A REMARK ON THE ESSENTIAL SPECTRA OF QUASI-SIMILAR DOMINANT CONTRACTIONS

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Introduction. We consider operators, i.e. bounded linear transformations, on an infinite dimensional separable complex Hilbert space H into itself. The operator A is said to be *dominant* if for each complex number λ there exists a number $M_\lambda(\geqslant 1)$ such that $\|(A-\lambda)^*x\| \leqslant M_\lambda \|(A-\lambda)x\|$ for each $x \in H$. If there exists a number $M \geqslant M_\lambda$ for all λ, then the dominant operator A is said to be *M-hyponormal*. The class of dominant (and M-hyponormal) operators was introduced by J. G. Stampfli during the seventies, and has since been considered in a number of papers, amongst then [7], [11]. It is clear that a 1-hyponormal is hyponormal. The operator A is said to be *quasi-normal* if A commutes with A^*A, and we say that A is *subnormal* if A has a normal extension. It is known that the classes consisting of these operators satisfy the following strict inclusion relation:

$$\text{quasi-normal} \subset \text{subnormal} \subset \text{hyponormal} \subset M\text{-hyponormal} \subset \text{dominant}.$$

We say that the operator X is a *quasi-affinity* if both X and X^* have dense range. Given operators A, B and X, let $C(A, B)X = AX - XB$. The operators A and B are said to be *quasi-similar*, denoted $A \sim B$, if there exist quasi-affinities X and Y such that $C(A, B)X = C(B, A)Y = 0$, and we say that $A \overset{d}{\sim} B$ if there exist operators X and Y with dense range such that $C(A, B)X = C(B, A)Y = 0$. Given the operator A, let $\sigma(A)$ and $\sigma_e(A)$ denote, respectively, the spectrum and the essential spectrum of A.

The problem of the equality of the spectra, and the essential spectra, of quasi-similar operators has been considered by a number of authors in the recent past (see [1], [2], [5], [8], [9], [10], [11]). Recall that each operator A has a direct sum decomposition $A = A_1 \oplus A_2$, where A_1 is normal and A_2 is pure. (The operator A_2 is said to be *pure* if there exists no non-trivial reducing subspace M of A_2 such that $A_2 \mid M$, the restriction of A_2 to M, is normal. It is to be noted here that either component in the direct sum $A = A_1 \oplus A_2$ may be absent.) Given quasi-similar dominant operators A and B such that $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, it is fairly easily seen that A_1 and B_1 are unitarily equivalent, and that $A_2 \overset{d}{\sim} B_2$ [11, Theorem 1.1]. The problem of determining whether $\sigma_e(A) = \sigma_e(B)$ (or, $\sigma(A) = \sigma(B)$) then reduces to that of determining whether $\sigma_e(A_2) = \sigma_e(B_2)$ (respectively, $\sigma(A_2) = \sigma(B_2)$). It is to be noted here that the operators (with dense ranges) intertwining A_2 and B_2 need not be quasi-affinities, even in the case in which A_2 and B_2 are quasi-normal [9]. However, the fact that these intertwining operators lack injectivity seems to play but a minor role. It is known that if A_2 and B_2 are quasi-normal, then $\sigma_e(A_2) = \sigma_e(B_2)$ [9] (and that if A_2 and B_2 are hyponormal, then $\sigma(A_2) = \sigma(B_2)$ [1], [8]): the problem, however, remains unsolved for subnormal A_2 and B_2. Assuming additional hypotheses, such as that one of the quasi-affinities intertwining A and B is compact, L. R. Williams [10] has shown that $\sigma_e(A) = \sigma_e(B)$ if A and B are hyponormal.

(That a similar result holds for k-quasihyponormal A and B has been shown by B. C. Gupta [**5**].) The result for hyponormal A and B extends to dominant A and B if A and B satisfy Dunford's Condition C [**11**, Theorem 2.4].

In this note we consider contractions A and B, and show that if A is injective, then the hypothesis that the dominant A and B satisfy Dunford's Condition C can be dispensed with in [**11**, Theorem 2.4]. Indeed we show more. It is known that the c.n.u. (=completely non-unitary) part of a dominant contraction is of the class $C_{.0}$ of contractions [**4**]. We show that if A and B are quasi-similar contractions with $C_{.0}$ c.n.u. parts, if one of the intertwining quasi-affinites is compact, and if A is injective, then A and B are hyponormal, and so $\sigma_e(A)=\sigma_e(B)$.

Results. A contraction A is said to be c.n.u. if there exists no non-trivial reducing subspace M of A such that $A \mid M$ is unitary. Recall that every contraction A has a direct sum decomposition $A=A_1 \oplus A_2$, where A_1 is unitary and A_2 is c.n.u. (and where either of the components may be absent). We say that the contraction A belongs to the class $C_{.0}$ (class $C_{.1}$) of contractions if $\|A^{*n}x\| \to 0$ as $n \to \infty$ (respectively, $0<\inf_n \|A^{*n}x\|$) for each non-trivial $x \in H$. The classes $C_{0.}$ and $C_{1.}$ are defined by considering A^* instead of A, and we say that $A \in C_{\alpha\beta}$, where α, $\beta=0$, 1, if $A \in C_{\alpha.} \cap C_{.\beta}$. In the following we shall denote the point spectrum of A by $\sigma_p(A)$, the closure of the range of A by $\overline{\operatorname{ran}}\, A$, and the orthogonal complement of the kernel of A by $\ker^{\perp} A$. We now prove our result for the case in which A and B are dominant: the case of general contractions with $C_{.0}$ c.n.u. parts will be seen to follow from the proof of this case.

THEOREM 1. *Let A and B be dominant contractions such that $0 \notin \sigma_p(A)$ and $A \sim B$. If one of the quasi-affinities intertwining A and B is compact, then $\sigma_e(A)=\sigma_e(B)$.*

Proof. Let X and Y be the quasi-affinities, with Y compact, such that $C(A,B)X=C(B,A)Y=0$. We show that A and B are hyponormal, and this is achieved by showing that the c.n.u. parts of A and B are hyponormal. Thus, decompose A and B into their unitary and c.n.u. parts by $A=A_1 \oplus A_0$ and $B=B_1 \oplus B_0$. Since A and B are dominant, A_0 and $B_0 \in C_{.0}$ [**4**], and so have triangulations

$$\begin{bmatrix} A_2 & * \\ 0 & A_3 \end{bmatrix} \text{ and } \begin{bmatrix} B_2 & * \\ 0 & B_3 \end{bmatrix} \text{ of the type } \begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix}$$

[**6**, p. 75]. Let X and Y have the corresponding matrix representations

$$X=[X_{ij}]_{i,j=1}^3 \quad \text{and} \quad Y=[Y_{ij}]_{i,j=1}^3.$$

Then, since A_1 and $B_1 \in C_{11}$, A_2 and $B_2 \in C_{00}$, and A_3 and $B_3 \in C_{10}$, it follows from the equations $0=C(A_1,B_2)X_{12}=C(B_1,A_2)Y_{12}=C(A_2,B_1)X_{21}=C(B_2,A_1)Y_{21}=C(A_3,B_1)X_{31}=C(B_3,\ A_1)_{31}=C(A_3,\ B_2)X_{32}=C(B_3,\ A_2)Y_{32}$ that $X_{12}=Y_{12}=X_{21}=Y_{21}=X_{31}=Y_{31}=X_{32}=Y_{32}=0$. (Sample argument: since A_1 is unitary and $B_3 \in C_{10}$, $\|Y_{31}^*x\|=\|A_1^{*n}Y_{31}^*x\|=\|Y_{31}^*B_3^{*n}x\| \leq \|Y_{31}^*\|\,\|B_3^{*n}x\| \to 0$ as $n \to \infty$.) Consequently, X and Y

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have the representations

$$\begin{bmatrix} X_{11} & 0 & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} \text{ and } \begin{bmatrix} Y_{11} & 0 & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix},$$

where X_{33} and Y_{33} have dense range (with Y_{33} compact). We now consider the equations $C(A_3, B_3)X_{33} = C(B_3, A_3)Y_{33} = 0$, and show that A_3 and B_3 are in fact non-existent.

Set $X_{33}Y_{33} = S_1$; then S_1 is a compact operator with dense range such that $C(A_3, A_3)S_1 = 0$. Since H is separable and $A_3 \in C_{10}$, there exists an isometry V and a quasi-affinity S_0 such that $C(V, A_3)S_0 = 0$ [**6**, Proposition II.3.5]. Set $S_0S_1 = S$; then S is a compact operator with dense range such that $C(V, A_3)S = 0$, and hence that $A_3^*S^*SA_3 = S^*S$. This implies that $\overline{\text{ran}}\, S^*S = \ker^\perp S$ reduces A_3 and $A_3/\ker^\perp S$ is unitary (see [**3**, Theorem 8 and Corollary 6.5]). Since A_3 is c.n.u., we conclude that A_3 is non-existent. By symmetry, B_3 is also non-existent. Consequently, $X_{13} = Y_{13} = 0$.

The preceding argument shows that $C(A_0, B_0)X_0 = C(B_0, A_0)Y_0 = 0$, where

$$X_0 = \begin{bmatrix} X_{22} & X_{23} \\ 0 & X_{33} \end{bmatrix} \text{ and } Y_0 = \begin{bmatrix} Y_{22} & Y_{23} \\ 0 & Y_{33} \end{bmatrix}$$

are quasi-affinities (with Y_0 compact). Set $X_0Y_0 = T$; then $(:H' \to H'$, say) is a compact quasi-affinity such that $C(A_0, A_0)T = 0$. Clearly, $0 \notin \sigma_p(A_0)$ and $0 \notin \sigma_p(B_0)$. Since $A_0 \in C_{.0}$, the sequence $\{A_0^n TT^* A_0^{*n}\}$ converges strongly to 0, and so, since $A_0TT^*A_0^* = TA_0A_0^*T^* \leq TT^*$, TT^* is a 'pure solution' (in the sense of [**3**, p. 24]) of $A_0TT^*A_0^* \leq TT^*$. Hence there exists a unilateral shift U (on a Hilbert space $\mathcal{H}$) and an operator C (on $H' \to \mathcal{H}$) such that $CA_0^* = U^*C$ and $TT^* = C^*C$ [**3**, Theorem 5]. Clearly, there exists an isometry $V : \mathcal{H} \to H'$ (indeed, V is a unitary on $\overline{\text{ran}}\, C \to H'$) such that $T^* = VC$, and so that

$$A_0^*T^* = T^*A_0^* = VCA_0^* = VU^*V^*VC = VU^*V^*T^*.$$

Since T^* is a quasi-affinity, this implies that $A_0^* = VU^*V^*$, and hence that A_0 is hyponormal. Consequently, A is hyponormal. By symmetry, B is also hyponormal.

It now follows that A and B are quasi-similar hyponormal contractions. Hence $\sigma_e(A) = \sigma_e(B)$, by [**10**], as was to be proved.

Remarks. (1). The proof of the theorem shows that $A_0 \sim B_0$: this does not in any way imply that the pure parts of A and B are quasi-similar. As mentioned earlier, it can be seen using a routine argument that the pure parts of A and B, say A_4 and B_4, satisfy $A_4 \overset{d}{\sim} B_4$.

(2). The argument in the proof above leading to the conclusion that A_0 is hyponormal fails for a general quasi-affinity. This is easily seen upon letting $\{e_n\}$, $-\infty < n < \infty$, be an orthonormal basis of H, (the dominant contraction) A_0 be the bilateral shift $A_0e_n = 2^{-|n|}e_{n+1}$ and T be the quasi-affinity defined by $Te_n = 2^{-1}e_n$.

(3). The hypothesis that A is injective seems to be required in Theorem 1 (just as it seems that some such hypothesis is required in [**3**, Theorem 5]) in the sense that the unilateral shift U may otherwise fail to exist. For suppose that A_0 is not injective and that $A_0 = VUV^*$. Then there exists a non-trivial x such that $A_0x = 0$, and hence that $V^*x = 0$. Since $T = C^*V^*$, we have that $Tx = 0$, i.e., T is not injective.

(4). A scrutiny of the proof above shows that the fact that A and B are dominant plays a part in the proof only through the fact that they have $C_{.0}$ c.n.u. parts. Hence Theorem 1 generalises to the following result.

THEOREM 2. *Let A and B be contractions with $C_{.0}$ c.n.u. parts such that $0 \notin \sigma_p(A)$ and $A \sim B$. If one of the quasi-affinities intertwining A and B is compact, then $\sigma_e(A) = \sigma_e(B)$.*

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