prove that G is finite. Conversely, if G is

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finite, then $\ell^\infty(G) = \ell^1(G)$, and so $\ell^\infty(G)$ is a Segal algebra on G. This shows that the convolution Banach algebra $L^\infty(G)$ is not always a Segal algebra on G.

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H. SAMEA, Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran
e-mail: h.samea@basu.ac.ir