4

How Can It Be True?

4.1 Introduction

After Chapter 3 showed the ubiquity of the failure of *proper* rigidity for noncompact locally symmetric manifolds, in this chapter we begin to examine the problem for closed manifolds. Rather than seriously engaging the question of how to prove the Borel conjecture, we focus on how it can possibly be true.

After all, every method for distinguishing manifolds is a potential obstacle that needs to be overcome. For example, in Chapter 3 we saw the import of characteristic classes, so a major focus of this chapter must be about why it is that (e.g. what are some mechanisms for) the characteristic classes of a manifold homotopy-equivalent to $K\backslash G/\Gamma$ must be the same as those of $K\backslash G/\Gamma$ (if we are in the compact case). This is essentially the topic of the Novikov conjecture[1] and it will be the main focus of this chapter and the next.

But there are invariants not at all related to the characteristic classes that can be used to distinguish manifolds.[2] The classical example is the theory of lens spaces: lens spaces are quotients of the sphere $\mathcal{S}^{2n-1}/\mathbb{Z}_k$ where $\mathcal{S}^{2n-1}$ is thought of as the unit sphere of $\mathbb{C}^n$, and $\mathbb{Z}_k$ acts via a unitary representation on $\mathbb{C}^n$. Each representation is a sum of irreducible one-dimensional representations, and to obtain a quotient *manifold* we assume that each of the irreducible pieces has $\mathbb{Z}_k$ acting *freely*, i.e. can be described as rotation by a *primitive* root of unity. These are called the rotation numbers in the definition of the lens space. A lens space

[1] It is important to note that Novikov was not at all thinking about the Borel conjecture when formulating his problem. It arose very naturally in the course of his work on the topological invariance of rational Pontrjagin classes as we will see in §4.5 below.

[2] Recall that in the situation of simply connected manifolds, the Browder–Novikov theorem (see the notes from Chapter 3) tells us that, beyond homotopy type, the G/Top characteristic class is a complete invariant (so that ordinary characteristic classes only lose a finite amount of information).

might be denoted by $\mathrm{Lens}_k(a_1, \ldots, a_n)$ or some such similar notation where the a_i are integers prime to k, and denote the rotation numbers.

Changing the order of the a_i is costless. Changing a_i to $-a_i$ is an equivalence of the underlying real representation, but not the complex one, and changes the natural orientation on the manifold. But one can do this an even number of times and keep the orientation.

If we just care about the underlying manifold, we can change the group action by multiplying all of the rotation numbers by the same s prime to k. Usually we will assume that we preserve an identification of the fundamental group with $\mathbb{Z}_k$, i.e. that we have a fixed homotopy class of maps to $K(\mathbb{Z}_k, 1)$ that we are preserving, equivalently that we are interested in conjugacy of the group actions.

The number of π_1 and orientation preserving homotopy types among lens spaces in a fixed dimension is $\varphi(k)$ (the Euler φ function) – with the homotopy type being determined by the product of the rotation numbers.[3] However,diffeomorphism (or homeomorphism, in this case) is exactly equivalent to orientation-preserving real linear equivalence, i.e. the changes we described above – according to a beautiful and deep theorem of de Rham (see Cohen, 1973).

In dimension 3, *all* orientable manifolds have trivial tangent bundles, and there is no more tangential information to be had.[4] (In higher dimensions, the Pontrjagin classes do distinguish some lens spaces from each other – they are essentially symmetric functions in the squares of the rotation numbers, but there aren't enough of these to determine these numbers themselves (see Milnor, 1966).

There are two essentially different proofs of this theorem; that is, proofs based on different principles: de Rham's original argument that it is now natural to view from the point of view of algebraic K-theory; and another argument, due to Atiyah, Bott, and Milnor (see (Atiyah and Bott, 1967), 1968) that involves (equivalence classes of) quadratic forms associated to the lens spaces defined either in terms of spaces that they bound (cobordism theory) or via some measure of how lopsided (around 0) the spectrum of some self-adjoint operators are ("spectral asymmetry") (Atiyah *et al.*, 1975a,b).

Both of these proofs pose challenges to the Borel conjecture; we will discuss

[3] A pair of lens spaces can be compared by a map that preserves their fundamental groups. Orienting them both, we can ask the degree of this map. A form of the Borsuk–Ulam theorem tells us that this degree is prime to k – the congruence class is independent of the map, and is the ratio of the product of the rotation numbers defining the two lens spaces.

[4] Actually, it is possible for the normal invariant of map between oriented 3-manifolds to be nontrivial, because of the extra information that goes beyond the tangent bundle itself – but this is information only at the prime 2, and it always vanishes for homotopy equivalences.

the Atiyah–Bott–Milnor argument later in the chapter, and de Rham's challenge in the next.

Let's start with the Novikov conjecture, the response to the challenge of characteristic classes.

4.2 The Hirzebruch Signature Theorem

Before discussing how the Borel conjecture can be true, it is worth asking, along the same lines, how the Poincaré conjecture can be true? After all, over S^{4k} there are an infinite number of vector bundles – distinguished by the Pontrjagin class p_k. Why aren't these the Pontrjagin classes of homotopy spheres?

The answer is given by the Hirzebruch signature theorem that gives a homotopy-theoretic interpretation of a certain combination of characteristic classes.

Definition 4.1 If M^{4k} is a closed oriented manifold, then the *signature* of M is defined as the *signature* of the quadratic (i.e. symmetric bilinear) form $H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \to H^{4k}(M;\mathbb{Q}) \to Q$; that is, it is the difference in dimensions between a maximal positive-definite subspace of $H^{2k}(M;\mathbb{Q})$ and a negative-definite subspace.

Note that Poincaré duality tells us that this quadratic form is non-singular. Such a form (over a field) can be diagonalized – and the signature is the number of positive eigenvalues – the number of negative ones.

By its definition it just depends on the oriented homotopy type of M.

Theorem 4.2 ((Hirzebruch, 1995)) *There are homogeneous graded polynomials $L_k(p_1,\ldots,p_k)$ in the Pontrjagin classes, so that*

$$L_k(p_1,\ldots,p_k) = 2^{2k}(2^{2k}-1)\mathrm{B}_k/(2k)!p_k + \text{terms involving the lower classes},$$

(where B_k is the kth Bernoulli number) so that $L = 1 + L_1 + \cdots + L_k + \cdots$ is multiplicative for sums of bundles, and

$$\operatorname{sign}(M^{4k}) = \langle L_k(p_1,\ldots,p_k), [M]\rangle,$$

where we have denoted by p_i the Pontrjagin classes of the tangent bundle of M.

The last statement is what gives the theorem its name, the Hirzebruch signature theorem. Hirzebruch actually gives a formula for the L from which the first statement follows.

Note that signature depends on orientation, exactly as the right-hand side does. An immediate consequence of the formula is that a manifold that is stably parallelizable (i.e. with trivial normal bundle) has signature equal to 0.

Another significant consequence of the theorem is that if $N \to M$ is an r-sheeted cover (not necessarily connected or regular), then $\operatorname{sign}(N) = r\operatorname{sign}(M)$, as the tangential information is the same for a manifold and its cover, just the fundamental classes are multiplied.

The signature can be defined for spaces more general than topological manifolds.[5] For example, for manifolds with boundary, the relevant quadratic form can be defined, but it has a torsion subspace that should be removed. In that case, for example, signature is not multiplicative in finite coverings.

Wherever one has Poincaré duality, there are signatures. And in all cases, one can ask the question about multiplicativity in finite covers.

For L^2-cohomology, the $*$ operator often gives rise to a form of Poincaré duality. For instance, for infinite regular covers, this gives rise to a generalization of multiplicativity: the L^2-signature (which is a kind of normalized signature of covers[6]) equals the signature of the base.[7] On the other hand, there are other complete manifolds where L^2-cohomology is self-dual not coming from covering spaces, and then one can define signature-type invariants which need not be multiplicative.

Intersection homology provides another example: for interesting classes of spaces such as compact complex algebraic varieties, it gives a form of Poincaré duality.[8]

Finally, whenever one has a representation $\rho\colon \pi_1(X) \to \mathrm{U}(n)$, there is an associated flat bundle on x^{4k}, and a Hermitian form $H^{2k}(X;\rho) \otimes H^{2k}(X;\rho) \to \mathbb{C}$, and hence a signature[9] $\in \mathbb{Z}$. A rather surprising consequence of the Atiyah–Singer index theorem is that for X a manifold, $\operatorname{sign}_\rho(X) = n\operatorname{sign}(X)$.

We shall see later (as a consequence of controlled topological ideas) that this is indeed a consequence of the fact that the Poincaré duality is a local state-

5 Aside from dimension 4, topological manifolds with trivial tangent bundle can be smoothed. And signature is multiplicative in finite covers of closed topological manifolds, as the reader should be able to prove by the end of the chapter.

6 And in the residually finite case, is a limit of normalized signatures of finite covers, *à la* Lück.

7 This is due to Atiyah, and resembles the statement we discussed in Chapter 3 about Euler characteristic, but it is somewhat deeper than it. The result about Euler characteristic is a statement about finite complexes, but this is one about manifolds. Atiyah's proof was based on the ideas of the Atiyah–Singer index theorem.

8 Which can sometimes, e.g. in the work of Cheeger on the Hodge theory of Riemannian pseudomanifolds and the work of many on the Zucker conjecture, be interpreted in L^2 terms.

9 In the Hermitian setting, there is not much of a difference between a Hermitian form and a skew-Hermitian form: you can go from one to the other by multiplying by i. As a result one can get get signature-type invariants in dimension $2 \bmod 4$. We will not see this playing a direct role for closed manifolds of dimension $4k + 2$ – but this does play a role in the Atiyah–Bott–Milnor story for lens spaces of dimension $1 \bmod 4$.

ment[10] and thus is true in the intersection homology and topological manifold settings as well.

Remark 4.3 (On the proofs) There are essentially two different proofs of the signature theorem. The original proof (Hirzebruch's) deduces the theorem axiomatically from three properties of signature:

(1) Signature is cobordism invariant, i.e. if $M = \partial W$ (where M and W are compact), then $\operatorname{sign}(M) = 0$.
(2) Multiplicativity $\operatorname{sign}(M \times N) = \operatorname{sign}(M)\operatorname{sign}(N)$.
(3) $\operatorname{sign}(\mathbb{CP}^{2k}) = 1$.

Then the result follows essentially from the work of Thom on cobordism theory. (As Hirzebruch (1971) writes: "How to prove it? After conjecturing it I went to the library of the Institute for Advanced Study (June 2, 1953). Thom's *Comptes Rendus* note had just arrived. This completed the proof.") This method[11] remains important in purely topological settings. Often it is important to make use of quantities over the form $\Omega_*(?) \otimes_{\Omega_*(*)} \mathbb{Z}$, where $\Omega_*(?)$ is the homology theory whose chains are maps of oriented manifolds in "?", and which is viewed as a module over $\Omega(*)$ by multiplication and $\mathbb{Z}$ is viewed as a module over $\Omega(*)$ by means of the signature (or some other invariant of manifolds, on some occasions).

These considerations, systematically employed[12] by Sullivan (and the key to his analysis of the structure of F/Top, for example) exist embryonically already in this work of Hirzebruch.

Moreover, the π–π theorem, at the core of the flexibility results in Chapter 3, gives a starring role to cobordism and "?" of the form $K(\pi, 1)$. (Please pause and think this through.)

All that being said, this method is hard to apply to the flat bundle result mentioned above. For example, naively, one runs into the fact that no multiple of S^1 with a nontrivial flat complex line bundle with non-root of unity monodromy bounds in a way that extends (flatly) over the surface: the bordism group of manifolds equipped with flat bundles is huge.

The other main method for proving the Hirzebruch signature theorem was motivated by it – it goes via the Atiyah–Singer index theorem. Here $\operatorname{sign}(M)$ or a twisted cousin of it is viewed as the index of an elliptic operator on M, and such an index can be calculated cohomologically.

Originally, this wasn't a completely disjoint proof in that the first proof of

[10] And is thus true in the intersection homology setting for varieties (or Witt spaces).
[11] Of spending time (at IAS or) in libraries.
[12] Or exploited?

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the index theorem went via cobordism theory (see Palais, 1965). However, subsequently two different proofs of the signature theorem have been found – one K-theoretic (Atiyah and Singer, 1968a, 1971) and one via study of the heat equation (see Atiyah *et al.*, 1975a,b; Gilkey, 1984). The K-theoretic perspective shall play a large role starting in Chapter 5 and will be discussed further there, but consequences of the index theorem shall already play an interesting role in this chapter, and the heat equation approach is also relevant to our story (e.g. in §4.9).

For now, so we can return to our story, let us be content with observing that the Hirzebruch theorem answers the question with which we started this section. If Σ is a homotopy sphere, then its signature is 0 (it has no middle cohomology at all), and so, since the coefficient of p_k in L_k is nonzero, we conclude that p_k must be 0 and none of those bundles we feared actually arise as tangent bundles of homotopy spheres.

4.3 The Novikov Conjecture

In the 1960s, Novikov (1966) suggested a generalization of one of the key consequences of the Hirzebruch theorem.

Conjecture 4.4 (Novikov conjecture; most primitive form) *Suppose that $\alpha \in H^i(\mathrm{B}\Gamma;\mathbb{Q})$ and let M be a closed oriented manifold of dimension $4k+i$, $f: M \to K(\Gamma,1)$ a map, then the quantity*

$$\operatorname{sign}_{\alpha}(M,[M]) = \langle f^*(\alpha) \cup L_k(M), [M]\rangle \in \mathbb{Q}$$

is an oriented homotopy-invariant.

To see its implication for the Borel conjecture, suppose that M and M' are closed aspherical manifolds with the same fundamental group Γ. Then if $h\colon M' \to M$ is a homotopy equivalence, and $h^*(p_i(M)) \neq p_i(M')$, then $h^*(L_i(M)) \neq L_i(M')$ and we can find a cohomology class in the cohomology of M' which equals that of M which equals that of $K(\Gamma,1)$ that pairs nontrivially on this difference (using Poincaré duality) and get a contradiction.

For many purposes the following equivalent dual formulation is useful:

Conjecture 4.5 (Novikov conjecture; dual formulation) *Suppose M is a closed oriented n-manifold, and a map $f\colon M \to K(\Gamma,1)$ is given. Then*

$$f_*(L(M) \cap [M]) \in \oplus H_{n-4i}(\mathrm{B}\Gamma;\mathbb{Q})$$

is an oriented homotopy-invariant (for manifolds with reference maps to $\mathrm{B}\Gamma$*).*

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Let's be concrete and consider the case of $\Gamma = \mathbb{Z}$ (imagining that Γ is $\pi_1 M$). In that case, there is a natural homotopy class of map $f\colon M \to K(Z,1) = \mathcal{S}^1$ to use. The conjecture calls attention to the invariant of M^{4k+1} given by $\langle f^*([s^1]) \cup L_k(M), [M]\rangle$. Playing with this a little and using the Hirzebruch formula, we see that the higher signature is given by sign $f^{-1}(*)$ for any regular value $*$.

Orientations are easily obtained from orientations on M and the circle. That this quantity is independent of the regular value is because of the cobordism invariance of the signature. The puzzle Novikov places before us is why this quantity is a homotopy invariant. After all, the property of being a homotopy equivalence is a global property, and does not descend to submanifolds – a homotopy equivalence $h\colon M' \to M$ does not need to induce a homotopy equivalence $h|_{h^{-1}f^{-1}(*)}\colon h^{-1}f^{-1}(*) \to f^{-1}(*)$.

Remark 4.6 This is a key difference with the problem of homeomorphisms, which are indeed hereditary homotopy equivalences. If $h\colon M' \to M$ is a homeomorphism, then restricted to any $U \subset M$, $h|_{h^{-1}(U)}\colon h^{-1}(U) \to U$ is a proper homotopy equivalence. For hereditary homotopy equivalences, Novikov's theorem on topological invariance of rational Pontrjagin classes holds (Siebenmann, 1972).

Remark 4.7 *This* case of the Novikov conjecture does have a straightforward algebraic topological explanation (as was first observed,[13] I believe, by Rochlin). The cohomology of the infinite cyclic cover of M has the structure of a module over $\mathbb{Q}[\mathbb{Z}]$, a p.i.d. The linking pairing on the torsion submodule on H^{2k} satisfies Poincaré duality over $\mathbb{Q}$, and its signature can be identified with the invariant under discussion. However, in §4.4, we will discuss other methods of much wider scope.

4.4 First Positive Results

We shall discuss two and a half methods that give some useful and interesting positive information about the problem. The first is "codimension-1 splitting" and is a high-dimensional variant of the powerful tools used in the pre-Thurston period of three-dimensional topology. It gives very good information about "Haken manifolds," even in high dimensions.

It starts by trying to answer the question we asked at the end of the last section, given that homotopy equivalence is not hereditary, why should the

[13] These ideas about infinite cyclic covers are the bread and butter of knot theory.

signature of certain submanifolds be unchanged? Splitting theorems show that the homotopy equivalence can (often) be homotoped to one which is hereditary on codimension-1 submanifolds. This will also be a first occasion to consider algebraic K-theory that arises as an obstruction.[14]

After this, we will turn to Lusztig's thesis, which introduced a nice family of flat line bundles on manifolds with free abelian fundamental group, and brought the Atiyah–Singer index theorem for families to bear on the problem. Finally, we will give a variant of the last method that gives a proof for high-genus surfaces, not based on their Haken nature, or any family of line bundles, but rather based on a beautiful surface fibration discovered by Atiyah and Kodaira (and also explain why we call it half a method).

4.4.1 The Splitting Problem

The splitting problem in its simplest form supposes we have $h\colon M' \to M$ a homotopy equivalence and V a locally separating codimension-1 submanifold of M. The problem is to homotop h to a new map that restricts to the inverse image of V as a homotopy equivalence.

Unfortunately the answer to this is sometimes negative (in high dimensions), and we will build up to it by explaining first the problem where the main obstruction to this first arose, fibering.[15] During our first run we will not be comprehensive, but will only introduce some of the *dramatis personae*.

It is reasonable to extend the range of the discussion of splitting to allow the target to be a non-manifold as follows: Let X^n be a Poincaré complex; that is, a space that satisfies Poincaré duality in a suitably strong form with respect to all local systems.[16] We suppose that Y^{n-1} is a locally two-sided $(n-1)$-dimensional Poincaré complex, Poincaré-embedded in X. This means that there is a (perhaps disconnected) complex Z, such that $Y \cup Y$ is a boundary for Z, i.e. $(Z, Y \cup Y)$ is a Poincaré pair, which simply means that Z with those two copies of Y satisfies the Poincaré duality appropriate to manifolds with boundary. We insist that X is the result of gluing together the two copies of Y.

The splitting problem can now be phrased as: given a homotopy equivalence $h\colon M \to X$ and $Y \subset X$ Poincaré-embedded, can we homotop h so that h is transverse to Y,[17] and restricts to a homotopy equivalence between Y and its inverse.

[14] But we will discuss this more seriously in the Chapter 5.

[15] Very closely related to the invariants introduced by de Rham in *his* proof of the classification of lens spaces that we will discuss in Chapter 5.

[16] Details can be found in Chapter 2 of Wall (1968).

[17] Note that for transversality one does not need manifolds: a "normal structure" to the sub-object that is a vector bundle (or perhaps somewhat weaker than that) is enough.

One example where this is relevant is the following fibering problem:

Problem 4.8 Suppose M is a manifold with a surjection $\pi_1 M \to \mathbb{Z}$ (i.e. a map $f\colon M \to S^1$ with connected homotopy fiber). When is there a fibration of M over the circle, i.e. a map to S^1 realizing this data (e.g. homotopic to this map)?

In the case when $\pi_1 M \to \mathbb{Z}$ is an isomorphism, a beautiful necessary and sufficient condition was given by Browder and Levine (1966): fibering is possible iff the associated infinite cyclic cover has finitely generated homology. (After all, if the manifold fibers, the fiber would be homotopy equivalent to this cover.)

The first hypothesis for the general problem should be an analogue of this kind of finiteness. It is convenient to ask that the infinite cyclic cover, written F, should have as cellular chain complex $C_*(F) \sim C_*$ a finitely generated projective complex.[18] A space with finitely presented fundamental group and with this property on its cellular chain complex is called *finitely dominated*.[19]

Under such conditions, we have the covering translate $\eta\colon F \to F$, and a homotopy equivalence $M \to T(\eta)$ from M to the mapping torus. This mapping torus, $(T\eta)$, is a Poincaré complex (it is homotopy equivalent to M) and with some effort one can show that F is too (it's finitely dominated). Splitting this map is part of homotoping the map to a fibration.

Theorem 4.9 (Siebenmann, 1970a; Farrell, 1971b) *If $f\colon M \to S^1$ is a map which is a surjection on fundamental groups, with* $\dim M > 5$*, then f is homotopic to a fibration iff the following hold:*

(1) *the associated infinite cyclic cover of M is finitely dominated;*
(2) *an obstruction that lies in* $\mathrm{Wh}(\pi)$ *vanishes.*

If M has boundary and its boundary already fibers, the same result holds for the problem of extending the fibration to M.

The Whitehead group $\mathrm{Wh}(\pi)$ is defined purely algebraically. Let $\mathbb{Z}\pi$ be the integral group ring of the group π (it consists of finite formal sums of symbols of the form $a_g g$, where the a_g are integers, and the g are elements of π – made into a ring in the only sensible way imaginable).

$\mathrm{GL}_n(\mathbb{Z}\pi)$ is the group of invertible $n \times n$ matrices over this ring. We can

[18] This is reasonable given the special role that projective modules play in homological algebra: checking projectivity is often much simpler than freeness – cohomological vanishing suffices for the former, but not the latter.

[19] This is equivalent to being a retract of a finite complex, just like a projective module is retract (i.e., factor) of a free module.

stabilize by adding an identity in the bottom right to an invertible matrix:

$$\mathrm{GL}_n(\mathbb{Z}\pi) \to \mathrm{GL}_{n+1}(\mathbb{Z}\pi) \to \cdots \text{ whose limit is } \mathrm{GL}(\mathbb{Z}\pi).$$

The first algebraic K-group is defined by $K_1(\mathbb{Z}\pi) = \mathrm{GL}(\mathbb{Z}\pi)/\mathrm{E}(\mathbb{Z}\pi)$, where $\mathrm{E}(\mathbb{Z}\pi)$ is the group generated by elementary matrices. A lemma of J.H.C. Whitehead (which can be found in any introduction to K-theory) tells us that this quotient is abelian; indeed the product of matrices AB is equivalent in K_1 to $A \oplus B$ and $\mathrm{E}(\mathbb{Z}\pi)$ is the commutator subgroups of $\mathrm{GL}(\mathbb{Z}\pi)$.

Finally set

$$\mathrm{Wh}(\pi) = K_1(\mathbb{Z}\pi)/(\pm\pi),$$

where we mod out by the obvious invertible 1×1 matrices $(\pm g)$ where, again, g denotes a group element in π.

Note that, if π is trivial, this group is trivial using row operations from linear algebra (and the Euclidean algorithm). When π is finite cyclic, $\mathrm{Wh}(\pi)$ contains the obstruction that de Rham used to distinguish homotopy equivalent lens spaces (see Cohen, 1973. The group $\mathrm{Wh}(\pi)$ has an important part to play in the Borel story and it is therefore important not to discuss it too early, since it also provides a possible obstruction to homotoping maps to homeomorphism. We will rely on a theorem of Bass *et al.* (1964) which tells us that $\mathrm{Wh}(z^k) = 0$.

The fibering theorem is an analogue of an earlier important theorem:[20]

Theorem 4.10 (h-cobordism) *Let M be a compact manifold,* dim > 4. *Then there is a one-to-one correspondence:*

$$\tau\colon \{W \mid \partial W = M \cup ? \textit{ and } W \textit{ deform retracts to both } M \textit{ and } ?\} \leftrightarrow \mathrm{Wh}(\pi).$$

In particular, if $\mathrm{Wh}(\pi) = 0$, then M and “?" are (Cat)-homeomorphic. In general, τ is called the torsion of the homotopy equivalence $M \to W$. A homotopy equivalence with zero torsion is called simple (see §5.5.3 for more discussion). The h-cobordism theorem asserts that s-cobordisms, i.e. h-cobordisms where the inclusion of one side is simple, are products, so finding s-cobordisms between M and “?” ends up being the same as finding (Cat)-homeomorphisms between these manifolds.

Almost all homeomorphisms constructed in high-dimensional topology make use of this theorem or ideas from its proof. In particular, the proof of the Borel conjecture for the torus $\mathbb{T}^k$ depends on the Bass–Heller–Swan calculation and the h-cobordism theorem in just this way.

[20] This is due to Smale in the simply connected case and is the backbone of his proof of the high-dimensional Poincaré conjecture. In general (for the PL and smooth categories) it is due to Barden, Mazur, and Stallings (see Kervaire, 1969); Rourke and Sanderson, 1982). The topological case is due to Kirby and Siebenmann.

The condition that W has its boundary components are deformation retracts is analogous to the finite domination of the infinite cyclic cover: it asserts the homotopical possibility of the geometric structure (i.e., a product structure or a fibering) we are seeking. In both cases, the obstruction lies in the same algebraic K-group, and they arise in both cases through "handlebody theory," the manipulation of handles to mimic geometrically that homotopy theory (which is a manipulation of cells) – except that one has to occasionally replace algebraic isomorphisms by geometric moves that require (freeness in the place of projectivity or) elementary matrices and their products.

In the presence of the h-cobordism theorem, the fibering theorem is then directly visible as a combination of two obstructions. The first is a splitting obstruction, which would give us a submanifold F' in M, homotopy equivalent to F. When we cut M open along F, we'd then get an h-cobordism from F to itself that also seems to involve a Wh obstruction to being a product. If it is a product, then we have exhibited M as a fiber bundle over $\mathcal{S}^1$.

Actually, and this is an important point, the π–π theorem enables one to work in reverse and prove the splitting theorem from the fibering theorem – the theorem of Farrell. The advantage of this is that the splitting theorem is relevant to many more manifolds and submanifolds than the fibering theorem.

In general, there are splitting theorems unrelated to fibering problems. For simplicity we shall just state a version adequate for our current purposes[21] that incorporates the vanishing of Whitehead groups.

Theorem 4.11 *Suppose M is a manifold with free abelian fundamental group $\mathbb{Z}^k$, and $V \subset M$ is a codimension-1 submanifold with fundamental group $\mathbb{Z}^{k-1}$. Then if* $\dim M > 5$, *any homotopy equivalence $f\colon M' \to M$ can be split along V.*

We can now use this to prove the Novikov conjecture and try to prove the Borel conjecture.

If we are dealing with manifolds W with free abelian fundamental group $\mathbb{Z}^k$, then the Novikov conjecture is essentially the statement about the homotopy invariance of the signatures of the inverse images of subtori $\mathbb{T}^i \subset \mathbb{T}^k$ (for the classifying map $h\colon M \to \mathbb{T}^k$).

The tori are inductively stacked $\mathbb{T}^i \subset \mathbb{T}^{i+1} \subset \mathbb{T}^{i+2} \subset \cdots \subset \mathbb{T}^k$, so we only have to deal with the codimension-1 situation. An important but not difficult lemma (that is essentially the same one that arises in the three-dimensional topology of Haken manifolds) is that we can homotop the map $h\colon M \to \mathbb{T}^k$

[21] Cappell (1974a,b) gave an essentially complete theoretical analysis of this problem. In some cases, his analysis contains a non-K-theoretic obstruction that we will return to when we discuss group actions on aspherical manifolds and the Farrell–Jones conjecture.

to one where $h^{-1}(\mathbb{T}^{k-1})$ has fundamental group $\mathbb{Z}^{k-1}$. Then we can split the homotopy equivalence $f\colon M' \to M$ along $h^{-1}(\mathbb{T}^{k-1})$ till we get all the way down to the inverse image of $\mathbb{T}^i$.

So, by the homotopy invariance of signature, we have proven the Novikov conjecture.

Except for one little point: there is a dimension condition in the theorem. The fibering and splitting theorems do fail in low dimensions. So we should get stuck when the codimension gets high enough.

However, for the purposes of the Novikov conjecture, this is irrelevant, by the multiplicativity property of (higher) signatures. We can cross our manifold by $\mathbb{CP}^2$ a number of times to increase dimension as much as we need, without changing any invariants ($\operatorname{sign}(\mathbb{CP}^2) = 1$) and then apply this argument.

For the Borel conjecture, it would be nice to argue inductively and get a decomposition of a homotopy torus as resembling homotopically the product of n copies of $(S^1, *)$ and then invoke the Poincaré conjecture (to handle these manifolds that inductively have boundary a sphere and are contractible). The dimension issue arises again but surgery theory gives a way to get around this. We will discuss the details of this later in this chapter. In the end, though, hidden behind the surgery method is a periodicity, which is an indirect application of the idea of crossing with $\mathbb{CP}^2$ still lurking behind the scenes.

4.4.2 Lusztig's Method

Recall that we had mentioned before that for any $\rho\colon \pi_1(M^{4k}) \to \mathrm{U}(n)$ we can define a signature $\operatorname{sign}_\rho(M)$ using $H^{2k}(M;\rho)$ thinking of ρ as defining a flat bundle.

The simple idea in Lusztig's (1972) was to do this in a family. Concretely, Lusztig considered $n = 1$, so let $\Lambda = \operatorname{Hom}[\pi_1(M^{4k}), \mathrm{U}(1)]$; it is a finite union of tori. (It is an abelian group under pointwise product.)

The idea is that, where before we thought of the signature as being an integer, one should reconsider it as a (virtual) vector space (and the integer as its dimension). Then, varying the construction over points of Λ, produces a vector bundle over Λ,[22] associated to M. So consider the bundle of $H^{2k}(M;\rho)$s over Λ; using an auxiliary Hermitian metric on this bundle, we can diagonalize the family of cup product pairings and obtain a virtual bundle: the difference between the positive and negative sub-bundles.

[22] There is an oversimplification here. The family of $H^{2k}(M;\rho)$ might not be a bundle, because of jumps in dimension. The same issue arises in the Atiyah–Singer index theorem for families where, for some values of the parameters, the dimension of kernel and cokernel might jump. One has to introduce some perturbations to the family to obtain genuine kernel and cokernel vector bundles. See Atiyah and Singer (1968b).

The Atiyah–Singer theorem for families gives a formula (for the Chern character of) $\mathrm{sign}_\pi(M) \in K^0(\Lambda)$. Note, by the way, that this invariant lies in a place that is covariant in π, as both Λ and K^0 are contravariantly functorial in π.

Now $\mathrm{sign}_\pi(M)$ *detects exactly the higher signatures associated to products of one-dimensional cohomology classes* (or dually the image of the higher-signature class from $\bigoplus H(K(\pi,1);\mathbb{Q})$ in $\bigoplus(H(K(H_1(\pi,\mathbb{Z})/\text{torsion}),1);\mathbb{Q}))$. The numerology of $K^*(\Lambda)$ makes this at least believable. For $\mathbb{Z}^k$, the space of U(1) representations, Λ is a torus $\mathbb{T}^k$, and K^* has a Künneth formula – giving us C_i^n copies of K^0, where C_i^n is the binomial coefficient; K^0 itself has a $\mathbb{Z}$ every fourth dimension (just the right size for a signature).[23]

This method is very nice in that not only does it prove that the relevant higher signature is a homotopy invariant, it also gives a formula that tells us why it's true. That is also true of the next variant that we will describe.

4.4.3 Using the Atiyah–Kodaira Fiber Bundle

A rather different verification of the Novikov conjecture is possible for surfaces of high genus[24] by making use of a different "representation." Atiyah and Kodaira have given a surface bundle over a surface with nonzero signature (see, for example, Atiyah, 1969).

The method is this: one takes a product of surfaces, and inside of it a subsurface that intersects each fiber the same way (i.e. in a fibered way). In this way the subsurface is a – perhaps disconnected – covering space of the base surface. If the subsurface is trivial as a class in mod 2 homology, then one can take the branched $\mathbb{Z}/2$ cover of the product along the surface. If the surface has nontrivial Euler class (e.g. its self-intersection is nontrivial integrally, or equivalently the cup square of its Poincaré dual is nontrivial), then it turns out that the signature of the total space of the branched cover is nontrivial.[25]

The details don't matter. What matters is this bundle whose total space has nonzero signature although the base does not. We denote the base of this bundle by Σ and the bundle itself by π.

Suppose now that we have $f\colon M^{4k+2} \to \Sigma^2$ a manifold with a map to a surface, we can *define*

$$\mathrm{sign}\,(f) := \mathrm{sign}\,(f * \pi).$$

[23] Actually, there's a $\mathbb{Z}$ every second dimension, but the ones that arise in $2 \bmod 4$ don't come up for these signature operators. Had we worked in KO, they wouldn't be observed.

[24] Note that, for any genus, we can reduce the Novikov conjecture to the special cases of $\mathbb{Z}$ and $\mathbb{Z}^2$ since all the cohomology of the surface is pulled back from one of these groups.

[25] These calculations can be done using the equivariant form of the signature theorem (Atiyah and Singer, 1968b).

Cobordism invariance of signature and its multiplicative properties show that this invariant of f only depends on the class that (M, f) represents in $\Omega_{4*+2}(\Sigma^2) \oplus_{\Omega^*(*)} \mathbb{Q}$, which is $f_*(L_k(M) \cap [M]) \in H_2(\Sigma; \mathbb{Q})$. In other words[26]

$$\operatorname{sign}(f) = C\langle f^*[\Sigma] \cup L_{4k}(M), [M]\rangle,$$

where C is determined by setting $f = \mathrm{id}$.

A by-product is that we now know the Novikov conjecture for high-genus surfaces by a homotopy-invariant formula (just as we had achieved for the case of $\mathbb{Z}$ in §4.3).

4.5 Novikov's Theorem

Gromov observed that the Atiyah–Kodaira example can be used to simplify[27] the proof of Novikov's theorem on topological invariance of rational Pontrjagin classes.

Theorem 4.12 *If $f : M' \to M$ is a homeomorphism between smooth manifolds, then $f^*(p(M)) = p(M') \in \bigoplus H^{4i}(M'; \mathbb{Q})$.*

Of course, we will prove the equivalent that $f * (L(M)) = L(M')$. We will first describe the argument if f is PL following Thom, Milnor, Rochlin, and Schwartz. Without loss of generality, we will assume that the dimension of M is odd, since we can cross M with an odd-dimensional sphere, without loss of information.

We would like to give a PL-invariant calculation of $\langle L(M), c\rangle$ for any homology class c. Note that its Poincaré dual is odd-dimensional. According to Serre's thesis, for every odd-dimensional cohomology class $\mathrm{PD}(c)$, there is a nonzero multiple $N\mathrm{PD}(c)$ and a map $f : M \to S^{2r-1}$ multiple (which is unique up to homotopy after a further multiple) so that $N\mathrm{PD}(c) = f^*([S])$.

We define $L(M)$ to be the unique cohomology class with

$$\langle L(M), c\rangle \operatorname{sign}(f^{-1}(*))/N,$$

where f is the map to the sphere defined above associated to the Poincaré dual of c, and $*$ is a regular value for f. Using cobordism invariance, this is well defined and linear, defining a unique rational cohomology class – which, if M is smooth, the Hirzebruch formula identifies with the usual L-class.

[26] Atiyah deduces this and a stronger formula from the index theorem for families, but our point here is to point out that this example is somewhat different from the Lusztig example, although one can unify them.

[27] Although it still does make use of a key trick of Novikov, the audacious introduction of fundamental group into a simply connected problem.

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Regular values exist by Sard's theorem in the smooth category; in the PL category, they exist for a less deep reason. Choosing triangulations so that f is locally affine, any point in the interior of a top simplex is a regular value. If we knew transversality in the topological category (which is indeed true, thanks to Kirby and Siebenmann), we could complete the argument in Top , as well, but that is a deeper result than Novikov's theorem.

What we have to prove is this:

Lemma 4.13 *If $h\colon U \to M \times \mathbb{R}^i$ is a homeomorphism between smooth manifolds, then* $\operatorname{sign}(M) = \langle L(U), h^*[M]\rangle$.

We shall actually make use of the fact that f is a hereditary homotopy equivalence: that is, f is a proper homotopy equivalence restricted to any open subset of the target.

Without loss of generality we will assume i is even, $i = 2l$. We note that there is a product of Atiyah–Kodaira bundles $\pi \times \pi \times \cdots \times \pi$ over $\Sigma \times \Sigma \times \cdots \times \Sigma$ the product of l-surfaces. This bundle can be used for proving the Novikov conjecture for the fundamental class of a product of these surfaces.

Note that the punctured manifold $(\Sigma\times\Sigma\times\cdots\times\Sigma-p)$ immerses[28] in $\mathbb{R}^{2l}$. Then $M\times(\Sigma\times\Sigma\times\cdots\times\Sigma-p)$ immerses in $M\times\mathbb{R}^{2l}$. We pull back the Atiyah–Kodaira bundle over this manifold and would like to take the signature of its total space (to recover $\operatorname{sign}(M)$). This is slightly tricky, because we are in a noncompact situation, so we have to see that the signature is what we expect it to be (now defined using signature where we mod out by the torsion). This is fairly easy because the bundle is trivialized at ∞ (which is the neighborhood of p – so we know that the homotopy type at infinity is that of $M \times S^{2l-1} \times$ Fiber, and can calculate the effect[29] of gluing or removing a plug of the form $M\times\mathcal{D}^{2l}\times$Fiber.

Now for U we can pull back using the homeomorphism h to obtain a homeomorphic smooth manifold, and associated bundles, etc. Since everything is proper homotopy equivalent to the other side, we get the same total signature. However, on computing the signature of this total manifold we get (a nonzero multiple of) $\langle L(U), h^*[M]\rangle$.

The executive summary is that we find a codimension-$2l$ signature by computing the signature of an associated $2l$-dimensional bundle over the manifold!

[28] This is not at all obvious, but it follows from immersion theory (often called Smale–Hirsch theory): any parallelizable open manifold immerses in Euclidean space of the same dimension. This can be found in almost any treatment of h-principles, since it's the prototype of such a theorem.

[29] If one glues two manifolds with boundary together along their complete boundary, one obtains the sum of the signatures, and the signature of $M \times \mathcal{D}^{2l} \times$ Fiber is zero. This formula is called Novikov additivity.

This is the trick of §4.4 for some cases of the Novikov conjecture, and it suffices for the current application to Novikov's theorem.

4.6 Curvature, Tangentiality, and Controlled Topology

Out goal in this section is to introduce the idea of doing topology with control and explain the proof of the following theorem:[30]

Theorem 4.14 (Ferry and Weinberger, 1991) *Suppose W is a complete non-positively curved manifold and $f: W' \to W$ is a homotopy equivalence which is a homeomorphism outside of a compact set. Then f is tangential, i.e. f pulls back tangent bundles (in a way compatible with the identification given by the homeomorphism outside of some larger compact set).*

This implies[31] that $W' \times \mathbb{R}^3$ is homeomorphic to $W \to \mathbb{R}^3$ so it's definitely progress. In §4.7, we will see this, as well as why this result implies the Novikov conjecture for $\pi_1 W$.

The use of the rel∞ condition was discussed in the "morals" section (§3.8) of Chapter 3. Without it, the theorem would be highly false, as we've seen.

A key role in the proof is played by the following important theorem due to Ferry:[32]

Theorem 4.15 (Ferry (1979)) *Let M^n be a compact topological manifold, endowed with a metric. Then there is an $\varepsilon > 0$ such that if $f: M \to N$ is a continuous map to a connected manifold of dimension less than n, with* $\operatorname{diam}(f^{-1}(n)) < \varepsilon$ *for all $n \in N$, then f is homotopic to a homeomorphism.*

The ε is related to the size of the smallest handle in a handle decomposition of M, so if M is noncompact, we can sometimes guarantee that the theorem holds anyway. There are also ε–δ statements that describe how far, in some sense, f has to be moved to make it into a homeomorphism. We'll need both kinds of refinement below when we apply the theorem.

[30] This was strongly inspired by earlier work of Kasparov proving the Novikov conjecture for $\pi_1 W$ (see §8.5) by analytic methods.

[31] This is a little white lie: the method of proof gives this improvement. Tangentiality by itself would not control what dimensional Euclidean space we'd need to cross with to obtain isomorphism. To get this dimension down to three, it's important that the tangentiality be "compatible with an identification of Spivak fibrations" so that one obtains vanishing normal invariant – not just the image of this under the map $[W/\infty : G/\mathrm{Top}] \to [W/\infty : \mathrm{BTop}]$, which is the assertion of the theorem. Once one has this, the π–π theorem quickly gives the homeomorphism.

[32] Actually, Ferry originally proved it for $n > 4$, but it's since been shown to be unconditional through advances in low-dimensional topology (the largest being the solutions of the three- and four-dimensional Poincaré conjectures by Perelman and Freedman).

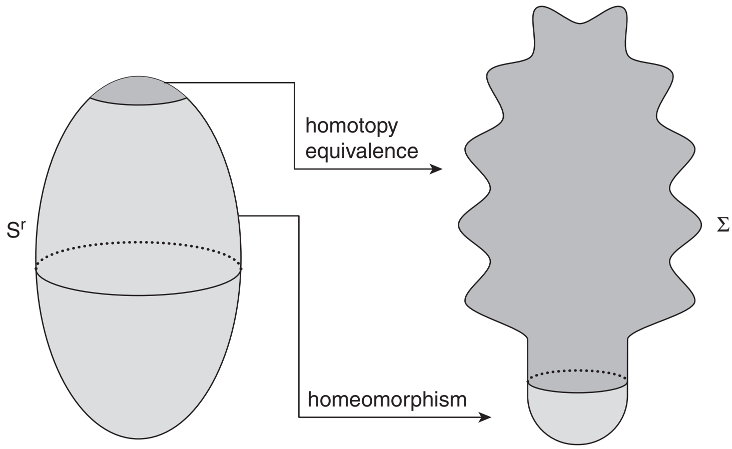

Figure 4.1 Proof of the Poincaré conjecture from Ferry's theorem.

This statement is of the form "an almost homeomorphism" is "almost a homeomorphism." Statements of this general type are sometimes trivial, sometimes trivially false, sometimes nontrivially false, and sometimes true, nontrivial, and useful. This theorem is of the fourth sort. To appreciate it, we shall give two examples:

Example 4.16 (Ferry $\Rightarrow$ Poincaré) Let S^n be the usual round n-sphere. And let ε be the epsilon guaranteed by Ferry's theorem. Let Σ be any homotopy n-sphere. I claim (assuming that Σ is a manifold!) that it is possible to build a map $f : S^n \to \Sigma$ where all point inverses have diameter less than ε.

Pick a point p and a neighborhood homeomorphic to $\mathbb{R}^n$. We map the complement of the $\varepsilon/2$ north polar ice cap in S^n homeomorphically to a ball in this neighborhood. The rest of Σ is contractible, so the map restricted to the $\varepsilon/2$-sphere around the north pole extends inwards (as a homotopy equivalence, although this is irrelevant to the application of Ferry's theorem) over the ice cap to Σ (lying entirely in the complement of the neighborhood of p; see Figure 4.1).

Let's examine the point inverses. If q lies in the neighborhood of p, then its inverse image $f^{-1}(q)$ is a single point, and has diameter equal to 0. If q lies outside the neighborhood, then its inverse image is constrained to lie in the $\varepsilon/2$ polar ice cap, and hence has diameter less than ε. Ferry's theorem then asserts that f is homotopic to a homeomorphism.

This proof shows the remarkable versatility of Ferry's theorem as a tool: the huge unexplored region in the range manifold is here shrunk to be in the $\varepsilon/2$

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polar ice cap, while the small coordinate chart around one point is expanded to be almost the whole sphere. It feels like a talk by a weak student who spends almost his full hour explaining trivialities, leaving only a couple of minutes to the whole essence of the matter! Nevertheless, in topology, Ferry's theorem says that this works! Of course, there's a price – the continuity of the map from the domain to the range.

Example 4.17 (Ferry $\Rightarrow$ a virtual Borel conjecture) Let V be a homotopy $\mathbb{T}^n$; then we shall see that every sufficiently large cover[33] of V, say with covering group $(\mathbb{Z}/k)^n$ for k large, is homeomorphic to the torus $\mathbb{T}^n$.

Let $f\colon \mathbb{T}^n \to V$ be a homotopy equivalence. Let's consider the map it induces between universal covers. Note that there is a universal bound C for all point inverses (for the map is automatically proper, and the bound for point inverses for one fundamental domain of the $\mathbb{Z}^n$-action works for all points by equivariance). Let ε be an ε appropriate to the torus $\mathbb{T}$. Suppose that $k > 2C/\varepsilon$, then we can identify the $(\mathbb{Z}/k)^n$ cover of $\mathbb{T}$ with $\mathbb{T}$, and we now have a map from $\mathbb{T}$ to a cover of V with point inverses of diameter less than ε. (The extra factor of 2 is to be in the range that the map from $\mathbb{R}^n \to \mathbb{T}^n$ is a local isometry.)

Remark 4.18 (On circularity) Of course, if the Poincaré conjecture and a virtual Borel conjecture for the torus were used in the proof of Ferry's theorem, this would be a circular argument. (Even so, the above should convince that the theorem is not vacuous!)

There are probably three different arguments for this theorem. They all go via the α-approximation theorem of Chapman and Ferry, which I will not describe – but its essential difference is that it measures sizes over the target, not the domain.

The original proof (Chapman and Ferry, 1979) is based on modifying the proof of a weaker version of the theorem, Siebenmann's CE-approximation theorem (Siebenmann, 1972). That theorem asserts:

Theorem 4.19 (CE-approximation) *A map $f\colon M \to X$ between manifolds is a limit of homeomorphisms (in the compact open topology) iff it is CE, i.e. if all $f^{-1}(x)$ are null-homotopic in arbitrarily small neighborhoods, i.e. iff f is a hereditary homotopy equivalence.*

This in turn used Kirby's torus trick, the basic tool in triangulation theory and requires a virtual Borel conjecture for tori. This argument would be indeed circular.

[33] That is, associated with any subgroup that intersects a metric ball of sufficiently large radius (depending on the original homotopy equivalence) only in the identity.

There is another proof, due to Quinn (1979, 1982b, 1982c, 1986), that is based on a controlled h-cobordism argument. This seems like it's essentially a generalization of the way the Poincaré conjecture was proved. However, the analogue of the fact that $\mathrm{Wh}(e) = 0$ is more difficult in the controlled situation and Quinn's proof uses the torus trick. This can be avoided by more recent methods of calculating.

However, finally, there is a third proof that is based on combining two approaches. The first is an engulfing argument (based loosely on Stallings's proof of the Poincaré conjecture, rather than Smale's) due to Chapman (1981) that reduces the α-approximation theorem to the CE-approximation theorem.

The CE-approximation theorem itself has an amazing extension to the situation where X is not assumed to be a manifold (and is hence very useful for proving that spaces are manifolds!) due to R.D. Edwards (it is the goal – achieved – of Daverman, 2007). In any case, Edwards's proof is purely geometric and does not rely on any algebraic tools, neither torus trickery nor surgery. So, finally – using this combination – this argument for the second application does not have to be viewed as circular.

Thus, the least generous view one could have is that Ferry's theorem somehow *is* a form of the Poincaré conjecture, but in liberating that problem from the sphere, we have obtained an extremely useful tool and perspective on the problem of homotoping maps to homeomorphisms. And, indeed, there certainly are many other arguments in the spirit of the ones above based on Ferry's theorem that are far from circular; indeed, much of the work on the Borel conjecture since the 1980s has this flavor (but are much more involved; see Chapter 8).

Now let us return to the proof of the main theorem of this section. For simplicity, we will first sketch our argument for W compact, where the result is actually considerably simpler – although not much simpler from the point of view that we adopt![34] We first note that the salient feature of the tangent bundle to a manifold is that it is a bundle of $\mathbb{R}^n$s – given a section – over a manifold, so that around the 0-section, the fiber direction is the same as the base direction. (In the smooth case, the exponential map sets up such an isomorphism.)

As a result, for an aspherical manifold $W = \widehat{W}/\Gamma$, we can consider the following quotient as a model for the tangent bundle:

$$\mathrm{T}W \approx (\widehat{W} \times \widehat{W})/\Gamma.$$

While we are used to saying "the universal cover" of a space, this notion actually

[34] The compact case was earlier proved by Farrell and Hsiang (1981) and doesn't need any version of Ferry's theorem.

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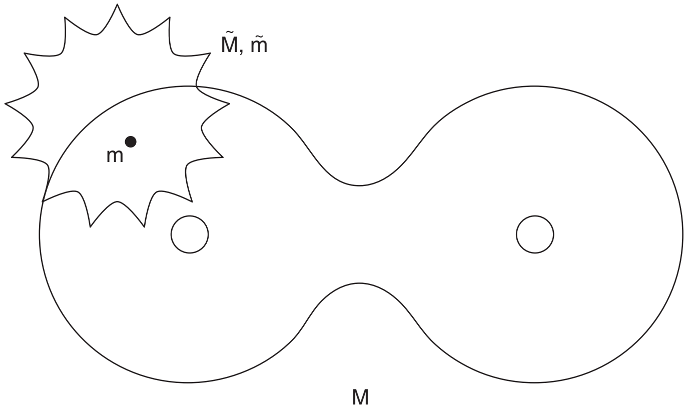

Figure 4.2 The moving family of universal covers as a tangent bundle.

requires a choice of base point, and as we vary the base point, this can indeed be a nontrivial bundle (see Figure 4.2).

In this model, the differential of the map $f\colon W \to W'$ (at the point w) is the lift of f to the universal cover $\widehat{W} \to \widehat{W}$ (based at the points w and $f(w)$, respectively). Notice that this map has bounded size point inverses (as in the case of the torus above). At this point let us use the Riemannian tangent bundle and the exponential map to a map $\mathrm{T}W_w \to \widehat{W}$. The fibers are now Euclidean spaces, and, *using non-positive curvature*, the point inverses are still of bounded size: as geodesics spread apart in non-positive curvature, the inverse of the exponential map is Lipschitz.

Now we apply Ferry's theorem to the family of maps $\mathrm{T}W_w \to \widehat{W'}$ to get a family of homeomorphisms.

In the closed case, one can actually "go all the way to ∞" and get an isomorphism between the ideal sphere bundles.[35] However, in the noncompact case we cannot do this, since we must interpolate between the "infinitesimal isomorphism" on tangent spaces coming from the homeomorphism outside of a compact, and the process we do in neighborhoods of points in $\widehat{W}$. As a result, Ferry and Weinberger (1991) instead argue about isotoping the family of balls of radius R, namely $\mathrm{B}\,(R, w)$, in $\mathrm{T}W_w$ to embeddings around $f(w)$ in $\widehat{W'}$.

Finally, as explained in §3.3, tangentiality is slightly less than we would want: we need the map to not change the identification of spherical fibrations guaranteed to us by Atiyah's theorem. This is essentially achieved by ensuring

[35] This method is due to Farrell and Hsiang (1981) and is reminiscent of one of the key steps in Mostow's (1968) proof of his rigidity theorem for closed hyperbolic manifolds.

that the isotopies are covered by homotopies that do not distort the function at ∞.

4.7 Surgery, Revisited

In Chapter 3, we discussed surgery in the special case where there are no obstructions, the π–π situation. In that case, the discussion ended up being essentially homotopy-theoretic, and the main results were the structure of classifying spaces, and therein lay the differences between the various categories.

We shall now discuss the case of closed manifolds and in a purely topological setting where the results seem to be in their most perfect form. The reader should treat these results as truths coming from on high: we shall not explain why they are true or take the form that they do.

On the other hand, since the moral of Chapter 3 was that functoriality is critical to our program, the reader should not object to a presentation of surgery theory in which functoriality plays a central role.

Let M be a compact manifold with ∂. We define the *structure set*:

$$S(M) = \{(M', \partial M', f) \mid f \colon (M', \partial M') \to (M, \partial M) \text{ a simple homotopy equivalence which restricts to a homeomorphism on the } \partial\}/s\text{-cobordism.}$$

For manifolds, s-cobordism, thanks to the h-cobordism theorem (see §4.4), is the same thing as being a product. However, to get the best properties of $S(M)$, it is convenient to allow in some non-manifolds in the definition of $S(M)$.

Definition 4.20 A homology manifold is a finite-dimensional absolute neighborhood retract, (ANR)[36] X so that, for all x, and all i,

$$H_i(X, X - x) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - 0).$$

Such an X satisfies Poincaré duality (in the noncompact sense), and therefore so does every open subset. A good example is the suspension of a homology sphere.[37] The cone points are the only non-manifold points, but they satisfy the hypothesis of the definition as their links are homology spheres.

Definition 4.21 A space X has the disjoint disk property (DDP) if, for any

[36] we are including this hypothesis in the definition of homology manifold, which is not a fully standard decision.

[37] A homology sphere here is a closed manifold with the $\mathbb{Z}$-homology of the sphere. Thanks to the Poincaré conjecture, such a space is the sphere (if $\dim > 1$) iff it is simply connected.

$f, g\colon \mathcal{D}^2 \to X$ and any $\varepsilon > 0$, there are perturbations, f' and g', such that $d(f, f') > \varepsilon$ and $d(g, g') < \varepsilon$ and $f'(\mathcal{D}^2) \cap g'(\mathcal{D}^2) = \varnothing$.

Note that manifolds of dimension 5 all satisfy DDP (as do many other spaces that are not homology manifolds). The DDP fails for the suspension example: any two $\mathcal{D}^2$s that go through a cone point, but whose boundaries map nontrivially in π_1(link) cannot be moved disjoint from the cone point.

Indeed DDP can fail quite dramatically: Daverman and Walsh gave an example (a *ghastly homology manifold*) of a homology manifold and a nice curve, so that every singular 2-disk it bounds contains an open set.

Edwards's theorem is a CE-approximation theorem for DDP homology manifolds:[38]

Theorem 4.22 (See Daverman, 2007) *A map $f\colon M \to X$ from a manifold to a homology manifold with the DDP is a limit of homeomorphisms (in the compact open topology) iff it is* CE*, i.e. all $f^{-1}(x)$ are null-homotopic in arbitrarily small neighborhoods, i.e. iff f is a hereditary homotopy equivalence.*

A homology manifold X for which a CE map $f\colon M \to X$ from some manifold exists is called *resolvable*. Quinn (1982a, 1987b) showed that, if X is a connected homology manifold and it contains any resolvable open subset (e.g. it has a manifold point), then it is resolvable! Quinn defined an invariant $I(X) \in Z$, which is 1 iff X is resolvable. We can compute $I(X)$ locally from a neighborhood of any point: it is $1 \bmod 8$, and it has the property that $I(X \times Y) = I(X)I(Y)$.

It turns out that using DDP homology manifolds in the definition of $S(M)$ above is the "right" thing to do. First of all, if $\partial M \neq \varnothing$, by Quinn's theorem combined with Edwards's, we are not actually allowing any nonmanifolds in.

Secondly, we will see that even allowing in homology manifolds does not affect the Borel conjecture: if it is true for manifolds, it is true for homology manifolds.

And, finally, it is with this more elaborate definition that $S(M)$ achieves its strongest functorial properties. We begin with that last point:

Theorem 4.23 (Bryant *et al.*, 1996) *If M is a manifold, $S(M) \cong S(M \times \mathcal{D}^4)$.*

This is an analogue of Bott periodicity. Actually, there is a version of the Thom isomorphism theorem: $S(M) \cong S(E)$ if E is an oriented $\mathcal{D}^{4k}$-bundle over M. However, even the periodicity statement has important consequences. Setting $M = \mathcal{S}^n$ we see the right-hand side is $\mathbb{Z}$,[39] so we "need" the sphere

[38] But the version where both M and X are DDP homology manifolds would be much better!
[39] $[\mathcal{S}^n \times \mathcal{D}^4/\partial\colon F/\mathrm{Top}]$ has a $\mathbb{Z}$ from $\pi^4(F/\mathrm{Top})$.

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to have nontrivial structures – seeming "counterexamples" to the Poincaré conjecture.

This $\mathbb{Z}$ is given by $(I(X)-1)/8$. There is a $\mathbb{Z}$′s worth of homotopy spheres that are really determined by their local structure. Conjecturally, there is a unique homotopy sphere for each integer. This would make a lot of sense: the proof of the Poincaré conjecture by Smale uses the h-cobordism theorem, and starts by choosing a small ball neighborhood of a point in a coordinate chart, using the manifold hypothesis. For other local indices, the very first trivial step seems to be the one that is obstructed.

In addition, note that on the right-hand side of this equation there is an abelian group structure: the structures we are using are homeomorphisms on the boundary. Thinking of $\mathcal{D}^4$ as a cube, we can glue along faces to "add" elements, and the usual proof that π_2 is abelian shows that this is a commutative group structure. (The elements of $S(M)$ that correspond to manifolds are a subgroup. The map $(M', f) \to (I(M) - I(M'))/8$ is a group homomorphism $S(M) \to \mathbb{Z}$.)

Using this, we can easily finish off the proof of the Borel conjecture for the torus we had sketched in §4.4: using periodicity there is no "low dimension" to push past. Of course, in this view, the vanishing of $S(*)$ should not be viewed as a triviality: it is – in light of periodicity – the Poincaré conjecture[40] in dimensions a multiple of 4.

A next formal point is that it then becomes reasonable to have additional groups, namely $S(M \times \mathcal{D}^i)$ for any i. Doing this systematically leads to the following definition for arbitrary finite CW-complexes X.

Definition 4.24 Let X be a finite complex. We define $S_n(X)$ as $S(M)$ where M is any compact oriented n-manifold with boundary simple homotopy equivalent to X. If there is none, define $S_n(X) = S_{n+4k}(X)$ where $4k$ is large. If X is infinite, take the limit over the finite subcomplexes of X.

Note that, thanks to periodicity, the groups $S_n(?)$ are actually covariantly functorial.[41] After a periodicity if necessary, given $f\colon X \to Y$, we can embed the manifold for X into the manifold for Y, and get the "pushforward" of the structure to be obtained by gluing in the annular region. Such an embedding, after further stabilization, is unique up to isotopy. This turns the $S_n(?)$ into homotopy functors.

Moreover, it does not take much to imagine the meaning of $S_n(Y, X)$ for a pair. We have to consider manifolds with boundary whose interiors are homotopy

[40] Here viewed as the vanishing of structures of the disk, which tacitly has the boundary condition – and therefore the manifold hypothesis – built in.

[41] This is completely analogous to the wrong way maps defined at the beginning of Atiyah and Singer (1968a) using Bott periodicity (interpreted as a Thom isomorphism kind of statement).

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equivalent to Y and whose boundaries contain a piece homotopy equivalent to X (and we work relative to the rest of the boundary).

Theorem 4.25 *The $S_n(?,?)$ are a sequence of covariant homotopy functors. They are 4-periodic and fit into an exact sequence of abelian groups:*

$$\cdots \to S_n(X) \to S_n(Y) \to S_n(Y,X) \to S_{n-1}(X) \to S_{n-1}(X) \to \cdots .$$

If M is an oriented n-manifold, then $S_n(M) \cong S(M)$.

(There is no difficulty in setting up an analogous theory for nonorientable manifolds, and a proper theory for the noncompact situation.)

Note that, from this perspective, the "cohomological term" that we had looked at in the π–π theorem actually is naturally a homology theory (the functoriality flipped!); the perspectives in these two approaches to the theory are Poincaré dual: $[M^n : \mathbb{Z} \times F/\mathrm{Top}] \cong H_n(M, \partial M; L(e))$, where $L(e)$ is the homology theory associated to $\mathbb{Z} \times F/\mathrm{Top}$.

Theorem 4.26 *If $K(\pi,1)$ is a finite complex, then the statement that, for all n, $S_n(K(\pi,1)) = 0$, follows from the Borel conjecture.*[42]

(If it is a manifold, then the vanishing of all the $S_n(K(\pi,1))$ follows from the vanishing of all of the $S(K(\pi,1) \times \mathbb{T}^k)$. Note that $S(K(\pi,1) \times D^4)$ is a summand of $S(K(\pi,1) \times \mathbb{T}^4)$, so if there is an extra $\mathbb{Z}$ arising from a homology manifold, it arises for manifolds four dimensions higher.)

The theorem follows immediately from functoriality together with the Davis construction:[43] it produces from $K(\pi,1)$ an aspherical manifold M (of any large enough dimension) that has $K(\pi,1)$ as a retract, and therefore $S_n(K(\pi,1))$ is a summand of $S_n(M) = S(M)$ (or if one is opposed to homology manifolds, a factor of $S_n(K(\pi,1) \times \mathbb{T}^4)$.

We can now think of the Borel conjecture as the statement for π torsion-free, $S(K(\pi,1)) = 0$ (in all dimensions).[44]

We now note that if the Borel conjecture is true for π, and $\pi_1(M^n) \to \pi$ is an isomorphism, then we get an exact sequence:

$$\cdots \to S_{n+1}\big(K(\pi,1)\big) \to S_{n+1}\big(K(\pi,1), M\big) \to S_n(M) \to S_{n-1}(K(\pi,1)) \to \cdots ,$$

which becomes a homological calculation of $S(M)$:

$$S(M) \cong S_n(M) \cong S_{n+1}(K(\pi,1), M) \to H_{n+1}(K(\pi,1), M; L(e)),$$

[42] The converse requires an additional statement about vanishing of Whitehead groups. We will discuss it in Chapter 5.

[43] As noticed by Davis.

[44] We note that groups with torsion do not have finite-dimensional $K(\pi,1)$s. Here we are taking a leap of faith that all torsion free groups behave the same way as those with a finiteness condition.

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the final isomorphism coming from the π–π theorem.

This isomorphism means that (when we take ∂) the (relevant) data comparing $M\prime$ to M, namely the difference of the L-classes, if we were working rationally, vanishes by the time we push further to $H_n(K(\pi,1); L(e))$. This is exactly (an integral form of) the Novikov conjecture.

Note then the philosophy that emerges from functoriality (conditionally on the Borel conjecture): *A manifold is exactly as rigid as it is homologically similar to* $K(\pi,1)$.

Note also, by a diagram chase, unconditionally, no homology in $H_n(M; L(e))$ that dies in $H_n(K(\pi,1); L(e))$ contributes to $S(M)$. (The cokernel in degree $n+1$ only contributes conditionally on Novikov-type statements.)

Let us be a bit more explicit and go back to a more classical view of surgery. It is high time that we mention the surgery exact sequence![45] This is an exact sequence that looks like:

$$\cdots \to L_{n+1}(\pi) \to S_n(M) \to H_n(M; L(e)) \to L_n(\pi) \to \cdots .$$

These groups, $L_n(\pi)$, are called "L-groups" or "Wall groups" and have a purely algebraic definition. They are 4-periodic, and describe the "obstruction to doing surgery to convert a degree-1 normal map into a (simple) homotopy equivalence."[46]

Classically,[47] this all was described as an exact sequence, valid in all categories:

$$\cdots \to L_{n+1}(\pi) \to S^{\mathrm{Cat}}(M) \to [M : F/\mathrm{Cat}] \to L_n(\pi),$$

with the following understanding: $S(M)$ is just a set with a distinguished element (the identity); and $[M : F/\mathrm{Cat}]$ is just a set.[48] Thus, exactness has to be interpreted appropriately, with "distinguished element" taking the place of 0. The map $L_{n+1}(\pi) \to S^{\mathrm{Cat}}(M)$ is then a group action. Given M and an element $\alpha \in L_{n+1}(\pi)$, there is a degree-1 normal map $W \to M \times [0,1]$, such that, on the bottom ∂ of W, one has a homeomorphism to M, and on the top, one has a homotopy equivalence. In that circumstance, there is a rel ∂ surgery obstruction

45 Ordinarily (and very properly) the centerpiece of presentations of surgery theory.

46 There are different theories of surgery based on whether one wants to obtain homotopy equivalences, and then the equivalence relation is h-cobordism; or whether one wants a simple homotopy equivalence, and then s-cobordism is the equivalence relation. The L-groups differ by 2-torsion in a way described by the Rothenberg sequence; see Shaneson (1969).

47 We refer to Wall (1968), Lees (1973), and Lück (2002a) for some expositions of the classical theory.

48 Although for Cat = PL or Top , Sullivan's H-space structure turns this into an abelian group and the map $[M : F/\mathrm{Cat}] \to L_n(\pi)$ is a homomorphism.

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of the map $W \longrightarrow M \times [0,1]$, which is α. The action then assigns to α the upper boundary homotopy equivalence.[49] One can show that this is well defined.

The groups $L_n(\pi)$ have purely algebraic definitions. The classical definition is given by Wall (1968), which makes it clear that $L_{2k}(\pi)$ is associated to $(-1)^k$-symmetric quadratic (or, better, Hermitian) forms over $\mathbb{Z}\pi$ and that $L_{2k+1}(\pi)$ are associated to their automorphisms. On the other hand, Ranicki (1980a,b) gave a very nice definition[50] in terms of chain complexes with duality, and their cobordism that makes the algebraic treatment "dimension-independent" and that is quite useful, because many more things end up directly defining elements in L-groups, and also it is more flexible for making constructions. Note that, by the algebraic definition, L-groups are 4-periodic. Indeed, this is the *source* of the periodicity of the S_n functors.

In any case, I should describe at least what happens in the simply connected case:

For $\pi = e$, then for n odd, $L_n(e) = 0$. For $4|n$, we have $L_n(e) \cong \mathbb{Z}$. The invariant is this: If $M \longrightarrow X$ is a degree-1 normal map (ignoring the bundle data), then $\operatorname{sign} M = \operatorname{sign} X$ is a necessary condition to be able to normally cobord f to a homotopy equivalence, since signature is both cobordism-invariant and homotopy-invariant. The isomorphism $L_0(e) \cong \mathbb{Z}$ is given by $1/8(\operatorname{sign} M - \operatorname{sign} X)$, the divisibility being a consequence of the bundle data that we suppressed: it ultimately forces the quadratic form on the cok $f*$ to have even numbers on the diagonal, which makes divisibility by eight automatic.

In dimensions that are $2 \bmod 4$, $L_2(e) \cong \mathbb{Z}/2$ with the isomorphism provided by the "Arf invariant." Here the antisymmetric bilinear form over $\mathbb{Z}$ is standard, but the refinement that a quadratic form has for $\lambda(x,x) = 2\mu(x)$ gives rise to an invariant $L_2(e) \cong L_{0,2}(F_2)$ (in the target, being characteristic 2, there is no difference between 0 and $2 \bmod 4$).

The homotopy group isomorphism $\pi_n(F/\mathrm{Top}) \cong L_n(e)$ is essentially a consequence of the Poincaré conjecture (although one needs special low-dimensional arguments for $n < 5$), and perhaps makes calling the spectrum whose homology theory we said was dual to $[?: \mathbb{Z} \times F/\mathrm{Top}]$ by the name $L(e)$ seem less peculiar.[51]

The classical surgery exact sequence continues infinitely to the left, but not

49 This is an action because, for any homotopy equivalence $M' \to M$, we can build such a W associated to α, with the given homotopy equivalence being the bottom boundary.

50 Strongly motivated by Chapter 9 of Wall (1968) that gives a cobordism treatment of relative L-groups that are complicated by the fact that manifolds with boundary are always both odd- and even-dimensional! An algebraic cobordism approach to L-groups was given first by Mischenko, but it was somewhat buggy at the prime 2.

51 The truth lies somewhat deeper than this – and arises either from blocked surgery or from controlled topology. This might already be clearer in the coming section. For more information about all the classifying spaces arising in surgery theory, see Madsen and Milgram (1979)).

to the right (unless one restricts to the setting we described: Top and including homology manifolds).

Using the π–π theorem, we can describe what we have written as an analysis of the map obtained by crossing with $\mathcal{D}^3$:

$$S_n(M) \to S_n(M \times \mathcal{D}^3, M \times \mathcal{S}^2) \cong H_{n+3}(M \times \mathcal{D}^3, M \times \mathcal{S}^2; L(e)) \cong H_n(M; L(e))$$

(the last isomorphism is a suspension isomorphism in a generalized homology theory) and, in these terms, $L_n(\pi)$ occurs as measuring the obstruction to solving a splitting problem. In any case, this perspective gives the normal invariants functoriality as well, and the whole surgery exact sequence becomes functorial. We note the special case:

$$\begin{array}{ccccccccc} \cdots \to & L_{n+1}(\pi) & \to & S_n(M) & \to & H_n\big(M; L(e)\big) & \to & L_n(\pi) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & L_{n+1}(\pi) & \to & S_n\big(K(\pi,1)\big) & \to & H_n\big(K(\pi,1); L(e)\big) & \to & L_n(\pi) & \to \cdots . \end{array}$$

This factors the surgery obstruction map through a universal functorial map between two group-theoretic objects $H_n(K(\pi,1); L(e)) \to L_n(\pi)$.

This map is called the assembly map. It has several interpretations, one of which has to do with assembling things. In §4.8 we will explain that it has another interpretation in terms of "forgetting control" in the sense of controlled topology. There are similar maps in algebraic K-theory and operator K-theory, and they will occupy us in Chapter 5.

Note that the Borel conjecture can be rephrased using the assembly map as follows:

Conjecture 4.27 *If π is a torsion-free group, then the assembly map is an isomorphism.*

The Novikov conjecture then is the following:

Conjecture 4.28 *If π is any group, then the assembly map is rationally an injection.*

We leave the verification that this is equivalent to the version involving higher signatures as an exercise. (Consider the map $[X\colon F/\mathrm{Top}\,] \to \Omega(X) \otimes_{\Omega*(*)} \mathbb{Q}$ given by $[M \to X] \to f_*\big(L(M) \cap [M]\big) - L(X) \cap [X]$.[52])

[52] We note that the right-hand side of the equation commutes with the periodicity isomorphism.

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4.8 Controlled Topology, Revisited

Having discussed briefly one result in controlled topology and then classical surgery theory, we would be remiss if we did not discuss their marriage. In general, the theme of controlled topology is to redo the problems solved in classical topology, but now with attention paid to the size of the constructions.

Size can be measured in various ways, and this theme has many incarnations and variants: indeed, we have tacitly already used two different kinds of controls. When discussing the Novikov conjecture, we used the fact that we had a uniform bound on the sizes of things, but this size was not made available – we were obliged just not to leave the category of maps that maintain that same property of uniform boundedness: this is called *bounded control*. Our discussion of Novikov's theorem was based on *epsilon control* because we used the fact thatc all point inverses become contractible in a small neighborhood of themselves.[53]

For stratified spaces, it is useful to use *continuously controlled at* ∞ techniques, and in Chapter 8 we will discuss the beautiful idea of *foliated control* introduced by Farrell and Jones.

In all settings, the basic idea is to:

(1) do what is classically done in topology, i.e. reduce the geometric problem to one of algebra – so we have already seen $\mathrm{Wh}(\pi)$, $L(\pi)$, etc. – so there should be some such structure associated to the problem and it will now be algebra associated to the "control space" as well as the fundamental groups involved; and then

(2) actually *do* the algebra. This point of view is mainly due to Quinn, who developed many consequences of it; the first nontrivial cases of the theory were already in place in advance: notably the work of Anderson and Hsiang (1976, 1977, 1980) and Chapman and Ferry (1979).

A simple example is this. The space of proper maps from $\mathbb{R}^n \to \mathbb{R}^n$ is highly nontrivial – for example, it has $\mathbb{Z}$ components given by degree. If we add the condition that $d\big(x, f(x)\big) < C$ (where C is allowed to vary), then the function space becomes contractible.

Another example is this (for the impatient, skip to the formal definition a few paragraphs from here). If we take a homotopy equivalence $f\colon X \to Y$, with

[53] There is room for a distinction here: one can also study *approximate control* where one tries to prove an ε–δ theorem, where one wants to move things by at most δ to a solution, willing to assume initial data which are "ε-controlled." Such results are called "squeezing theorems" and the prototype might be Chapman's proof of the α-approximation theorem: a squeezing theorem reduces an approximate problem to an ε-controlled one – in our terminology. Quinn's (1979, 1982b, 1982c, 1986) papers deal with both issues, the squeezing and the ε-controlled, simultaneously.

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X and Y polyhedral, and ask that, after taking an open cone,[54] Cf is still a homotopy equivalence in the category of maps where no point is moved more than a bounded amount (measured in CY), then we can deduce that, for every open set U in Y, $f^{-1}U \to U$ is a proper homotopy equivalence. If X and Y were manifolds of the same dimension, then f would be a uniform limit of homeomorphisms!

Of course, the benefits of having control are not always this dramatic. Having various controlled categories provides us with language and tools to scaffold incremental progress towards building homeomorphisms (or other useful geometric maps).

To give the idea and stay close to our roots, we will focus on the bounded category.

Definition 4.29 Let X be a metric space. Then the bounded category $\mathrm{Bdd}(X)$ is the category whose objects are spaces with maps, $(Z, f\colon Z \to X)$, and morphisms consisting of continuous maps $g\colon Z \to Z'$ so that $d(gf', f) < C$ for some C. Note that (Z, f) and (Z, f') are canonically equivalent in this category by the "identity map" if $d(f, f') < C$ via the identity, and that we do *not* insist that f be continuous.

Because of the last point, many metric spaces give rise to equivalent bounded categories.

Definition 4.30 If X and Y and are metric spaces, then a map $\varphi\colon X \to Y$ is a *coarse quasi-isometry* if (1) there are constants such that $A^{-1}d(x, x') - B < d(\varphi(x), \varphi(x') < Ad(x, x') + B)$; and (2) $\varphi(X)$ is C-dense in Y, i.e. every point in Y is within C of some point of X.

Coarse quasi-isometric metric spaces clearly have equivalent bounded categories.[55] As a trivial example, any bounded diameter metric space is equivalent (i.e., coarse quasi-isometric) to a point.

A very important example is that the universal cover of a compact manifold is coarse quasi-isometric to its fundamental group (with the word metric). The subject of geometric group theory is largely the study of this equivalence relation on finitely generated (or finitely presented) groups.

With the above terminology in place, it makes sense to raise questions such as the h-cobordism problem or the surgery question in $\mathrm{Bdd}(X)$. If X is a point, then this is just the old question and the obstructions involve $\mathrm{Wh}(\pi)$ and $L_n(\pi)$,

[54] We metrize so that the cone on a simplex is a Euclidean octant.

[55] Actually, there is a looser equivalence relation that also gives isomorphisms of bounded categories, wherein one replaces the linear upper and lower bounds by non-decreasing functions that go to ∞ (such as log and exponential).

etc. So we will need to take the fundamental groups of the objects into account, in particular the system of fundamental groups of inverse images of large balls of X.[56] But, for simplicity at this point, let's stick to the simply connected case.

Given that there is freedom in choosing the X when considering the category $\mathrm{Bdd}(X)$, we can try to choose the best possible model for X. For example $\mathbb{R}^n$ is in some ways a better model than $\mathbb{Z}^n$, because every object over the former can be replaced by one where the reference map f is *continuous*. We leave this an exercise, but observe that the key property that allows this (when the underlying space of the object is finite-dimensional) is the following:

Definition 4.31 A metric space X is *uniformly contractible* if there is a function $u(R)$ from $\mathbb{R}^+ \to \mathbb{R}^+$ such that any point $x \in X$, the ball $B(x,R)$ is *null-homotopic* in a larger concentric ball, given by $B(x,R)$ included in $B\big(x,u(R)\big)$.

One example is the open cone of a compact ANR, as we leave to the reader.

Another very good (and *important*) example is the universal cover of a compact $K(\pi,1)$.

Indeed, uniformly contractible metric spaces generally seem like good analogues of $K(\pi,1)$s. They are terminal objects in the bounded category of spaces over X that are coarse quasi-isometric to X – just as $K(\pi,1)$s are the terminal objects in the category of spaces that have the same 1-type as X.

Given the philosophy we have espoused in the previous chapter's morals section (§3.8), we should conjecture some type of bounded rigidity for uniformly contractible manifolds.[57]

Conjecture 4.32 (Bounded Borel) *If M is a uniformly contractible manifold, and $f\colon M' \to M$ is a homotopy equivalence in the bounded category over M, then f is homotopic (in this category) to a homeomorphism.*

[56] As a pro-system: the fundamental groups themselves have no real meaning, but 'system', as we allow larger and larger balls, does make sense.

[57] Unlike the Borel conjecture itself, the following conjecture is known to be false (Dranishnikov *et al.*, 2003). The example is a rather pathological Riemannian manifold that is abstractly a Euclidean space. It is based on an amazing example of Dranishnikov of a space of finite cohomological dimension but infinite covering dimension, and requires a violation of "bounded geometry." For example, if M has a triangulation with all simplices of bounded size, *and* a uniform bound on the valence of any vertex (or even a lower bound on injectivity radius and bounds on curvature), such as the universal cover of a finite $K(\pi,1)$-complex then the methods of that paper do not apply.

This example could suggest that the Borel and allied conjectures are not as well founded for groups of infinite cohomological dimension. However, there are many cases where the conjectures do seem to be correct even in this setting and the question requires a lot more thought.

Remark 4.33 (1) With a bit of care, one can show that the CE-approximation theorem is essentially a verification of a special case of this conjecture.

(2) Actually taking the fundamental group into account suggests a unification of these conjectures, the bounded rigidity of a *uniformly aspherical* manifold. Good examples of these are $K\backslash G/\Gamma$ for lattices – and they are indeed rigid (in high dimensions as observed in Chang and Weinberger, 2007).

Let us consider what the surgery exact sequence should suggest for this problem:

$$\cdots \underset{=0}{S_{n+1}(M \downarrow M)} \to H^{\mathrm{lf}}_{n+1}(M; L(e)) \to L_{n+1}(M \downarrow M) \to \underset{=0}{S_n(M \downarrow M)} \to \cdots .$$

So the bounded Borel conjecture asserts that *bounded L-groups are a homology theory of the control space* in the uniformly contractible case.

And, indeed frequently this is the case. For example, for the cone on a finite polyhedron this can be verified in a couple of ways. One can deduce this from the α-approximation theorem – rather like the way the L-groups of the trivial group are identified with the homotopy of F/Top via the Poincaré conjecture[58] (Ferry, 2010). Or one can show that this is a homology theory in the space being coned – with a codimension-1 splitting argument being used for the critical verification of the excision (or, equivalently, the Mayer–Vietoris) axiom (Carlsson and Pederson, 1995; Ferry and Pederson, 1995).

Versions of this principle are true for all the types of control we had mentioned and such results are central to controlled topology's many geometric consequences.

We end this section with a discussion of the following points.

Remark 4.34 (1) What to do when we don't have (or know) a uniformly contractible model.

(2) How to formulate a bounded Novikov conjecture.

(3) Then observe that Novikov's theorem on topological invariance of rational Pontrjagin classes is a special case of the latter.

(1) What can we do when we don't have (or know) a uniformly contractible model? This is completely analogous to the situation in ordinary surgery when we don't have a finite complex $K(\Gamma, 1)$.[59] What we do there is choose a sequence of finite complexes that approximates it. The limit is a space that we call $K(\Gamma, 1)$,

[58] With the opposite goal in mind. When we made this identification before, the goal was to learn about the homotopy type of F/Top, because the groups $L(e)$ were under control. Here, the bounded L-groups are the objects we are interested in learning about.

[59] Or, indeed, whenever a category doesn't have a terminal object. One takes a limit in a pro-category.

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but had our ideology allowed only finite complexes we wouldn't be able to call it a space. The homology that we use is cognizant of this fact: we take the homology with finite chains, not the locally finite chains.

So if Z is a metric space, we can form a simplicial complex $R_1 Z$ by having the points of X (or, for technical convenience, a discrete, 1/2-dense subset of Z) be vertices, and then adding edges when both points have distance ≤ 1, adding 2-simplices when all three vertices have distance ≤ 1, and so on. Then we can include this complex as a subcomplex of the same construction with 2 replacing 1, $R_2 Z$, and so on. We take the direct limit of these complexes. We also take the limit of the locally finite homologies:

$$\mathrm{HX}_i(Z; L(e)) = \lim H_i^{lf}(R_n Z; L(e)).$$

The X in HX reminds us that we are working at a large scale: all finite-scale phenomena have been wiped away (e.g. a cycle present at size t is killed in $R_{t+1} Z$).

(2) In these terms, which higher signatures should we expect to be homotopy invariant in $\mathrm{Bdd}(Z)$? Exactly the pushforwards in $\bigoplus HX^{lf}_{n-4i}(Z; \mathbb{Q})$ of $L(M) \cap [M]$, of course.

(Moreover, we can conjecture that there are integral versions in more refined theories, just like the Novikov conjecture has integral refinements for torsion free groups, as we will emphasize in Chapter 5. We will see that, frequently, these conjectures are indeed correct.)

(3) As for Novikov's theorem? Note that the bounded Novikov conjecture for $\mathbb{R}^n$ implies Novikov's theorem as we explained in §4.5. The map $U \to V \times \mathbb{R}^n$ considered there is surely a bounded homotopy equivalence over $\mathbb{R}^n$ (as it's a homeomorphism). The bounded Novikov conjecture implies that the transverse inverse image of V has the same signature as V.

4.9 The Principle of Descent

We can now reformulate the argument in §4.6 more abstractly. It shares with the Lusztig argument the idea of taking a naive homotopy invariant and then varying it in a family to get more information out of it.

For $\mathbb{Z}^n$ there is an important map:

$$L_i(\mathbb{Z}^n) \to L_i^{\mathrm{Bdd}}(e; \mathbb{Z}^n).$$

The left-hand side is isomorphic to a sum of $n!/j!(n-j)!$ copies of $L_{i-j}(\mathbb{Z}^n)$ (varying j). These are essentially the simply connected parts of the surgery obstructions you'd see along all of the codimension-j tori. The right-hand side we discussed in the previous section; it is just the codimension-n surgery obstruction, and we have seen that one way to obtain this is by Ferry's theorem. (It is $\mathrm{HX}_i(\mathbb{Z}^n; L(e))$.)

This map is a "transfer."[60] If we have a normal invariant over a space with fundamental group $\mathbb{Z}^n$, then when we take the universal cover it is a normal invariant of the cover, and we can try to perform surgery on it to make it into a bounded homotopy equivalence over $\mathbb{R}^n$. This obstruction is powerful: it gave us Novikov's theorem in Remark 4.33(3).

Using it and projection to all of the smaller tori gives us a proof of the Novikov conjecture for free abelian groups (although not essentially more elementary than the ones we've already discussed).

Now, if W is a complete simply connected manifold with nonnegative sectional curvature, the inverse of the exponential map gives a Lipschitz diffeomorphism:

$$\mathrm{Log}\colon W \to \mathbb{R}^n,$$

and hence a map $L^{\mathrm{Bdd}}(W) \to L^{\mathrm{Bdd}}(\mathbb{R}^n)$. Direct application of this map is clearly strong enough to prove the Novikov conjecture for the fundamental class of W/Γ for any cocompact group of isometries of W.

What we did in §4.6, though, was much stronger in that we made use of the fact that there were "logarithm maps" at *all* points w of W, and made a family of bounded surgery obstructions. These were strong enough to detect the whole $H_W\big(W/\Gamma; L(e)\big)$.

Setting up the formalism of these families can be done in more than one way. Two of the most popular are: (1) blocked surgery; and (2) homotopy fixed-point sets. Both of these depend on "spacification," which means finding spaces whose homotopy groups are the L-groups, normal invariants, and structure sets, and so that the surgery exact sequence becomes the exact sequence of a fibration. This process is similar to viewing indices as vector spaces and then being able to associate bundles to families of operators.

There is an important difference between index theory and manifold theory in this "spacification." In index theory, one uses genuine families of operators. In surgery, we do not need to. Genuinely parameterized surgery is a much more complicated subject than we need for these purposes.

Recall that, when we discussed the Farrell fibering theorem, although we fibered a manifold over the circle, the actual process was different: we found a single fiber (via a splitting theorem, but never mind). In other words, we solved a problem over a vertex in a triangulation of S^1. (Solving it over other vertices

60 Transfer (for covariant functors) is generally a map that passes from an invariant of a quotient to one of the original space (or perhaps an intermediate quotient).

is no additional problem.) Then we cut open the circle and had to see that we were OK over the resulting interval.

In general, in spacification, we want the families to not involve more complicated objects than arise in the vertices.[61] if we can. We are being conservative in the type of difficulties that ever need to be considered. This can achieved because when we did surgery there was no difference in the theory of closed manifolds and manifolds with boundary if we work relative to the boundary.

And a similar method can be done with manifolds with corners.

One is thus led to consider a simplicial complex, where the k-simplices are surgery problems that are "modeled" on Δ^k. In this space, $L_n(\pi)$, vertices are n-dimensional surgery problems with fundamental group π; 1-simplices are cobordisms between such, i.e. they are $(n+1)$-dimensional surgery problems with two boundary components (labeled by 0 and 1), everything with fundamental group π, and the boundary of the problem determines the boundary of the 1-simplex. And so on for defining the higher simplices. Doing this gives a space, and $\pi_i\big(L_n(\pi)\big) = L_{i+n}(\pi)$. Moreover, there are homotopy equivalences $\Omega^4\big(L_n(\pi)\big) \cong L_{n+4}(\pi)$ that are an analogue of the 4-fold periodicity of L-groups.

It is worth saying a word about this equivalence, since it is the prototype of the notion of an assembly map, and it explains the use of the word "assembly," which often mystifies people who see other maps that are also called assembly maps because they somehow resemble this one. An element of $\Omega^4(L_n(\pi))$ (or a vertex in a simplicial model of it) is a map from $\mathcal{S}^4$ into $L_n(\pi)$). So, we can think of $\mathcal{S}^4$ as being triangulated, and each simplex in that triangulation being assigned one of the defining simplices in the space $L_n(\pi)$. We can "assemble" all of these together to define a vertex in the space $L_{n+4}(\pi)$.

Of course, if one is fibered over a base, one has this situation, but this "blocked" theory is much simpler: it satisfies our desideratum that no object is more complicated than which occurs already over a vertex.[62] So, above, we exactly have this from a surgery problem:

$$N' \to N \to W/\Gamma.$$

We can take universal covers and lift

$$\tilde{N}' \to \tilde{N} \to W,$$

giving the family $(\tilde{N}' \to \tilde{N}) \times_\Gamma W$, which fibers over W/Γ. Over each Δ in W/Γ one has the product $(\tilde{N}' \to \tilde{N}) \times \Delta$ (although identifications change as one

[61] Except combinatorially. An object over a simplex will need the combinatorial complexity of a simplex (at least).

[62] That is, that surgery on manifolds with boundary relative to a solution on the boundary has exactly the same type of obstruction that already arises on closed manifolds.

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moves around). This is a simplex is a space of surgery problems of the form $L^{\mathrm{Bdd}}(W)$. If one thinks through all the identifications made, then one realizes that one has detected $H_i\big(W/\Gamma; L(e)\big)$ in $L_i(\Gamma)$ via the composite

$$L_i(\Gamma) \to H^0\big(W/\Gamma; L^{\mathrm{Bdd}}(W)\big) \to H^0\big(W/\Gamma; L^{\mathrm{Bdd}}(\mathrm{T}_w W)\big) \cong H_i\big(W/\Gamma; L(e)\big),$$

where $H^0\big(W/\Gamma; L^{\mathrm{Bdd}}(W)\big)$ denotes the *twisted* cohomology of the spectrum $L^{\mathrm{Bdd}}(W)$ over W/Γ and $\mathrm{T}W$ denotes the tangent bundle of W, called the *family bounded transfer*.[63]

A very nice alternative description of this method is to recall the notion of homotopy fixed set. If G is a group acting on a space X, then X^G, the fixed-point set of the action, can be thought of as the equivariant mapping space $\mathrm{Map}_G[pt\colon X]$. There is a map of this space into a more homotopical object[64]

$$X^{\mathrm{hG}} = \mathrm{Map}_G[\mathrm{EG}\colon X].$$

Unlike fixed points, that are quite sensitive to a space being acted upon, if $f\colon X \to Y$ is an equivariant map that is a homotopy equivalence, then f induces a homotopy equivalence $X^{\mathrm{hG}} \to Y^{\mathrm{hG}}$.

Moreover, there is clearly a map $X^G \to X^{\mathrm{hG}}$.

If one unravels the notation in the family bounded transfer, one sees that one has the map

$$L_i(\Gamma) \to L_i^{\mathrm{Bdd}}(\Gamma)^{\mathrm{h}\Gamma}$$

and thus one can interpret our proof in the non-positively curved situation as using a *homotopy fixed set* for the purpose of splitting an assembly map.

In any case, these ideas lead to a method of descent, wherein a suitable Borel-type conjecture (or maybe a little less) for Γ as a metric space gives rise to the Novikov conjecturec for Γ itself as a group – again, these are little different from what we did by hand in §4.6, but this interpretation, for example, makes sense for many functors other than L, and also is now suitable for situations where we get our bounded information from any source, not only from the Ferry ε-map theorem.

63 This discussion ignores the aspect where we put support conditions on the homotopy equivalence and changed the cohomological term to one with compact supports.

64 Below, EG is the universal contractible space on which G acts freely, and hG is homotopy fixed set of the G (or Γ) action.

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4.10 Secondary Invariants

. . . And a Little More Surgery

It is time to return to lens spaces, those remarkable explicit manifolds which, while homotopy equivalent, can frequently not be distinguished by their tangential information (say in dimension 3). Recall de Rham's theorem that lens spaces are only diffeomorphic[65] when they are linearly so – it is time to understand why this is.

This discussion contains embryonically one of the main keys to understanding closed manifolds whose fundamental groups have torsion. If we think functorially, the key question is:

Problem 4.35 What does $S(K(\pi, 1))$ look like when π has torsion?

(Indeed, *when the Novikov conjecture is true*,[66] we always have, rationally, a decomposition of

$$\begin{aligned} S_n(M) \otimes \mathbb{Q} &\cong S_n(K(\pi,1)) \otimes \mathbb{Q} \times S_{n+1}(K(\pi,1), M) \otimes \mathbb{Q} \\ &\cong S_n(K(\pi,1)) \otimes \mathbb{Q} \times H_{n+1}(K(\pi,1)), (M; L(e)) \otimes \mathbb{Q} \\ &\cong S_n(K(\pi,1) \otimes \mathbb{Q} \times \bigoplus H_{n\pm 4i+1}(K(\pi,1)M;\mathbb{Q}).) \end{aligned}$$

We shall see[67] that $S_3(K(\pi,1)) \otimes \mathbb{Q}$ is never 0 for groups with torsion[68] (and that, furthermore, $S_3(M) \otimes \mathbb{Q}$ is nonzero for any closed orientable manifold whose fundamental group has torsion[69]). For general groups, the Farrell–Jones conjecture (to be discussed in Chapter 8) gives a conjectural answer.[70]

To begin answering this, we must first consider the important case of π finite.

For finite groups, the homology term is irrelevant (rationally), so we need to think about the L-groups. Wall (1974, 1976a) showed that, for finite groups π, the $L_n(\pi)$ are finitely generated abelian groups, with 2-primary torsion. Moreover, the groups for n odd are finite, so we shall concentrate on n even. Indeed, for future reference, let me go so far as to actually define these groups.[71]

65 Indeed, homeomorphic, although de Rham could not have known that!

66 This is a diagram chase, when one takes into account that the assembly map, as a map of spectra, being rationally injective on homotopy groups, has a rational splitting as a map. For integral analogues see, e.g., Weinberger *et al.* (2020).

67 Following Chang and Weinberger (2003).

68 But it can be 0 in the setting of groups with nontrivial orientation character.

69 This is not a formal consequence of the previous remark, as it was conditional.

70 When we discuss the equivariant version of the Borel conjecture, we will be led to a more straightforward geometric conjecture for $S(K(\pi,1)) \otimes \mathbb{Z}[1/2]$. However, the "correct formula" for $S(M)$ will then be a purely homological variation of the first summand in the decomposition above.

71 I have always been amazed at how much is possible to do in surgery theory without a definition of the obstruction groups, and only a modicum of their properties. However, for

Definition 4.36 Let π be a group, and $w\colon \pi \to \mathbb{Z}/2$ a homomorphism (called the orientation character). Then $L_{2k}(\pi, w)$ is the group generated by 3-tuples (M, λ, μ), where M is a free finitely generated $\mathbb{Z}\pi$-module, $\lambda\colon M \times M \to \mathbb{Z}\pi$ is bilinear and nonsingular[72] over $\mathbb{Z}\pi$, Λ is $(-1)^k$ Hermitian (the conjugacy on $\mathbb{Z}\pi$ generated by sending g to $w(g)g^{-1}$), and $\mu\colon M \to \mathbb{Z}\pi/(u - (-1)^k\bar{u})$ is a quadratic refinement of λ so that $\mu(x + y) - \mu(x) - \mu(y) = \lambda(x, y)$ and $\mu(ux) = \bar{u}\mu(x)u$ for u a multiple of a group element.

An element is trivial if M contains a subspace K such that λ and μ restrict trivially to K, and $\lambda\colon K \to M/K$ is an isomorphism.

One can define variants using projective modules rather than free, or free-based modules, and then impose a condition on $\det(\lambda)$. All of these just affect L at the prime 2, so we will not worry about them here.

Ranicki (1979a) showed that the map $L_n(\mathbb{Z}\pi) \to L_n(\mathbb{Q}\pi)$ is an isomorphism away from 2 for *any* group π. Moreover, with 2 inverted in the coefficient ring, the μ is irrelevant (i.e. determined by the λ).

So we now have a stripped-down picture of the kind of invariant we are seeking: a Witt class of a quadratic form over $\mathbb{Q}\pi$.

That is a straightforward invariant to try to get: whenever the finite group π acts on an oriented space[73] X^{2k} satisfying Poincaré duality (with orientation properties given by w), we get such a structure on $H^k(X; \mathbb{Q})$. Let $\langle x, y\rangle = \sum\langle(x \cup g_* y), [X]\rangle g \in \mathbb{Q}\pi$. We shall call this invariant π-sign $(X) \in L_{2k}(\mathbb{Q}\pi) \otimes \mathbb{Q}$.

For π trivial, this invariant is trivial for k odd (every skew-symmetric form over $\mathbb{Q}$ is determined by its dimension as a vector space: it is a symplectic vector space, and every Lagrangian in it defines an equivalence to the trivial element). For k even, $L_{2k}(\mathbb{Q}) \cong \mathbb{Z} \oplus T$, where T is an infinite sum of $\mathbb{Z}/2$s and $\mathbb{Z}/4$s. The $\mathbb{Z}$ is just the signature of the quadratic form.

In general, we can analyze $L_{2k}(\mathbb{Q}\pi)$ in a few equivalent ways. The invariants we will be discussing are representations, and therefore can be thought of as characters – which means that we only need pay attention to cyclic groups.[74] For $k = 0$, we can diagonalize the quadratic form, and then consider the difference of positive and negative definite parts $[H^+] - [H^-]$ in $\mathrm{RO}(\pi)$ as an invariant. When k is odd, and π is cyclic, we can take a complex representation and (after multiplying by i) get a Hermitian inner product. It has a signature.

almost anything involving groups with torsion, the definitions are necessary, and hands made dirty by calculation cannot be avoided.

72 That is, λ defines an isomorphism $M \to M^*$ (where $M^* = \mathrm{Hom}\,(M : \mathbb{Z}\pi)$.

73 We are reserving the right to allow X not to be a manifold, and the action not to be free – since these affect nothing. Moreover, by allowing the modules to be projective in the definition of the L-group, we really have a very transparent invariant.

74 Of course, the L-groups themselves are more complicated than this. The reader might wish to think about the cases of Q_8 and the symmetric groups to see various phenomena.

Proposition 4.37 *If π is a finite group that acts freely on M^{4k}, then π-sign (M) is a multiple of the regular representation (i.e. its character is trivial for all $g \neq e$). If π acts freely on M^{4k+2}, then π-sign (M) vanishes.*

This can be proved in several ways. First of all, it is a consequence of the Atiyah–Singer G-signature theorem (Atiyah and Singer, 1968b). It can also be easily proved by cobordism considerations: bordism of free π-actions is equivalent to $\Omega(K(\pi, 1))$, but after $\otimes\mathbb{Q}$ this is the image of $\Omega(*)$, i.e. every bordism class is induced from a trivial action, immediately giving the result.

Note that this proposition includes our earlier observation that signature is multiplicative for finite covers of closed manifolds.

Now we can define a basic invariant of an odd-dimensional manifold with finite-order fundamental group.

Definition 4.38 Suppose M^{2k-1} is a closed manifold with finite fundamental group π. Let W be such that km = boundary of W; we define $\rho(M) \in L_{2k}(\mathbb{Q}\pi)/L_{2k}(\mathbb{Q}) \otimes \mathbb{Q}$ as follows. Some multiple, kM, of M bounds with fundamental group $\pi \cdot kM = \partial W$:

$$\rho(M) = (1/k)\, \pi\text{-sign}\,(W) \in L_{2k}(\mathbb{Q}\pi)/L_{2k}(\mathbb{Q}) \otimes \mathbb{Q}.$$

Remark 4.39 We have been quite cavalier in ignoring integrality and torsion issues. With more care,[75] one need not be.

For lens spaces we can make this completely explicit.

Start with the following $\mathbb{Z}/n$ action of a surface. Take the branched cover of S^2 branched at n points so that one gets a surface with a semifree $\mathbb{Z}/n$ action, so that all n fixed points have the same tangential representation – say t which equals rotation by $2\pi/n$. Changing the generator gives an analogous $\mathbb{Z}/n$ equivariant surface but with any rotation number one wishes. A product of k such surfaces will give a manifold of dimension $2k$ with a semi-free $\mathbb{Z}/n$ action, with n^k isolated fixed points, all with the same given normal representation. If we removed a deleted neighborhood of the fixed point set, and take quotients, we obtain n^k lens spaces all with our chosen set of rotation numbers. Moreover, the ρ-invariant can be computed from the calculation for the original branched cover using Galois invariance and multiplicativity of signatures. In any case, the calculation is done (by another method) in Atiyah and Bott (1968) and furthermore they explain the quite nontrivial and interesting proof that this invariant is strong enough to distinguish the lens spaces from one another.[76]

75 Using the technology of assembly maps (and using calculations of equivariant Witt groups).

76 It is interesting that both this proof and de Rham's original proof both rely on the same number-theoretic fact: the Franz independence lemma (see Milnor, 1966; Atiyah and Bott, 1968; Cohen, 1973).

In any case, this invariant ρ now presents a challenge to the Borel conjecture. Using our surgical description of ρ, however, we have:

Conjecture 4.40 *If Γ is torsion-free and π is finite, then $L(\Gamma) \to L(\pi)/L(e)$ has finite image.*[77]

Theorem 4.41 *The Borel conjecture*[78] *implies the above conjecture.*

The theory of ρ-invariants is susceptible to a nice generalization by Atiyah *et al.* (1975a,b) where one assigns an invariant for every finite-dimensional unitary representation of Γ (whether finite, torsion-free, or anything at all!).

This analytic method does not have, as far as I know, a purely topological approach. It is a descendent of the above-mentioned fact that, for closed manifolds (but not for manifolds with boundary), the signature with coefficients in any flat unitary bundle is the same as the ordinary signature (times the dimension of the representation).

Definition 4.42 Let D be a self-adjoint elliptic operator on an odd-dimensional manifold. Associated to D we form the series

$$\eta(s) = \sum |\mathrm{sign}\,\lambda|\lambda^{-s}$$

(summed over the *nonzero* eigenvalues λ of D) and, via analytic continuation, form the real number $\eta(0)$.

The η-invariant enters as a correction term from the boundary in an Atiyah–Singer theorem for manifolds with boundary. Therefore, as we are in a situation where relationships that hold for closed manifolds do not apply to manifolds with boundary, the η-invariant arises – without a choice of cobounding manifold – to give an invariant of M itself.

For M^{2k-1} there is a "signature operator" B on forms of even degrees $(2p)$ given by

$$\mathrm{B}\phi = i^k(-1)^{p+1}(*d - d*)\phi.$$

If $\alpha\colon \pi_1 M \to \mathrm{U}(n)$ is a unitary representation, then we can also consider B with coefficients in the flat bundle determined by α.

Definition/Theorem 4.43 (Atiyah *et al.*, 1975b) *The invariant $\rho_\alpha(M)$ is*

[77] Actually, most of the 2-torsion should also not be hit, as one can see using more detailed information about the assembly map for finite groups, i.e. the problem of which surgery obstructions arise from problems involving closed manifolds. This is called the "oozing problem" for historical reasons, with important contributions being Wall (1976b), Cappell and Shaneson (1979), Morgan and Pardon (unpublished), and Hambleton *et al.* (1988).

[78] As well as the Baum–Connes conjecture.

defined as the difference of the η-invariants for the signature operator with coefficients in the trivial bundle and that with coefficients in α:

$$\rho_\alpha(M) = n\eta_{\mathrm{B}}(0) - \eta_{\mathrm{B}\alpha}(0).$$

This invariant is independent of the Riemannian metric on M. If $M = \partial W$ so that the flat bundle extends, then

$$\rho_\alpha(M) = n\mathrm{sign}\,(W) - \mathrm{sign}_{\,\alpha}(W).$$

So, of course, all the flat bundles and APS invariants give potential obstructions to the Borel conjecture. We can turn this around and make a theorem:

Theorem 4.44 *If the Borel conjecture is true for (the torsion-free group) Γ, then, for all α, ρ_α is a homotopy invariant.*

Remark 4.45 See Weinberger (1988b, 1989) for the details.

(1) While the numbers $\rho_\alpha(M)$ can be arbitrary real numbers if we make no assumption about the unitary representation α (even for the circle, this is a non-constant continuous function of the representation), the non-homotopy invariance $\rho_\alpha(M') - \rho_\alpha(M)$, for homotopy-equivalent manifolds, is always an element of $\mathbb{Q}$. Note that for cobordant manifolds for which the flat bundle extends, this difference is an integer. It stands to reason, then, that the cobordism information implicit in the Novikov conjecture could lead to this rationality at the level of conjecture. That it is true unconditionally is based on ideas developed through work on the Novikov conjecture.
(2) It is not hard to see that if Γ is residually finite and has torsion, then there is an α which detects the infinitude of $S_3(K(\Gamma, 1))$.[79] This method actually gives more information, because ρ_α is an invariant of manifolds, so it can be used to implies that the manifolds are different from each other, not only that some given homotopy equivalence is not homotopic to a homeomorphism.

This theorem is related, but not quite equivalent, to the following obstacle to the surjectivity of the assembly map.

If $\alpha\colon \Gamma \to \mathrm{U}(n)$ is a unitary representation, then it induces a homorphism $\mathbb{R}\Gamma \to \mathrm{GL}_n(\mathbb{C})$, compatible with conjugation, and thus a map (by Morita invariance – i.e., viewing a matrix of matrices as a larger matrix with ordinary entries) $\mathrm{sign}_{\,\alpha}\colon L_{2k}(\Gamma) \to L_{2k}(\mathbb{C}) = \mathbb{Z}$. By the index theorem, if the assembly map is surjective (so that every element of $L_{2k}(\Gamma)$ comes from a closed manifold), we must have $\mathrm{sign}_{\,\alpha} \equiv n\mathrm{sign}$ on $L_{2k}(\Gamma)$. Note that if this weren't true,

[79] This infinitude is true even if Γ is not residually finite, as can be seen (Chang and Weinberger, 2006) using an L^2-variant of the η-invariant introduced by Cheeger and Gromov (1986).

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then we could use such an element and the Wall realization theorem to give a counterexample to the Borel conjecture.

The proof of the theorem also used the Novikov (i.e. injectivity) half of the Borel conjecture.

In the following notes we describe a less classical invariant based on these considerations.

4.11 Notes

In §4.1, the proof of de Rham's theorem was based on calculations of Reidemeister torsions of the lens spaces. The Reidemeister torsion can be defined for any space that has an acyclic flat bundle on it. Torsions are definable more generally when one has a situation where a finitely generated free chain complex is acyclic (and, crucially, the chain groups have given bases): such as a homotopy equivalence between finite complexes (where the chain complex of the mapping cylinder has this nature).

The torsion measures the determinant of the underlying geometric chain complexes. For a finite $\mathbb{Q}$-acyclic complex, the torsion is essentially the alternating product of the orders of the integral homology groups. Importantly and by contrast, in non-simply connected situations the torsion is *not* a homotopy invariant. Of course, the non-homotopy invariance is often an obstacle to calculation.

The torsion also occurs in the h-cobordism theorem (see §4.4) as the obstruction of an h-cobordism being a product. An h-cobordism is a manifold that deform retracts to its boundary components. The basic obstruction to being $\partial \times [0, 1]$ is that the torsion of the inclusion of (either) one of the ∂ components is trivial and the h-cobordism theorem asserts that (in dimension greater than 5) this is the only obstruction.

See §5.5.3 for more discussion.

As mentioned in the body of §4.2, the Hirzebruch signature theorem has two rather different proofs. There is the cobordism-theoretic proof and the index-theoretic proof. Both of these are subject to extensive and important generalization.

The cobordism-theoretic proof can be modified to allow cobordism of more algebraic or singular objects than merely manifolds. Doing so, the Hirzebruch theorem then becomes a calculation of where smooth manifolds fit into those cobordism theories. One such version is to consider the bordism of "controlled algebraic Poincaré complexes" (Yamasaki, 1987) – where control here is as in controlled topology introduced in §§4.6 and 4.8. This turns out to be the

$H_*(X; L(R))$, where X is the control space, and R is the ring (and L is the spectrum discussed in §4.9). Doing this gives a most satisfying proof of Novikov's theorem – the L-classes have been topologically defined as "the controlled symmetric chain complex of M over M."

The work has been hidden. As we discussed in §4.8, proving that a controlled algebraic functor is a homology theory boils down to a statement like the α-approximation theorem and we've already seen such a result of the correct level of depth for this type of applications.

The Atiyah–Singer index theorem has had numerous extensions and variants. Many of these are subsumed by very some general theorems (and even more by philosophies) in the setting of noncommutative geometry and C^*-algebras. A reference to the Connes (1994) book is surely necessary, but not sufficient. An excellent introduction is Higson and Roe (2000).

Broadening one's viewpoint in this way, besides enabling the proofs of the most advanced known results on the Novikov conjecture, also significantly expands its scope of application – as I hope will become clearer as we continue.

Section 4.4 explains two of the early approaches to the Novikov conjecture.

Probably the most misleading aspect of my treatment is viewing the vanishing of the Whitehead group of free abelian groups as an exogenous fact, that we just exploit, rather than an integral part of the "Borel package" of conjectures. Indeed, for the Borel conjecture to be true, one must have such vanishing, and one should view the vanishing of the Whitehead groups of torsion-free groups as being completely analogous to the conjectured isomorphism statement for L-groups that we take as the algebraic version of the conjecture in §4.7. We will rectify this failure in Chapter 5.

Besides the h-cobordism theorem that involved $K_1(\mathbb{Z}\pi)$, one had Wall's 1965 paper which showed how $K_0(\mathbb{Z}\pi)$ regulates whether a finitely dominated complex is homotopy equivalent to a finite complex. Siebenmann's 1965 thesis showed the bearing of Wall's work on the question of when noncompact manifolds are the interiors of manifolds with boundary (i.e. the compactification problem). The first approach to the fibering problem by Farrell (1996) had multiple obstructions involving various K-groups and a nil-type group.[80] Siebenmann (1965) gave another approach to the problem where all the obstructions were unified into one – the connections between the pieces in Farrell's approach being given by a nonabelian generalization of the Bass–Heller–Swan theorem (Farrell and Hsiang, 1970). Farrell (1971b) gave another very elegant approach to fibering in his ICM talk.

80 Nil groups are Grothendieck groups of nilpotent matrices. The connection to the K-theory of Laurent series is straightforward: if N is nilpotent over $\mathbb{R}$, then $I + t^{\pm 1}N$ are invertible over $\mathbb{R}[t, t^{-1}]$. See Bass and Murthy (1967) for some calculations of these for abelian groups.

More general splitting was developed by Cappell and applied to the Novikov conjecture by him (Cappell, 1976a,b). It turns out that it has aspects that are not attributable to algebraic K-groups. A consequence of Cappell's theorem is that the problem of being a connected sum is homotopy invariant (in dimension greater than 4) if the fundamental group has no 2-torsion (but not in general).

Developing the relevant algebraic K-theory for amalgamated free products by Waldhausen (1968) – motivated by the work he did on 3-manifolds, as one of the authors of "Haken–Waldhausen theory" – was an important step in the development of higher algebraic K-theory. In any case, this work led to consideration of the "Cappell–Waldhausen class" of groups, which are accessible from the trivial group by amalgamated free products and HNN extensions any number (including transfinite) of times. For these, the assembly map is an isomorphism after $\otimes\mathbb{Z}[1/2]$. In low dimensions, this includes many of the fundamental groups that seem important, but, in light of Property (T), no high-rank lattices in simple groups lie in this class.

On the other hand, after introducing the ideas of bounded and controlled topology, the splitting methods return, as we split the control spaces (spaces can be broken up into pieces much more easily than groups can) and thus this method is implicit in many of the subsequent topological (and many of the analytic) approaches.

An exception is Lusztig's method. Extension of this to non-positively curved situations (and beyond) was taken up fairly soon after by Mischenko and Kasparov. Mischenko, besides using infinite-dimensional bundles, also introduced the formalism of algebraic Poincaré complexes and their cobordism to get invariants of manifolds (essentially elements of $L(\mathbb{Q}\pi)$). A useful exposition of Mischenko's work can be found in Hsiang and Rees (1982).

Kasparov (1988) developed an extensive new technology, KK-theory, for the problem, which he applied to give the first proof of the Novikov conjecture for fundamental groups of complete non-positively curved manifolds (we gave a geometric approach in §4.6). It is fundamental for most of the subsequent analytic results. A useful reference for Kasparov theory is Blackadar (1998).

(From a noncommutative geometry perspective, Lusztig's method uses a family of operators parameterized by a commutative space, and one can look for families parameterized by a noncommutative space.)

In §4.5, the survey paper of Ferry *et al.* (1995) translates one of Novikov's papers and helps track his train of thought. Novikov's master stroke of using non-simply connected manifolds to get information about the topology of $\mathbb{R}^n$ was commented on, for example, in Atiyah's citation of Novikov for his Fields Medal. The approach I took is based on the idea of Kirby's torus trick (a somewhat different trick that has the same crude description) and is a variation of one of Gromov (1993).

In the original version of the torus trick, the matter of filling in the "hole" was accomplished using Siebenmann's completion (or end, or boundary) theorem in his thesis (Siebenmann, 1965). A nice aspect of using the signature of fiber bundle approach is that this is unnecessary.

The torus trick, or alternatively controlled topology, is used in proving the annulus conjecture and the other foundational theorems for the topological category (see Kirby, 1969; Kirby and Siebenmann, 1977; Quinn, 2010).

It is important to realize that in the equivariant setting all of the basic tools of the topological category that the above work fashions, dramatically fail. Handlebody structures neither exist nor are unique; equivariant Whitehead torsion is not topologically invariant; and transversality fails. Nevertheless, the equivariant signature operator is a topological invariant. We will discuss these matters in Chapter 6.

In §4.6 the Novikov conjecture for closed non-positively curved manifolds was first proved by Mischenko (1974). Farrell and Hsiang (1981) gave a direct geometrical proof (that includes the stable homeomorphism statement). Their method uses the compactification of the fibers and Alexander tricks rather than the use of the α-approximation theorem that we do. The result of this section is the main result of Ferry and Weinberger (1991) and is a slight improvement – from the topological perspective – of Kasparov's result.

Kasparov's theorem was important for the philosophical reason that it did not *seem* to require a hypothesis on the quasi-isometry type of the group, while in the closed case one immediately sees the "sphere at ∞" implicated in the solution: an idea already present explicitly in Mostow's work. When one has infinite volume, there was a strong psychological presentiment that "almost anything is possible." Indeed, we will take up this theme in Chapter 8.

Ferry's theorem was the solution to a problem of Siebenmann from his CE-approximation paper. I consider it one of the high points of twentieth-century topology. It is based on his joint result with Chapman on the α-approximation theorem, which says, in modern terminology, that a homotopy equivalence $M' \to M$ that is controlled over M is controlled homotopic to a homeomorphism.[81]

Quinn's (1979, 1982b, 1982c, 1986) papers essentially liberated the place where the control was being measured from the space where the problem was

[81] And, here, we can have ε-δ control in this theorem. In fact, the name α-approximation means that one can use any cover α for an open manifold, and then refine it to a β, so that β-homotopy equivalences are α-homotopic to homeomorphisms. So, oddly enough, the α is just the name chosen for a variable.

solved. This simple idea has proved to be enormously important, as the many applications in that series already showed. And there have been very many more; we use this type of reformulation in §§4.8 and 4.9.

A homology manifold can be thought of as a space that is a controlled Poincaré duality space, controlled over itself. This was a critical insight that led to Quinn's obstruction to resolution (Quinn, 1982a, 1987b), and the construction of nonresolvable homology manifolds and their classification (Bryant *et al.*, 1993).

In §4.7, the topological form of surgery is decidedly less elementary than the smooth theory, but it has the much better features described in this section.

The use of periodicity is an elaboration of the idea, which occurred first in Shaneson (1969), of using the periodicity of L-groups to do an end-run around the problem that low-dimensional problems cause for studying high-dimensional problems. (He showed that there is a smooth manifold homotopy equivalent to $\mathbb{T}^2 \times \mathcal{S}^3$ that looks like a product of $\mathbb{T}^2$ with a counterexample to the Poincaré conjecture despite the fact that one can't really unwrap those circles via Farrell's theorem.) In Wall (1968) and Hsiang and Shaneson (1970), this idea is used to prove the Borel conjecture for the torus.

Nowadays, the inclusion of periodicity into the functoriality means that we do not have to consciously think about these issues. On the other hand, having included homology manifolds into our structure sets, the objects we study are even less elementary than topological manifolds. In Bryant *et al.* (1993), the paper in which they were constructed, they are also classified up to s-cobordism (under some technical conditions)[82].

Homology manifolds were initially studied as places where sheaf theory behaved similarly to the theory of manifolds, and then, later, in the Bing school as cousins to manifolds with interesting topological properties in their own right (e.g. being manifold factors or fixed sets of group actions). Edwards's theorem (and the earlier work of Cannon and Edwards on the double suspension problem[83]), and Sullivan's observation that Novikov's theorem applies to CE maps, made their study central to geometric topology, and Freedman's proof of the four-dimensional Poincaré conjecture was perhaps their crowning triumph – showing that even those tame souls just interested in manifolds could not ignore these spaces as pathological.

While the resolution conjecture, asserting that high-dimensional homology

[82] The argument in that paper is only correct as given for homology manifolds that are $L^*(Z)$-orientable, which includes any that are homotopy equivalent to a manifold. An erratum (in preparation) will show how to deal with more general homology manifolds.

[83] Is the suspension of the suspension of a homology sphere a manifold (and therefore the sphere)? Yes.

manifolds are all resolvable, is false (and therefore the characterization of manifolds cannot be expressed just in terms of DDP), it is conceivable that DDP homology manifolds are in all regards just as beautiful as manifolds. However, the most basic properties of these spaces, e.g. whether they are topologically homogenous, whether the h-cobordism theorem is true for them, etc., remain open (see, e.g., Weinberger, 1995).

An ideal situation would be the true extension of the CE-approximation theorem to the setting of DDP homology manifolds: any CE map between ANR homology manifolds with the DDP should be a uniform limit of homeomorphisms – this would lead to homogeneity and the s-cobordism theorem, but can it really be that Edwards's theorem is around the same depth as homogeneity? (Surely for manifolds this isn't the case.)

Section 4.8 had as its goal to explain in more detail how controlled topology works, more formally and systematically than by example. I had tried once before in Weinberger (1994). Other (more) useful references are Chapman (1981, 1983), Quinn (1987a), Anderson *et al.* (1994), and Ferry and Pederson (1995). It is possibly fair to say that the prehistory of controlled topology began with the work of Kirby and Siebenmann on topological manifolds, and Chapman and Ferry on the α-approximation theorem and metric criteria for simplicity of a homotopy equivalence, but was consciously and effectively developed by Quinn (1979, 1982b,c, 1986) and turned into a systematic tool (wherein the control space gained its independence from the formulation of the problem).

The bounded Borel conjecture was, I think, in the air with controlled topology and this whole circle of problems. Its formulation using the Rips complex (i.e. the direct limit of the nerve of coverings by bigger and bigger balls) was natural given that uniformly contractible models do not always seem to exist. See Block and Weinberger (1992, 1997), Gersten (1993), and Roe (1993) for early uses of the Rips construction and its homology.

That this substitute should work out better than a uniformly contractible model was a great surprise to me, and this was the source of the example in Dranishnikov *et al.* (2008). (Although, I guess, the moral is that functorial constructions that work in great generality, substituting for objects that don't necessarily exist, will occasionally beat those objects, even when they do exist.) The phenomenon itself was a derivative of Dranishnikov's (1988) discovery that if $X \to Y$ is a CE map, then, while it induces an isomorphism on ordinary homology, it does not necessarily induce one on non-connective (and, in particular, periodic) homology theories, when Y has infinite covering dimension.

In the C^*-algebra setting, Yu (1998) gave other more dramatic failures of even the coarse Novikov conjecture (discussed in Chapter 5 in the setting of positive scalar curvature). And, even in the presence of bounded geometry, expander graphs give rise to other examples in this setting, as we will discuss in Chapter 8.

In §4.9, the principle of descent seems to have been developed and rediscovered multiple times. Its job is to explain the miracle (not present in our treatment of the torus) of how understanding, say, hyperbolic space, extremely well is enough for the understanding of how *every* cohomology class of every hyperbolic manifold (and we do not really understand very well this cohomology!) enters as a potential obstruction to homotopy equivalence.

Besides the work of Kasparov on the Novikov conjecture mentioned above, Gromov and Lawson (1983) used a variant in their beautiful paper on positive scalar curvature (see Section 13 therein). Our treatment here is based on Ferry and Weinberger's 1995 reformulation of Ferry and Weinberger (1991); Carlsson (1995) developed it in the guise of homotopy fixed sets (and extended its reach in papers with Pedersen, for example Carlsson and Pederson, 1995). An excellent explanation of its C^*-algebra version appears in Roe (1993).

The technique used here can be used to prove the Novikov conjecture for groups of finite asymptotic dimension. This was first done by analytic methods by Yu (1998). But methods based on the squeezing properties of a finite complex (i.e. α-approximation type results) together with descent have been successfully applied to give this result in Bartels (2003), Carlsson and Goldfarb (2004), and Chang *et al*. (2008). A completely different topological approach (based on the existence of appropriate acyclic completions of the $E\Gamma$) is given in Dranishnikov *et al.* (2008).[84]

Spacification was introduced by Casson (1967) to get information about fibering a manifold over S^2. Quinn's thesis (see Quinn, 1970) developed it systematically (see also Nicas, 1982). Other treatments can be found in Burghelea *et al.* (1975), Weinberger (1994), and Cappell and Weinberger (1995).

Finally, in §4.10, W. Neumann (1979) was the first to show that, for the case of $\mathbb{Z}^n$, Atiyah–Patodi–Singer invariants are homotopy invariants by means of an explicit homotopy-invariant formula for them. In Weinberger (1985a) I explained how the Borel conjecture implies that for torsion-free groups these are homotopy invariant. Keswani (2000) showed how a version of the Baum–Connes conjecture implies this as well, and therefore, for torsion-free amenable groups, the work of Higson and Kasparov implies this conclusion.

The fact that APS invariants are homotopy invariant up to rational numbers

[84] The Higson corona used in that paper is a variant of the Stone–Čech compactification. The utility of a generalization of the boundary of hyperbolic space for rigidity purposes is a central theme in Mostow, Tits, Gromov, and through to the present.

was something I had worked on, off and on, for almost a decade. My final proof (Weinberger, 1988a) used a deformation argument (whose key point was a calculation of Farber and Levine) to reduce to subgroups of GL_n with algebraic entries. This argument ended up being only a couple of pages long. Later, Higson and Roe (2005a,b,c, 2010) gave a more direct argument.

Mathai (1992) was the first to study the homotopy invariance properties of the Cheeger–Gromov reduced L^2-η invariant. Keswani related this to a variant of the Baum–Connes conjecture (one true for all amenable groups, but false for groups with Property (T)). Chang (2004) showed that the Borel conjecture implies homotopy invariance in the torsion-free case. Chang and Weinberger (2003) showed the non-homotopy invariance for all groups with torsion.

Given that any nontrivial torsion in π gives rise to the infinitude of $S_3(M)$, it seems reasonable to believe that the size of $S_3(M)$ (and of similar invariants) should be larger when the fundamental group of M has more torsion. This has not been shown unconditionally, but, for very many fundamental groups, lower bounds in terms of the number of orders of torsion elements (or even on the number of conjugacy classes of torsion elements) have been given in Weinberger and Yu (2015).

There is a general philosophy of secondary invariants that comes out of the Novikov conjecture and, complementary to these, homotopy-invariant secondary invariants. These were explicitly introduced in Weinberger (1999b) and are "higher ρ-invariants" (although they were implicitly used already in Weinberger (1988a)). Like Reidemeister torsion, they require some amount of acyclicity[85] to define (and examples show that this is actually necessary). Typical places that they take values in is a quotient of $S(K(\Gamma, 1))$ or of an L-group or a some kind of homological (or K-theory) invariant related to the fundamental group – where the quotient is determined by the type of Novikov technology that we will discuss in Chapter 6. I am being vague about this because there isn't yet an overarching general theory that includes all others. Weinberger (1999b), for example, does not deal at all with torsion issues – although some of the later literature (Higson and Roe, 2005a,b,c; Piazza and Zenobi, 2016; Weinberger *et al.*, 2020) does – and the context of their definition is "up for negotiation," essentially in terms of what kind of acyclicity hypothesis is necessary, or what is its source.

[85] Actually the relevant acyclicity is only necessary around the middle dimension. That manifolds with this property are special and can be more easily understood than general manifolds was first realized by Jean-Claude Hausmann, who studied (in unpublished work sometime in the 1970s) them under the slightly less general but more geometric condition of having no middle-dimensional handles in a handle decomposition.

This invariant is adequate for distinguishing lens spaces after crossing with aspherical manifolds for which the Novikov conjecture is known.

The reason that they are somewhat subtle is that, unlike higher signatures that are signatures of appropriate submanifolds associated to cycles in $K(\Gamma, 1)$-, ρ-invariants are very definitely *not* cobordism invariant, and therefore it is *prima facie* unclear that any higher version of an such invariant should be definable. Interestingly, acyclicity solves this and this makes some cycles more canonical than others, in a way that is not apparent to straightforward transversality.

The precise definition uses the fact that sufficiently acyclic manifolds are both algebraically (tautologously) and geometrically null-cobordant[86] (using the Novikov conjecture), and that one gets an interesting invariant by comparing these two nullcobordisms.

These invariants have a number of interesting applications. The first is that it gives a way of showing that manifolds are not homeomorphic, not only that a certain map is not homotopic to a homeomorphism. Nabutovsky and I used this to show that, even among homotopy equivalent manifolds, the homeomorphism problem can be algorithmically undecidable (Nabutovsky and Weinberger, 1999).

Another, more recent, application is to Gromov–Hausdorff space. Recall (Gromov, 1999) that Gromov–Hausdorff space is a compact metric space of compact metric spaces, and that spaces are close if they can be approximately "aligned" like two fairly dense subsets of a third metric space. Gromov–Hausdorff space and limits in it have become an important tool in comparison differential geometry.

One can hope to find strong geometric restrictions on sets of manifolds in Gromov–Hausdorff space that have a contractibility function.[87] It turns out, for example, that sufficiently close manifolds of this sort have the same simple homotopy type and rational Pontrjagin classes (Ferry, 1994). However, nevertheless, there are infinite families of manifolds M_i that are pairwise distinct, but which can all be made arbitrarily close to each other. See Dranishnikov *et al.* (2020) for how this goes, and how higher ρ-invariants are used.

In this example, it is important that the contractibility function (including the ε that describes a threshold at which balls are null-homotopic) is allowed to vary with i. For a fixed contractibility function f, Ferry (1994) proved a

[86] In some sense; for example, in the Witt cobordism sense.

[87] A contractibility function for X is a function $f\colon [0,\varepsilon) \to \mathbb{R}$ such that f is continuous, $f(t) \leq t$ and $f(0) = 0$, so that for each point x in X, the ball around x of radius t in null-homotopic in the ball of radius $f(t)$. It is a generalization of the notion of injectivity radius for a manifold, which corresponds to the case of the $f(x) = x$ on $[0, \text{inj}]$, where inj is the injectivity radius. It is an easy exercise that, given a local contractibility function f and a dimension n, there is a δ such that δ-close n-dimensional ANRs are homotopy equivalent.

contrasting finiteness theorem: the number of manifolds in a precompact part of Gromov–Hausdorff space with any specified contractibility function is finite.

The same technology that defines the higher ρ-invariant is used in Leichtnam *et al.* (2000) to define higher signatures for noncompact complete manifolds under the same type of middle-acyclicity condition at ∞. (These higher signatures involve the cohomology of the fundamental group of the manifold, and nothing further about ∞. Assuming the Novikov conjecture, they are proper homotopy invariant.) Leichtnam *et al.* (2002) used a variant of this idea to study how higher signatures of closed manifolds change if one cuts them open along submanifolds and glues back differently.