SEMI - HAUSDORFF SPACES

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It is well-known that in a Hausdorff space, a sequence has at most one limit, but that the converse is not true. The condition that every sequence have at most one limit will be called the semi-Hausdorff condition. We will prove that the semi-Hausdorff condition is strictly stronger than the T_1-axiom and is thus between the T_1 and T_2 axioms. In this note, we investigate into some properties of the spaces satisfying the semi-Hausdorff condition.

DEFINITION 1. A topological space X is said to be <u>semi-Hausdorff</u> if and only if every sequence in X has at most one limit.

THEOREM 1. Every Hausdorff space is semi-Hausdorff, but not conversely.

The first part is too well-known to need a proof. The second is proved by considering an uncountable set with the countable complement topology.

THEOREM 2. Every semi-Hausdorff space is T_1, but not conversely.

<u>Proof.</u> Let X be semi-Hausdorff and $x, y \in X$ be distinct points. The sequence $(x, x, x, \ldots)$ converges to x and hence cannot converge to y. Hence there exists a neighborhood U_y which does not contain x.

For the second part, let X be an infinite set, with the finite complement topology. This space is T_1, but every sequence of distinct points converges to every point in the space.

THEOREM 3. The semi-Hausdorff property is hereditary.

<u>Proof.</u> Obvious.

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THEOREM 4. The semi-Hausdorff property is invariant under every one-one onto open mapping.

Proof. Let $f: X \to Y$ be a 1-1 onto open mapping, with X semi-Hausdorff. Suppose (y_n) is a sequence in Y converging to y' and y''. Since f^{-1} is continuous, the sequence $(f^{-1}(y_n))$ converges to $f^{-1}(y')$ and $f^{-1}(y'')$. This implies $f^{-1}(y') = f^{-1}(y'')$ and hence $y' = y''$.

COROLLARY 4.1. The semi-Hausdorff property is a topological invariant.

THEOREM 5. The semi-Hausdorff property is productive and projective.

Proof. Let $X = \times X_\alpha$ be the product of non-empty spaces X_α. Suppose each X_α is a semi-Hausdorff space, and let (f_n) be a sequence in X converging to distinct limits f and g. Then there exists an index β such that $f(\beta) \neq g(\beta)$. If p is the projection from X to X_β, it follows from the continuity of p that the sequence $(f_n(\beta))$ has distinct limits $f(\beta)$ and $g(\beta)$, a contradiction.

Conversely, let X be a semi-Hausdorff space. Each X_α is homeomorphic with a subspace of X. By Theorem 3 and Corollary 4.1, X_α is semi-Hausdorff.

COROLLARY 5.1. If Y is semi-Hausdorff, so is Y^X with the pointwise convergence topology.

The semi-Hausdorff property is not divisible, in fact not invariant even under open quotient mappings. Let X be the real space (with the usual topology) and $R = \{(x, y) \mid x-y$ rational$\}$. Then X/R is an indiscrete space and hence not semi-Hausdorff.

THEOREM 6. Every first countable semi-Hausdorff space is Hausdorff.

Proof. Let x and y be distinct points of a first-

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countable semi-Hausdorff space X. Let $U_1 \supset U_2 \supset U_3 \supset \ldots$ be a local basis at x, and $V_1 \supset V_2 \supset V_3 \supset \ldots$ be a local basis at y. Suppose that U_i intersects V_i for every i. Choose $x_i \in U_i \cap V_i$, $i = 1, 2, 3, \ldots$. The sequence (x_n) then converges to both x and y, which is a contradiction. Hence $U_i \cap V_i = \emptyset$ for some i.

DEFINITION 2. Let X be a topological space and $S \subset X$. S is <u>sequentially compact</u> if and only if every sequence in S has a subsequence converging to a point in S. S is <u>sequentially closed</u> if and only if no sequence in S converges to a point in X - S.

THEOREM 7. In a semi-Hausdorff space, every set which is sequentially compact is sequentially closed.

<u>Proof.</u> Let S be sequentially compact and suppose there is a sequence (x_n) in S converging to $x \notin S$. Then (x_n) has a subsequence (x_{n_k}), converging to a point $y \in S$. But as a subsequence of a convergent sequence, $(x_{n_k}) \to x$. Contradiction. Hence S is sequentially closed.

We conclude with two characterizations of semi-Hausdorff spaces.

THEOREM 8. A space X is semi-Hausdorff if and only if the diagonal set Δ is sequentially closed in $X \times X$.

<u>Proof.</u> If X is semi-Hausdorff and a sequence (x_n, x_n) in Δ converges to $(x, y) \notin \Delta$, then $x_n \to x$, $x_n \to y$, $x \neq y$, which is a contradiction. Hence Δ is sequentially closed.

Conversely, if Δ is sequentially closed and X is not semi-Hausdorff, then there exists a sequence (x_n) in X such that $x_n \to x$, $x_n \to y$, $x \neq y$. Then (x_n, x_n) converges to $(x, y) \notin \Delta$. Contradiction. Hence X is semi-Hausdorff.

THEOREM 9. Let $f : X \to Y$ and $g : X \to Y$ be se-

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quentially continuous functions. If Y is semi-Hausdorff, then the set $A = \{x \mid f(x) = g(x)\}$ is sequentially closed. Conversely, if A is sequentially closed for all X, f and g, then Y is semi-Hausdorff.

Proof. Let Y be semi-Hausdorff and suppose there is a sequence (x_n) in A converging to $x \notin A$. Then $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. Hence $f(x) = g(x)$ and $x \in A$. Contradiction.

The converse follows trivially from Theorem 8 on taking $X = Y \times Y$ and f, g to be the projections.

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