Canad. Math. Bull. Vol. **58** (2), 2015 pp. 225–232
http://dx.doi.org/10.4153/CMB-2014-064-2

Characterizing Distinguished Pairs by Using Liftings of Irreducible Polynomials

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Abstract. Let v be a henselian valuation of any rank of a field K and let $\overline{v}$ be the unique extension of v to a fixed algebraic closure $\overline{K}$ of K. In 2005, we studied properties of those pairs (θ, α) of elements of $\overline{K}$ with $[K(\theta):K] > [K(\alpha):K]$ where α is an element of smallest degree over K such that

$$\overline{v}(\theta - \alpha) = \sup\{\overline{v}(\theta - \beta) \mid \beta \in \overline{K},\ [K(\beta):K] < [K(\theta):K]\}.$$

Such pairs are referred to as distinguished pairs. We use the concept of liftings of irreducible polynomials to give a different characterization of distinguished pairs.

1 Introduction

Throughout this paper, v is a henselian valuation of any rank of a field K and $\overline{v}$ is the unique extension of v to a fixed algebraic closure $\overline{K}$ of K with value group $\overline{G}$. For an overfield K' of K contained in $\overline{K}$, we shall denote by $G(K')$ and $R(K')$ respectively the value group and the residue field of the valuation v' of K' obtained by restricting $\overline{v}$ to K'. By the degree of an element $\alpha \in \overline{K}$, we shall mean the degree of the extension $K(\alpha)/K$ and denote it by $\deg \alpha$. For a finite extension $(K', v')/(K, v)$ (or briefly K'/K), $\operatorname{def}(K'/K)$ will stand for the defect of the valued field extension K'/K, *i.e.,* $\operatorname{def}(K'/K) = [K':K]/ef$, where e and f are respectively the index of ramification and the residual degree of v'/v. For any β in the valuation ring of $\overline{v}$, β^* will denote its $\overline{v}$-residue, *i.e.,* the image of β under the canonical homomorphism from the valuation ring of $\overline{v}$ onto its residue field.

An extension w of v to a simple transcendental extension $K(x)$ of K is called *residually transcendental* if the residue field of w is a transcendental extension of the residue field of v. Alexandru et al. characterized all residually transcendental extensions of v by means of minimal pairs (see [3, 4]). Recall that a pair (α, δ) in $\overline{K} \times \overline{G}$ is said to be minimal (with respect to (K, v)) if whenever $\beta \in \overline{K}$ satisfies $\overline{v}(\alpha - \beta) \geq \delta$, then $\deg \alpha \leq \deg \beta$. It is clear that when $\alpha \in K$, (α, δ) is a minimal pair for each $\delta \in \overline{G}$; however, as can be easily seen, a pair (α, δ) in $(\overline{K} \setminus K) \times \overline{G}$ is minimal if and only if δ is strictly greater than each element of the set $M(\alpha, K)$ defined by

$$M(\alpha, K) = \{\overline{v}(\alpha - \beta) \mid \beta \in \overline{K},\ [K(\beta):K] < [K(\alpha):K]\}.$$

Received by the editors May 27, 2013.
Published electronically February 10, 2015.
AMS subject classification: **12J10**, 12J25, 12E05.
Keywords: valued fields, non-Archimedean valued fields, irreducible polynomials.

This led to the main invariant $\delta_K(\alpha)$, defined by $\delta_K(\alpha) = \sup M(\alpha, K)$ for those $\alpha \in \overline{K} \setminus K$ for which $M(\alpha, K)$ has an upper bound in $\overline{G}$, where for the sake of supremum, $\overline{G}$ may be viewed as a subset of its Dedekind order completion. It may be pointed out that the supremum of $M(\alpha, K)$ being in $\overline{G}$ does not necessarily imply that it belongs to $M(\alpha, K)$. In [2], Aghigh and Khanduja studied properties of those pairs (θ, α) of elements of $\overline{K}$ with $\deg\theta > \deg\alpha$ where α is an element of smallest degree over K such that $\overline{v}(\theta - \alpha) = \delta_K(\theta)$. Such pairs are called *distinguished pairs* (more precisely (K, v)-distinguished pairs) and were introduced in [13]. In other words, a pair (θ, α) of elements of $\overline{K}$ is a distinguished pair if the following three conditions are satisfied.

(i) $\overline{v}(\theta - \alpha) = \delta_K(\theta)$;
(ii) $\deg\theta > \deg\alpha$;
(iii) if γ belonging to $\overline{K}$ has degree less than that of α, then $\overline{v}(\theta - \gamma) < \overline{v}(\theta - \alpha)$.

Distinguished pairs give rise to distinguished chains in a natural manner. A chain $\theta = \theta_0, \theta_1, \dots, \theta_r$ of elements of $\overline{K}$ will be called a *complete distinguished chain* for θ if (θ_i, θ_{i+1}) is a distinguished pair for $0 \le i \le r-1$ and $\theta_r \in K$. It is worthwhile mentioning that complete distinguished chains for an element θ in $\overline{K} \setminus K$ give rise to several invariants associated with θ that are the same for all K-conjugates of θ and hence are invariants of the minimal polynomial of θ over K (see [2]). They are important tools of valuation theory that are used extensively in studying the properties of irreducible polynomials with coefficients in a valued field (K, v) (see [6, 8] for example).

The concept of lifting of a polynomial is another important tool for investigating the properties of irreducible polynomials with coefficients in valued fields (see [7, 10, 11] for example). We briefly recall a survey of it.

If $f(x)$ is a fixed nonzero polynomial in $K[x]$, then using the Euclidean algorithm, each $F(x) \in K[x]$ can be uniquely represented as a finite sum $\sum_{i\ge 0} F_i(x) f(x)^i$, where for any i, the polynomial $F_i(x)$ is either 0 or has degree less than that of $f(x)$. The above representation will be referred to as the f-expansion of $F(x)$.

For a pair $(\alpha, \delta) \in \overline{K} \times \overline{G}$, the valuation $\overline{w}_{\alpha,\delta}$ of $\overline{K}(x)$ defined on $\overline{K}[x]$ by

$$\overline{w}_{\alpha,\delta}\Big(\sum_i c_i (x-\alpha)^i\Big) = \min_i\{\overline{v}(c_i) + i\delta\}, \quad c_i \in \overline{K}, \tag{1.1}$$

will be referred to as the valuation defined by the pair (α, δ). The description of $\overline{w}_{\alpha,\delta}$ on $K(x)$ is given by the already known theorem stated below (see [3]).

Theorem 1.1 *Let $\overline{w}_{\alpha,\delta}$ be the valuation of $\overline{K}(x)$ defined by a minimal pair (α, δ) and $w_{\alpha,\delta}$ be the valuation of $K(x)$ obtained by restricting $\overline{w}_{\alpha,\delta}$. If $f(x)$ is the minimal polynomial of α over K of degree n and λ is an element of $\overline{G}$ such that $w_{\alpha,\delta}(f(x)) = \lambda$, then the following hold.*

(i) *For any $F(x)$ belonging to $K[x]$ with f-expansion $\sum_i F_i(x) f(x)^i$, we have*

$$w_{\alpha,\delta}(F(x)) = \min_i\{\overline{v}(F_i(\alpha)) + i\lambda\}.$$

(ii) *Let e be the smallest positive integer such that $e\lambda \in G(K(\alpha))$ and $h(x)$ belonging to $K[x]$ be a polynomial of degree less than n with $\overline{v}(h(\alpha)) = e\lambda$. Then* *the $w_{\alpha,\delta}$-residue $\big(\frac{f(x)^e}{h(x)}\big)^*$ of $\big(\frac{f(x)^e}{h(x)}\big)$ is transcendental over $R(K(\alpha))$, and the residue field of $w_{\alpha,\delta}$ is canonically isomorphic to $R(K(\alpha))\big(\big(\frac{f(x)^e}{h(x)}\big)^*\big)$.*

Using the canonical homomorphism from the valuation ring of v onto its residue field, one can lift any monic irreducible polynomial having coefficients in $R(K)$ to yield a monic irreducible polynomial with coefficients in K. The description of the residue field of $w_{\alpha,\delta}$ given in Theorem 1.1(ii) led Popescu and Zaharescu to generalize the usual notion of lifting (see [13]). In fact, they introduced the notion of lifting of a polynomial belonging to $R(K(\alpha))[Y]$ (Y an indeterminate) with respect to a minimal pair (α,δ) as follows.

For a (K,v)-minimal pair (α,δ), let $f(x)$, n, λ, and e be as in Theorem 1.1. As in [13], a monic polynomial $F(x)$ belonging to $K[x]$ is said to be a lifting of a monic polynomial $Q(Y)$ belonging to $R(K(\alpha))[Y]$ having degree $m \geq 1$ with respect to (α,δ) if there exists $h(x) \in K[x]$ of degree less than n such that

(i) $\deg F(x) = emn$;
(ii) $w_{\alpha,\delta}(F(x)) = mw_{\alpha,\delta}(h(x)) = em\lambda$;
(iii) the $w_{\alpha,\delta}$-residue of $F(x)/h(x)^m$ is $Q((f^e/h)^*)$.

To be more precise, the above lifting will be referred to as the one with respect to (α,δ) and h. This lifting is said to be *trivial* if $\deg F(x) = \deg f(x)$. Note that if (α,δ) is the minimal pair $(0,0)$, then the corresponding valuation $w_{0,0}$ is the Gaussian extension of v to $K(x)$ given by $w_{0,0}(\sum_i a_i x^i) = \min_i(v(a_i))$ with residue field $R(K)(x^*)$.

In this paper, we show that liftings and distinguished pairs are closely related to each other. We give a characterization of distinguished pairs using liftings of irreducible polynomials. Indeed, we shall prove the following theorem.

Theorem 1.2 *Let θ, α be elements in the algebraic closure $\overline{K}$ of a henselian valued field (K,v) with respective minimal polynomials $g(x)$, $f(x)$ over K. Suppose that $\deg g(x) > \deg f(x)$. Let e be the smallest positive integer such that $e\overline{v}(f(\theta))$ is in $G(K(\alpha))$ with $e\overline{v}(f(\theta)) = \overline{v}(h(\alpha))$. Then the following three statements are equivalent.*

(i) *(θ,α) is a distinguished pair.*
(ii) *$(\alpha,\overline{v}(\theta-\alpha))$ is a minimal pair and $g(x)$ is a non-trivial lifting of the minimal polynomial of $(f(\theta)^e/h(\alpha))^*$ over $R(K(\alpha))$ with respect to this minimal pair.*
(iii) *$(\alpha,\overline{v}(\theta-\alpha))$ is a minimal pair and $g(x)$ is a lifting of some irreducible monic polynomial $Q(Y) \neq Y$ belonging to $R(K(\alpha))[Y]$ with respect to $(\alpha,\overline{v}(\theta-\alpha))$.*

2 Preliminary Results

In 1999, Khanduja and Saha generalized the fundamental principle stated in [13, Remark 3.3] to henselian valued fields of arbitrary rank (see [12, Theorem 1.1]). They proved the following theorem.

Theorem 2.1 *Let (K,v) be a henselian valued field of any rank. Let $\alpha,\beta \in \overline{K}$ be such that $\overline{v}(\alpha-\beta) > \overline{v}(\alpha-\gamma)$ for any $\gamma \in \overline{K}$ satisfying $[K(\gamma):K] < [K(\alpha):K]$. Then*

(i) $G(K(\alpha)) \subseteq G(K(\beta))$;
(ii) $R(K(\alpha)) \subseteq R(K(\beta))$;
(iii) $\operatorname{def}(K(\alpha)/K)$ *divides* $\operatorname{def}(K(\beta)/K)$.

Aghigh and Khanduja, in the course of investigation of the main invariant of elements algebraic over henselian valued fields, proved a useful lemma (see [1, Lemma 2.3]).

Lemma 2.2 *Let (K,v) be henselian and θ be an element of $\overline{K}\backslash K$ such that $\delta_K(\theta)$ belongs to $M(\theta,K)$. If $\alpha \in \overline{K}$ is an element of smallest degree over K such that $\overline{v}(\theta-\alpha) = \delta_K(\theta)$, then*

(i) *$(\alpha, \delta_K(\theta))$ is a minimal pair;*
(ii) *$\overline{w}_{\alpha,\delta}(G(x)) = \overline{v}(G(\theta))$ for any polynomial $G(x) \in K[x]$ of degree less than the degree of θ over K, where the valuation $\overline{w}_{\alpha,\delta}$ is as defined by (1.1) with $\delta = \delta_K(\theta)$.*

The following lemmas, which were actually obtained during the course of the proof of [2, Theorem 1.1], are also immediate consequences of it.

Lemma 2.3 *Suppose that (θ,α) is a (K,v)-distinguished pair, $f(x)$ is the minimal polynomial of α over K, and e is the smallest positive integer such that $e\overline{v}(f(\theta))$ is in $G(K(\alpha))$ with $e\overline{v}(f(\theta)) = \overline{v}(h(\alpha))$. Then $e(deg\alpha)$ divides deg θ.*

Lemma 2.4 *With the notations of Lemma 2.3, denote* $\deg\theta$ *and deg α by m and n respectively; then $\left(\frac{f(\theta)^e}{h(\alpha)}\right)^*$ is algebraic of degree m/en over $R(K(\alpha))$.*

Moreover, the following lemma proved in [2, Lemma 2.3] will be used in the sequel.

Lemma 2.5 *Let $f(x)$ and $g(x)$ be respectively two monic irreducible polynomials over a henselian valued field (K,v) of degree n and m such that $f(\alpha) = g(\beta) = 0$. Then $m\overline{v}(f(\beta)) = n\overline{v}(g(\alpha))$.*

We also need the following proposition, which is already known (see [5, Proposition 2.3]). Its proof is omitted.

Proposition 2.6 *Let (α,δ) be a (K,v)-minimal pair and let $f(x)$, λ, e, and $h(x)$ be as in Theorem 1.1. Let $g(x) \in K[x]$ be a monic polynomial that is a lifting of a monic polynomial $Q[Y]$ not divisible by Y belonging to $R(K(\alpha))[Y]$ of degree m with respect to (α,δ) and h. Then we have that*

(i) $\overline{v}(\theta_i - \alpha) \le \delta$ for each root θ_i of $g(x)$;
(ii) there exists a root θ of $g(x)$ such that $\overline{v}(\theta-\alpha) = \delta$;
(iii) if θ is as in (ii), then $Q((f(\theta)^e/h(\alpha))^*) = 0$.

Finally, we will employ the following known lemma in the proof of Theorem 1.2 (see [9, Lemma 2.1]). For the sake of completeness, we prove it here.

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Lemma 2.7 *Let $\overline{w}_{\alpha,\delta}$ be the valuation of $\overline{K}(x)$ with respect to a minimal pair (α,δ) defined by (1.1). If $F(x)$ belonging to $K[x]$ is such that for each root β of $F(x)$, $\overline{v}(\alpha-\beta)<\delta$, then $\overline{w}_{\alpha,\delta}\big(\frac{F(x)}{F(\alpha)}-1\big)>0$.*

Proof Write $F(x)=b\prod_i(x-\beta_i)$. Hence we have

$$\frac{F(x)}{F(\alpha)}=\prod_i\Big(\frac{x-\beta_i}{\alpha-\beta_i}\Big)=\prod_i\Big(1+\frac{x-\alpha}{\alpha-\beta_i}\Big).$$

By (1.1), $\overline{w}_{\alpha,\delta}(\frac{x-\alpha}{\alpha-\beta_i})=\delta-\overline{v}(\alpha-\beta_i)$. Since $\overline{v}(\alpha-\beta_i)<\delta$ for every i, it follows that $\overline{w}_{\alpha,\delta}(\frac{x-\alpha}{\alpha-\beta_i})>0$. Therefore, one can obtain that $\overline{w}_{\alpha,\delta}\big(\frac{F(x)}{F(\alpha)}-1\big)>0$. ∎

3 Proof of Theorem 1.2

For simplicity of notation, we shall denote $\overline{v}(\theta-\alpha)$ by δ. Let $\overline{w}_{\alpha,\delta}$ be the valuation of $\overline{K}(x)$ defined by the pair (α,δ).

(i) ⇒ (ii). Suppose first that (θ,α) is a distinguished pair. By Lemma 2.3, $\deg g/e(\deg f)$ is an integer, say l. So the f-expansion of g can be written as $g(x)=f(x)^{el}+g_{el-1}(x)f(x)^{el-1}+\cdots+g_0(x)$, $\deg g_i<\deg f$. We will prove that $g(x)$ is a lifting of an irreducible polynomial of degree l over $R(K(\alpha))$ with respect to (α,δ), which is a minimal pair by virtue of Lemma 2.2(i). For this we first prove that

$$\overline{w}_{\alpha,\delta}(g(x))=el\overline{w}_{\alpha,\delta}(f(x)). \tag{3.1}$$

Keeping in view Lemma 2.2, $\overline{v}(f(\theta))=\overline{w}_{\alpha,\delta}(f(x))=\lambda$ (say). Since (K,v) is henselian, for any K-conjugate θ_i of θ, there exists a K-conjugate α' of α such that $\overline{v}(\theta_i-\alpha)=\overline{v}(\theta-\alpha')\le\delta_K(\theta)=\overline{v}(\theta-\alpha)$; consequently

$$\overline{w}_{\alpha,\delta}(x-\theta_i)=\min\{\delta,\overline{v}(\alpha-\theta_i)\}=\overline{v}(\alpha-\theta_i).$$

Therefore $\overline{w}_{\alpha,\delta}(g(x))=\overline{v}(g(\alpha))$. Applying Lemma 2.5, we see that

$$\overline{v}\big(g(\alpha)\big)=el\overline{v}\big(f(\theta)\big)=el\lambda.$$

The desired assertion (3.1) now is obtained.

By virtue of Theorem 1.1 and (3.1), we have (on taking $g_{el}(x)=1$),

$$\overline{w}_{\alpha,\delta}(g)=\min_{0\le i\le el}\{\overline{v}(g_i(\alpha))+i\lambda\}=el\lambda.$$

Recall that e is the smallest positive integer such that $e\lambda\in G(K(\alpha))$. It now follows that

$$\begin{aligned}&\overline{v}(g_i(\alpha))+i\lambda\ge el\lambda \quad \text{for } 0\le i\le el, \text{ and}\\ &\overline{v}(g_i(\alpha))+i\lambda> el\lambda \quad \text{if } e \text{ does not divide } i.\end{aligned} \tag{3.2}$$

Fix a polynomial $h(x)\in K[x]$ of degree less than n with $\overline{v}(h(\alpha))=e\lambda$. We shall denote $f(x)^e/h(x)$ by $r(x)$. Observe that by virtue of Lemma 2.7, $\overline{w}_{\alpha,\delta}(h(x))=\overline{v}(h(\alpha))$, and hence (3.1) implies that $\overline{w}_{\alpha,\delta}(r(x))=0$.

Keeping in view (3.2) and the fact that $\overline{v}(g_i(\alpha)) = \overline{w}_{\alpha,\delta}(g_i(x))$, we quickly conclude that

$$\overline{w}_{\alpha,\delta}\Big(\frac{g_i(x)f^i(x)}{h(x)^l}\Big) \geq 0, \quad 0 \leq i \leq el,$$
$$\overline{w}_{\alpha,\delta}\Big(\frac{g_i(x)f^i(x)}{h(x)^l}\Big) > 0, \quad \text{if } e \text{ does not divide } i.$$

On passing to the residue field of $\overline{w}_{\alpha,\delta}$, we obtain

$$\Big(\frac{g(x)}{h(x)^l}\Big)^* = (r(x)^*)^l + \Big(\frac{g_{e(l-1)}(x)}{h(x)}\Big)^*\big(r(x)^*\big)^{l-1} + \cdots + \Big(\frac{g_0(x)}{h(x)^l}\Big)^*.$$

Let us denote $g_{e(l-j)}(x)/h(x)^j$ by $B_{l-j}(x)$. Keeping in mind that $B_{l-j}(x)^* = B_{l-j}(\alpha)^*$ by virtue of Lemma 2.7, we see that $g(x)$ is a lifting of the polynomial $H(Y) = Y^l + B_{l-1}(\alpha)^* Y^{l-1} + \cdots + B_0(\alpha)^*$ with respect to the minimal pair (α,δ).

It remains to be shown that $H(Y)$ is the minimal polynomial of $\xi^* = \big(\frac{f(\theta)^e}{h(\alpha)}\big)^*$ over $R(K(\alpha))$. As asserted by Lemma 2.4, ξ^* is algebraic over $R(K(\alpha))$ of degree $l = (\deg g)/e(\deg f)$. So the desired assertion is proved once we show that ξ^* is a root of the polynomial $H(Y)$.

Taking the image of the equation

$$0 = \frac{g(\theta)}{h(\alpha)^l} = \sum_{i=0}^{el} \frac{g_i(\theta)}{h(\alpha)^l} f(\theta)^i$$

in the residue field, we conclude, using (3.2), that

$$\xi^{*^l} + \Big(\frac{g_{e(l-1)}(\theta)}{h(\alpha)}\Big)^* \xi^{*^{l-1}} + \cdots + \Big(\frac{g_0(\theta)}{h(\alpha)^l}\Big)^* = 0. \tag{3.3}$$

On the other hand, for any polynomial $q(x) \in K[x]$ of degree less than n, one may write $q(x) = c\prod_j (x-\beta_j)$. With the same method as the proof of Lemma 2.7, we get $\big(\frac{q(\theta)}{q(\alpha)}\big)^* = 1$.

Therefore (3.3) can be rewritten as

$$\xi^{*^l} + (B_{l-1}(\alpha))^* \xi^{*^{l-1}} + \cdots + (B_0(\alpha))^* = 0$$

which shows that ξ^* is a root of $H(Y)$. This completes the proof of (i) ⇒ (ii).

(ii) ⇒ (iii). This is trivial.

(iii) ⇒ (i). Suppose that $g(x)$ is a lifting of a monic irreducible polynomial $Q(Y) \neq Y$ belonging to $R(K(\alpha))[Y]$ of degree s with respect to the minimal pair (α,δ).

If $\beta \in \overline{K}$ and $\deg\beta < \deg\alpha$, then $\overline{v}(\theta-\beta) < \delta$, for otherwise $\overline{v}(\alpha-\beta) \geq \delta$, which is impossible as (α,δ) is a minimal pair. So to prove that (θ,α) is a distinguished pair, it is enough to show that whenever γ belonging to $\overline{K}$ satisfies $\overline{v}(\theta-\gamma) > \delta$, then $\deg\gamma \geq \deg\theta$. Let $\gamma \in \overline{K}$ be such that $\overline{v}(\theta-\gamma) > \delta$, then $\overline{v}(\alpha-\gamma) = \delta$. Since (α,δ) is a minimal pair, it follows that for any $\beta \in \overline{K}$ with $\deg\beta < \deg\alpha$, $\overline{v}(\alpha-\gamma) > \overline{v}(\alpha-\beta)$. Therefore, by Theorem 2.1,

$$G(K(\alpha)) \subseteq G(K(\gamma)), \quad \operatorname{def}(K(\alpha)/K) \mid \operatorname{def}(K(\gamma)/K), \quad R(K(\alpha)) \subseteq R(K(\gamma)). \tag{3.4}$$

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Let e, h, and f be as in Theorem 1.1. We next show that $\xi^* = \left(\frac{f(\theta)^e}{h(\alpha)}\right)^*$ belongs to $R(K(\gamma))$. Write

$$\frac{f(\gamma)}{f(\theta)} = \prod_{\alpha'}\left(\frac{\gamma-\alpha'}{\theta-\alpha'}\right) = \prod_{\alpha'}\left(1+\frac{\gamma-\theta}{\theta-\alpha'}\right).$$

Since $\overline{v}(\theta-\gamma) > \delta$, and by Proposition 2.6(i), $\overline{v}(\theta-\alpha') \le \delta$, it follows from the above expression for $f(\gamma)/f(\theta)$ that $\left(\frac{f(\gamma)}{f(\theta)}\right)^* = 1$; in particular,

$$\overline{v}(f(\gamma)) = \overline{v}(f(\theta)), \quad \left(\frac{f(\gamma)^e}{h(\alpha)}\right)^* = \left(\frac{f(\theta)^e}{h(\alpha)}\right)^* = \xi^*. \tag{3.5}$$

By Proposition 2.6(iii), ξ^* is a root of the polynomial $Q(Y)$ belonging to $R(K(\alpha))[Y]$, which is given to be irreducible. So we conclude from (3.4) and (3.5) that e divides $[G(K(\gamma)):G(K(\alpha))]$ and s divides $[R(K(\gamma)):R(K(\alpha))]$. As $\operatorname{def}(K(\alpha)/K)$ divides $\operatorname{def}(K(\gamma)/K)$, we see that $es(\deg f)$ divides $\deg\gamma$. In particular, $\deg\gamma \ge es(\deg f)$. But by definition of lifting, $\deg g = es\deg f = \deg\theta$. It now follows that $\deg\gamma \ge \deg\theta$, as desired. Hence, (θ,α) is a distinguished pair.

4 An Example

Let v^x be the Gaussian extension of any henselian valuation v of a field K to $K(x)$ defined by $v^x(\sum_i a_i x^i) = \min_i\{v(a_i)\}$, $a_i \in K$. Let $f(x)$ be a monic polynomial with coefficients in the valuation ring of v such that the corresponding polynomial $f^*(x)$ (*i.e.*, the polynomial obtained by replacing the coefficients of f by their corresponding v-residues) belonging to $R(K)[x]$ is irreducible and separable over $R(K)$. Let $F(x) \in K[x]$ be a polynomial whose f-expansion given by $F(x) = \sum_{i=0}^{s} F_i(x)f(x)^i$ satisfies

$$F_s(x) = 1, \quad \frac{v^x(F_i(x))}{s-i} \ge \frac{v^x(F_0(x))}{s} > 0, \quad 0 \le i \le s-1$$

and that there does not exist any rational integer $r > 1$ dividing s such that $\frac{v^x(F_0(x))}{r} \in G(K)$. Let θ be a root of $F(x)$. Since $F^*(x) = (f^*(x))^s$, it follows that there exists a (unique) root α of $f(x)$ such that $\theta^* = \alpha^*$. We claim that (θ,α) is a distinguished pair.

As shown in the proof of [11, Theorem 1.1], $F(x)$ is a lifting of the polynomial $Y+1$ with respect to the minimal pair (α,δ), where $\delta = v^x(F_0(x))/s > 0$. So by Proposition 2.6, there exists a root α' of $f(x)$ such that $\overline{v}(\theta-\alpha') = \delta$. Observe that $\alpha' = \alpha$, for otherwise

$$\overline{v}(\alpha-\alpha') \ge \min\{\overline{v}(\alpha-\theta), \overline{v}(\theta-\alpha')\} > 0,$$

which is impossible in view of the hypothesis that f^* is a separable polynomial. So $\overline{v}(\theta-\alpha) = \delta$. Now the claim follows from Theorem 1.2.

Let us assume that $\deg\alpha > 1$. Then we show that $(\alpha, 1)$ is a distinguished pair and hence θ, α, 1 is a complete distinguished chain of length 2. Note that $\delta_K(\alpha) = 0$, because if $\beta \in \overline{K}$ has degree less than $\deg\alpha$, then $\overline{v}(\alpha-\beta) \le 0$, for otherwise $\alpha^* = \beta^*$ would lead to

$$[K(\beta):K] \ge [R(K)(\beta^*):R(K)] = [R(K)(\alpha^*):R(K)] = [K(\alpha):K].$$

Since $\overline{v}(\alpha-1) = 0 = \delta_K(\alpha)$, it follows that $(\alpha,1)$ is a distinguished pair.

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