

## CHEVALLEY SUPERGROUPS OF TYPE $D(2, 1; a)$

F. GAVARINI

*Dipartimento di Matematica, Università di Roma ‘Tor Vergata’,  
Via della Ricerca Scientifica 1, 00133 Roma, Italy  
(gavarini@mat.uniroma2.it)*

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*Abstract* We present a construction ‘à la Chevalley’ of connected affine supergroups associated with Lie superalgebras of type  $D(2, 1; a)$ , for any possible value of the parameter  $a$ . This extends the results by Fioresi and Gavarini, in which all other simple Lie superalgebras of classical type were considered. The case of simple Lie superalgebras of Cartan type is dealt with in a previous paper by the author, so this work completes the programme of constructing connected affine supergroups associated with any simple Lie superalgebra.

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### 1. Introduction

In his work of 1955, Chevalley provided a combinatorial construction of all simple affine algebraic groups over any field. In particular, his method led to an existence theorem for simple affine algebraic groups: one starts with a simple (complex, finite-dimensional) Lie algebra and a simple module  $V$  for it, and realizes the required group as a closed subgroup of  $\mathrm{GL}(V)$ . This can also be recast so as to provide a description of all simple affine groups as group schemes over  $\mathbb{Z}$ .

In [6] the philosophy of Chevalley was revisited in the context of supergeometry. The outcome is a construction of affine supergroups whose tangent Lie superalgebra is of *classical type*. However, the cases when the Lie superalgebra is of type  $D(2, 1; a)$  and the parameter  $a$  is not an integer number were omitted; the present work fills in this gap. As the case of simple Lie superalgebras of Cartan type is solved in [8], the present paper completes the programme of constructing connected affine supergroups associated with any simple Lie superalgebra.

By ‘affine supergroup’, we mean a representable functor from the category (salg) of commutative superalgebras, over some fixed ground ring, to the category (groups) of groups, in other words, an affine supergroup scheme, identified with its functor of points. In [6], one first constructs a functor from (salg) to (groups), recovering Chevalley’s ideas

to define the values of such a group functor on each superalgebra  $A$ , i.e. to define its  $A$ -points. Then, one proves that the sheafification of this functor is representable; hence, it is an affine supergroup scheme.

For the case  $D(2, 1; a)$ , with  $a \notin \mathbb{Z}$ , one needs to make a careful modification to the general procedure of [6]; thus, the presentation hereafter details those steps which need changes, and will simply refer to [6] for those where the original arguments still work unchanged.

The initial datum is a simple Lie superalgebra  $\mathfrak{g} = D(2, 1; a)$ .

We start with basic results on  $\mathfrak{g}$ : the existence of *Chevalley bases* (with nice integrality properties) and a Poincaré–Birkhoff–Witt (PBW) theorem for the Kostant  $\mathbb{Z}$ -form of the universal enveloping superalgebra  $U(\mathfrak{g})$ .

Next we take a faithful, finite-dimensional  $\mathfrak{g}$ -module  $V$ , and we show that it has suitable lattices  $M$  invariant by the Kostant superalgebra. This allows us to define, functorially, additive and multiplicative one-parameter (super)subgroups of operators acting on scalar extensions of  $M$ . The additive subgroups are just like in the general case: there exists one of them for every root of  $\mathfrak{g}$ . The multiplicative ones are instead associated with elements of the fixed Cartan subalgebra of  $\mathfrak{g}$ , and are of two types: those of *classical* type, modelled on the group functor  $A \mapsto U(A_0)$ , the *group of units* of  $A_0$ ; and those of *a-type*, modelled on the group functor  $A \mapsto P_a(A)$ , the group of elements of  $A_0$ , *which may be raised to the  $a^k$ th power, for all  $k$* . The second type of multiplicative one-parameter subgroup, not used in [6], is now needed because one has to consider the ‘operation’  $t \mapsto t^a$ , defined just for  $t \in P_a(A)$ ; this marks a difference from the case  $a \in \mathbb{Z}$ .

We then consider the functor  $G: (\text{salg}) \rightarrow (\text{groups})$ , whose value  $G(A)$  on  $A \in (\text{salg})$  is the subgroup of  $\text{GL}(V(A))$ , with  $V(A) := A \otimes M$ , generated by all the homogeneous one-parameter supersubgroups mentioned above. This functor is a presheaf; hence, we can take its sheafification  $\mathbf{G}_V = \mathbf{G}: (\text{salg}) \rightarrow (\text{groups})$ . These  $\mathbf{G}_V$  are, by definition, our ‘Chevalley supergroups’.

Acting as in [6], one defines a ‘classical affine subgroup’  $\mathbf{G}_0$  of  $\mathbf{G}_V$ , corresponding to the even part  $\mathfrak{g}_0$  of  $\mathfrak{g}$  (and to  $V$ ), and then finds a factorization  $\mathbf{G}_V = \mathbf{G}_0 \mathbf{G}_1 \cong \mathbf{G}_0 \times \mathbf{G}_1$ , where  $\mathbf{G}_1$  corresponds instead to the odd part  $\mathfrak{g}_1$  of  $\mathfrak{g}$ . Actually, one even has a finer factorization

$$\mathbf{G}_V = \mathbf{G}_0 \times \mathbf{G}_1^{-, <} \times \mathbf{G}_1^{+, <}$$

with  $\mathbf{G}_1^{\pm, <}$  being totally odd superspaces associated with the positive or negative odd roots of  $\mathfrak{g}$ . Thus,  $\mathbf{G}_1 = \mathbf{G}_1^{-, <} \times \mathbf{G}_1^{+, <}$  is representable, and  $\mathbf{G}_0$  is representable too; hence, the above factorization implies that  $\mathbf{G}_V$  is also representable, so it is an affine supergroup. The outcome is then that our Chevalley supergroups are affine supergroups.

Finally, one also proves that our construction is functorial in  $V$  and that  $\text{Lie}(\mathbf{G}_V)$  is just  $\mathfrak{g}$ , as one expects, as in [6] (no special changes are needed).

## 2. Preliminaries

### 2.1. Superalgebras, superspaces, supergroups

Let  $\mathbb{k}$  be a unital, commutative ring.

We call a  $\mathbb{k}$ -superalgebra any associative, unital  $\mathbb{k}$ -algebra  $A$  that is  $\mathbb{Z}_2$ -graded. Thus,  $A$  splits as  $A = A_0 \oplus A_1$ , and  $A_a A_b \subseteq A_{a+b}$ . The  $\mathbb{k}$ -submodule  $A_0$  and its elements are called *even*, while  $A_1$  and its elements are called *odd*. We denote by  $p(x)$  the *parity* of any homogeneous element  $x \in A_{p(x)}$ . A superalgebra  $A$  is said to be *commutative* if and only if  $xy = (-1)^{p(x)p(y)}yx$ , for all homogeneous  $x, y \in A$ , and  $z^2 = 0$  for all odd  $z \in A_1$ . Clearly,  $\mathbb{k}$ -superalgebras form a category, whose morphisms are those in the category of algebras that preserve the unit and the  $\mathbb{Z}_2$ -grading. We denote by  $(\text{salg})$  the full subcategory of commutative superalgebras; we also write  $(\text{salg})_{\mathbb{k}}$  to stress the role of  $\mathbb{k}$ .

Finally, for any  $n \in \mathbb{N}$  we call  $A_1^n$  the  $A_0$ -submodule of  $A$  spanned by all products  $\vartheta_1 \cdots \vartheta_n$ , with  $\vartheta_i \in A_1$  for all  $i$ , and  $A_1^{(n)}$  the unital  $\mathbb{k}$ -subalgebra of  $A$  generated by  $A_1^n$ .

We consider the notions of *superspace*, *(affine) superscheme* and *(affine) supergroup* as defined in detail in [4, 6]; here, we just recall them rather quickly. Roughly speaking, a *superspace* is a locally ringed space whose structure sheaf is made of commutative superalgebras, the stalks being local; the morphisms among superspaces are then morphisms of locally ringed spaces, respecting the parity on sections of the structure sheaf. In any superspace  $S$ , the structure sheaf  $\mathcal{O}_S$  has natural ‘even’ and ‘odd’ parts, say  $\mathcal{O}_{S,0}$  and  $\mathcal{O}_{S,1}$ : the latter is a sheaf of modules over the former, which in turn is just a sheaf of commutative algebras (whose stalks are local).

A *superscheme* is just a superspace  $S$  for which  $\mathcal{O}_{S,1}$  is quasi-coherent as a  $\mathcal{O}_{S,0}$ -module.

For any  $A \in (\text{salg})$ , the prime spectrum  $\text{Spec}(A_0)$  of its even part bears the natural structure of a superscheme, denoted by  $\text{Spec}(A)$ , induced by the fact that  $A$  itself is an  $A_0$ -module. Any superscheme that is isomorphic to such a  $\text{Spec}(A)$  is said to be *affine*.

If  $X$  is any superscheme, its *functor of points* is the functor  $h_X : (\text{salg}) \rightarrow (\text{sets})$  defined on objects by  $h_X(A) := \text{Hom}(\text{Spec}(A), X)$  and on arrows by  $h_X(f)(\phi) := \phi \circ \text{Spec}(f)$ . If  $(\text{groups})$  is the category of groups and  $h_X$  is a functor  $h_X : (\text{salg}) \rightarrow (\text{groups})$ , then we say that  $X$  is a *supergroup*. When  $X$  is affine, this is equivalent to the fact that  $\mathcal{O}(X)$  (the superalgebra of global sections of the structure sheaf on  $X$ ) is a (commutative) *Hopf superalgebra*. More generally, by *supergroup functor* we mean any functor  $G : (\text{salg}) \rightarrow (\text{groups})$ .

Any representable supergroup functor is the same as an affine supergroup; indeed, the former corresponds to the functor of points of the latter. Thus, we use the same letter to denote both a superscheme and its functor of points. See [4, Chapters 3–5] for more details.

**Examples 2.1.** (1) The *affine superspace*  $\mathbb{A}_{\mathbb{k}}^{p|q}$ , also denoted by  $\mathbb{k}^{p|q}$ , is defined, for each  $p, q \in \mathbb{N}$ , as

$$\mathbb{A}_{\mathbb{k}}^{p|q} := \text{Spec}(\mathbb{k}[x_1, \dots, x_p] \otimes_{\mathbb{k}} \mathbb{k}[\xi_1 \cdots \xi_q]);$$

in what follows,  $\mathbb{k}[\xi_1 \cdots \xi_q]$  denotes the exterior (or ‘Grassmann’) algebra generated by  $\xi_1, \dots, \xi_q$ , and  $\mathbb{k}[x_1, \dots, x_p]$  is the polynomial algebra in  $p$  indeterminates.

(2) Let  $V$  be a free  $\mathbb{k}$ -supermodule. Set  $V(A) := (A \otimes V)_0 = A_0 \otimes V_0 \oplus A_1 \otimes V_1$  for any  $A \in (\text{salg}_{\mathbb{k}})$ ; this yields a representable functor in the category of superalgebras, represented by the  $\mathbb{k}$ -superalgebra of polynomial functions on  $V$ . Hence,  $V$  can be seen as an affine superscheme.

(3) *GL(V) as an affine supergroup*: let  $V$  be a free  $\mathbb{k}$ -supermodule of finite (super)rank  $p|q$ . For any  $A \in (\text{salg})_{\mathbb{k}}$ , let  $\text{GL}(V)(A) := \text{GL}(V(A))$  be the set of isomorphisms  $V(A) \rightarrow V(A)$ . If we fix a homogeneous basis for  $V$ , we see that  $V \cong \mathbb{k}^{p|q}$ ; in other words,  $V_0 \cong \mathbb{k}^p$  and  $V_1 \cong \mathbb{k}^q$ . In this case, we also denote  $\text{GL}(V)$  by  $\text{GL}_{p|q}$ . Now,  $\text{GL}_{p|q}(A)$  is the group of invertible matrices of size  $(p+q)$  with diagonal block entries in  $A_0$  and off-diagonal block entries in  $A_1$ . It is known that the functor  $\text{GL}(V)$  is representable (see [15, Chapter 3] for details).

## 2.2. Lie superalgebras

The notion of Lie superalgebra is well known, at least over a field of characteristic neither 2 nor 3. To take into account *any ground ring*, we consider a modified formulation: a ‘correct’ notion of a Lie superalgebra given by the standard notion enriched with an additional ‘2-mapping’, an analogue to the  $p$ -mapping in a  $p$ -restricted Lie algebra over a field of characteristic  $p > 0$ .

**Definition 2.2 (Bahturin *et al.* [1]; M. Duflo (personal communication, 2011)).** Let  $A \in (\text{salg})_{\mathbb{k}}$ . We call a *Lie A-superalgebra* any  $A$ -supermodule  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  endowed with a *(Lie super)bracket*  $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $(x, y) \mapsto [x, y]$ , and with a *2-operation*  $(\cdot)^{(2)}: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ ,  $z \mapsto z^{(2)}$ , such that (for all  $x, y \in \mathfrak{g}_0 \cup \mathfrak{g}_1$ ,  $w \in \mathfrak{g}_0$ ,  $z, z_1, z_2 \in \mathfrak{g}_1$ )

- (a)  $[\cdot, \cdot]$  is  $A$ -superbilinear (in the obvious sense),  $[w, w] = 0$ ,  $[z[z, z]] = 0$ ,
- (b)  $[x, y] + (-1)^{p(x)p(y)}[y, x] = 0$  (antisymmetry),
- (c)  $(-1)^{p(x)p(z)}[x, [y, z]] + (-1)^{p(y)p(x)}[y, [z, x]] + (-1)^{p(z)p(y)}[z, [x, y]] = 0$  (Jacobi identity),
- (d)  $(\cdot)^{(2)}$  is  $A$ -quadratic, i.e.  $(a_0z)^{(2)} = a_0^2z^{(2)}$ ,  $(a_1w)^{(2)} = 0$  for  $a_0 \in A_0$ ,  $a_1 \in A_1$ ,
- (e)  $(z_1 + z_2)^{(2)} = z_1^{(2)} + [z_1, z_2] + z_2^{(2)}$ ,  $[z^{(2)}, x] = [z, [z, x]]$ .

All Lie  $A$ -superalgebras form a category, whose morphisms are the  $A$ -superlinear (in the obvious sense) graded maps preserving the bracket and the 2-operation.

A Lie superalgebra  $\mathfrak{g}$  is called *classical* if it is simple, i.e. it has no non-trivial (homogeneous) ideals, and  $\mathfrak{g}_1$  is semisimple as a  $\mathfrak{g}_0$ -module. Classical Lie superalgebras of finite dimension over algebraically closed fields of characteristic 0 were classified by V. G. Kac (see [11, 13]), to whose work we refer for the standard terminology and notions.

**Examples 2.3.** (a) Let  $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$  be any associative  $\mathbb{k}$ -superalgebra. There exists a canonical structure of Lie superalgebra on  $\mathcal{A}$  given by  $[x, y] := xy - (-1)^{p(x)p(y)}yx$  for all homogeneous  $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$  and by the 2-operation  $z^{(2)} := z^2 = zz$  (the associative square in  $\mathcal{A}$ ) for all odd  $z \in \mathcal{A}_1$ .

(b) Let  $V = V_0 \oplus V_1$  be a *free*  $\mathbb{k}$ -supermodule, and consider  $\text{End}(V)$ , the endomorphisms of  $V$  as an ordinary  $\mathbb{k}$ -module. This is, again, a free super  $\mathbb{k}$ -module,  $\text{End}(V) = \text{End}(V)_0 \oplus \text{End}(V)_1$ , where  $\text{End}(V)_0$  are the morphisms that preserve the parity, while  $\text{End}(V)_1$  are the morphisms that reverse the parity. By the recipe in (a),

$\text{End}(V)$  is a Lie  $\mathbb{k}$ -superalgebra with  $[A, B] := AB - (-1)^{p(A)p(B)}BA$ ,  $C^{(2)} := C^2$ , for all  $A, B, C \in \text{End}(V)$  homogeneous, with  $C$  odd.

The standard example is for  $V$  of finite rank, say  $V := \mathbb{k}^{p|q} = \mathbb{k}^p \oplus \mathbb{k}^q$ , with  $V_0 := \mathbb{k}^p$  and  $V_1 := \mathbb{k}^q$ ; in this case we also write that  $\text{End}(\mathbb{k}^{m|n}) := \text{End}(V)$  or  $\mathfrak{gl}_{p|q} := \text{End}(V)$ . Choosing a basis of homogeneous elements for  $V$  (writing first the even ones), we identify  $\text{End}(V)_0$  with the set of all diagonal block matrices, and  $\text{End}(V)_1$  with the set of all off-diagonal block matrices.

### 2.3. The Lie superalgebra $D(2, 1; a)$

Let  $\mathbb{K}$  be an algebraically closed field of characteristic 0, and let  $a \in \mathbb{K} \setminus \{0, -1\}$ . Then, let  $\mathbb{Z}[a]$  be the unital subring of  $\mathbb{K}$  generated by  $a$ . Clearly,  $\mathbb{Z}[a] = \mathbb{Z}$  if and only if  $a \in \mathbb{Z}$ .

According to Kac’s work, we can realize  $\mathfrak{g} := D(2, 1; a)$  as a contragredient Lie superalgebra; in particular, it admits a presentation by generators and relations with a standard procedure (detailed for the general case in [11]). In order to do that, we first fix a specific choice of Dynkin diagram and corresponding Cartan matrix, as in [7, §2.28] (first choice), namely,

$$\begin{array}{c} \textcircled{2} \text{---} \textcircled{1} \text{---} \textcircled{3} \\ \quad \quad \quad \times \\ \quad \quad \quad a \end{array}, \quad (a_{i,j})_{i,j=1,2,3} := \begin{pmatrix} 0 & 1 & a \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

We define  $\mathfrak{g} = D(2, 1; a)$  as the Lie superalgebra over  $\mathbb{K}$  with generators  $h_i, e_i, f_i$  ( $i = 1, 2, 3$ ), with degrees  $p(h_i) := 0$ ,  $p(e_i) := \delta_{1,i}$ ,  $p(f_i) := \delta_{1,i}$  ( $i = 1, 2, 3$ ) and relations (for  $i, j = 1, 2, 3$ )

$$\begin{aligned} [h_i, h_j] &= 0, & [e_1, e_1] &= 0, & [f_1, f_1] &= 0, \\ [h_i, e_j] &= +a_{i,j}e_j, & [h_i, f_j] &= -a_{i,j}f_j, & [e_i, f_j] &= \delta_{i,j}h_i, \\ e_1^{(2)} &= 0, & f_1^{(2)} &= 0. \end{aligned}$$

The subspace  $\mathfrak{h} := \sum_{i=1}^3 \mathbb{K}h_i$  is a Cartan subalgebra of  $\mathfrak{g}$  (included in  $\mathfrak{g}_0$ ). The adjoint action of  $\mathfrak{h}$  splits  $\mathfrak{g}$  into eigenspaces, namely,

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \quad \text{with } \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$$

for all  $\alpha \in \mathfrak{h}^*$ . We then define the roots of  $\mathfrak{g}$  by  $\Delta := \Delta_0 \amalg \Delta_1 = \{\text{roots of } \mathfrak{g}\}$ , with

$$\begin{aligned} \Delta_0 &:= \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq \{0\}\} = \{\text{even roots of } \mathfrak{g}\}, \\ \Delta_1 &:= \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq \{0\}\} = \{\text{odd roots of } \mathfrak{g}\}. \end{aligned}$$

Now,  $\Delta$  is called the root system of  $\mathfrak{g}$ , and for each root  $\alpha$  we call  $\mathfrak{g}_\alpha$  its root space. Moreover, every non-zero vector in a root space is called a root vector. Note that every root space is one dimensional, so any root vector forms a basis of its own root space. Any  $\mathbb{K}$ -basis of  $\mathfrak{h}$  together with any choice of a root vector for each root will then provide a  $\mathbb{K}$ -basis of  $\mathfrak{g} = D(2, 1; a)$ .

There exists an even, non-degenerate, invariant bilinear form on  $\mathfrak{g}$ , whose restriction to  $\mathfrak{h}$  is in turn an invariant bilinear form on  $\mathfrak{h}$ . We denote this form by  $(x, y)$ , and we use it to identify  $\mathfrak{h}^*$  to  $\mathfrak{h}$ , via  $\alpha \mapsto H_\alpha$ , and then to define a similar form on  $\mathfrak{h}^*$ , such that  $(\alpha', \alpha'') = (H_{\alpha'}, H_{\alpha''})$ . In particular, we fix normalizations such that  $\alpha(H_\alpha) = 2$  and  $\alpha(H_{\alpha'}) = 0$  for all  $\alpha, \alpha' \in \Delta_0$  such that  $(\alpha, \alpha') = 0$  (in short, we adopt the normalizations as in [9]). Moreover, if  $\alpha$  is any (even, odd, etc.) root, we call the vector  $H_\alpha$  the (even, odd, etc.) *coroot* associated with  $\alpha$ .

Actually, we can explicitly describe the root system of  $\mathfrak{g} = D(2, 1; a)$  (after [7, Table 3.60]) as

$$\begin{aligned} \Delta &= \{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm 2\varepsilon_3, \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}, \\ \Delta_0 &= \{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm 2\varepsilon_3\}, \\ \Delta_1 &= \{\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}, \end{aligned}$$

where  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is an orthogonal basis in a  $\mathbb{K}$ -vector space with inner product  $(\cdot, \cdot)$  such that  $(\varepsilon_1, \varepsilon_1) = -(1 + a)/2$ ,  $(\varepsilon_2, \varepsilon_2) = 1/2$ ,  $(\varepsilon_3, \varepsilon_3) = a/2$ .

We then fix a distinguished system of *simple roots*, say  $\{\alpha_1, \alpha_2, \alpha_3\}$ , namely,  $\alpha_1 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3$ ,  $\alpha_2 = 2\varepsilon_2$ ,  $\alpha_3 = 2\varepsilon_3$ , associated with this choice. In terms of these, we call the roots

$$\begin{aligned} 2\varepsilon_1 &= 2\alpha_1 + \alpha_2 + \alpha_3, & 2\varepsilon_2 &= \alpha_2, & 2\varepsilon_3 &= \alpha_3, \\ \varepsilon_1 - \varepsilon_2 - \varepsilon_3 &= \alpha_1, & \varepsilon_1 + \varepsilon_2 - \varepsilon_3 &= \alpha_1 + \alpha_2, \\ \varepsilon_1 - \varepsilon_2 + \varepsilon_3 &= \alpha_1 + \alpha_3, & \varepsilon_1 + \varepsilon_2 + \varepsilon_3 &= \alpha_1 + \alpha_2 + \alpha_3 \end{aligned}$$

*positive* (those in the first line being even, the others odd), and denote their set by  $\Delta^+$ , so

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3\}.$$

We call the roots in  $\Delta^- := -\Delta^+$  *negative* instead. So the root system is given by  $\Delta = \Delta^+ \amalg \Delta^-$ . We also set  $\Delta_0^\pm := \Delta_0 \cap \Delta^\pm$ ,  $\Delta_1^\pm := \Delta_1 \cap \Delta^\pm$ .

It is worth stressing at this point that the coroots  $H_\alpha \in \mathfrak{h}$  associated with the positive roots are  $H_{2\varepsilon_1} = (1 + a)^{-1}(2h_1 - h_2 - ah_3)$ ,  $H_{2\varepsilon_2} = h_2$ ,  $H_{2\varepsilon_3} = h_3$ ,  $H_{\varepsilon_1 - \varepsilon_2 - \varepsilon_3} = h_1$ ,  $H_{\varepsilon_1 + \varepsilon_2 - \varepsilon_3} = h_1 - h_2$ ,  $H_{\varepsilon_1 - \varepsilon_2 + \varepsilon_3} = h_1 - ah_3$  and  $H_{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} = h_1 - h_2 - ah_3$ ; then, the formula  $H_{-\alpha} = H_\alpha$  yields the coroots associated with negative roots out of those associated with positive ones.

We now introduce the following elements:

$$\begin{aligned} e_{1,2} &:= [e_1, e_2], & e_{1,3} &:= [e_1, e_3], & e_{1,2,3} &:= [e_{1,2}, e_3], & e'_{1,1,2,3} &:= [e_1, e_{1,2,3}], \\ f_{2,1} &:= [f_2, f_1], & f_{3,1} &:= [f_3, f_1], & f_{3,2,1} &:= [f_3, f_{2,1}], & f'_{3,2,1,1} &:= [f_{3,2,1}, f_1]. \end{aligned}$$

All these are root vectors for the positive roots  $\alpha_1 + \alpha_2$ ,  $\alpha_1 + \alpha_3$ ,  $\alpha_1 + \alpha_2 + \alpha_3$  and  $2\alpha_1 + \alpha_2 + \alpha_3$ , respectively (for the first line vectors), and similarly for the negative roots (for the second line vectors). Moreover, by definition the generators  $e_i$  and  $f_i$  ( $i = 1, 2, 3$ ) are root vectors for the roots  $+\alpha_i$  and  $-\alpha_i$  ( $i = 1, 2, 3$ ), respectively. As  $\{h_1, h_2, h_3\}$  is a

$\mathbb{K}$ -basis of  $\mathfrak{h}$ , we conclude that all these root vectors, along with  $h_1, h_2$  and  $h_3$ , form a  $\mathbb{K}$ -basis of  $\mathfrak{g}$ .

The relevant new brackets among all these basis elements, dropping the zero ones, those given by the very definition of  $D(2, 1; a)$ , those coming from others by (super)skew-commutativity and those involving the  $h_i$  (given by the fact that all involved vectors are  $\mathfrak{h}$ -eigenvectors), are

$$\begin{aligned}
 [e_1, e_2] &= e_{1,2}, & [e_1, e_3] &= e_{1,3}, & [e_1, e_{1,2,3}] &= e'_{1,1,2,3}, \\
 [e_1, f_{2,1}] &= f_2, & [e_1, f_{3,1}] &= af_3, & [e_1, f'_{3,2,1,1}] &= -(1+a)f_{3,2,1}, \\
 [e_2, e_{1,3}] &= -e_{1,2,3}, & [e_2, f_{2,1}] &= f_1, & [e_2, f_{3,2,1}] &= f_{3,1}, \\
 [e_3, e_{1,2}] &= -e_{1,2,3}, & [e_3, f_{3,1}] &= f_1, & [e_3, f_{3,2,1}] &= f_{2,1}, \\
 [f_1, f_2] &= -f_{2,1}, & [f_1, f_3] &= -f_{3,1}, & [f_1, f_{3,2,1}] &= f'_{3,2,1,1}, \\
 [f_1, e_{1,2}] &= e_2, & [f_1, e_{1,3}] &= ae_3, & [f_1, e'_{1,1,2,3}] &= (1+a)e_{1,2,3}, \\
 [f_2, f_{3,1}] &= f_{3,2,1}, & [f_2, e_{1,2}] &= -e_1, & [f_2, e_{1,2,3}] &= -e_{1,3}, \\
 [f_3, f_{2,1}] &= f_{3,2,1}, & [f_3, e_{1,3}] &= -e_1, & [f_3, e_{1,2,3}] &= -e_{1,2}, \\
 [e_{1,2}, e_{1,3}] &= -e'_{1,1,2,3}, & [e_{1,2}, f_{2,1}] &= h_1 - h_2, \\
 [e_{1,2}, f_{3,2,1}] &= af_3, & [e_{1,2}, f'_{3,2,1,1}] &= (1+a)f_{3,1}, \\
 [e_{1,3}, f_{3,1}] &= h_1 - ah_3, & [e_{1,3}, f_{3,2,1}] &= f_2, & [e_{1,3}, f'_{3,2,1,1}] &= (1+a)f_{2,1}, \\
 [f_{2,1}, f_{3,1}] &= -f'_{3,2,1,1}, & [f_{2,1}, e_{1,2,3}] &= ae_3, & [f_{2,1}, e'_{1,1,2,3}] &= -(1+a)e_{1,3}, \\
 [f_{3,1}, e_{1,2,3}] &= e_2, & [f_{3,1}, e'_{1,1,2,3}] &= -(1+a)e_{1,2}, \\
 [e_{1,2,3}, f_{3,2,1}] &= h_1 - h_2 - ah_3, & [e_{1,2,3}, f'_{3,2,1,1}] &= -(1+a)f_1, \\
 [f_{3,2,1}, e'_{1,1,2,3}] &= -(1+a)e_1, & [e'_{1,1,2,3}, f'_{3,2,1,1}] &= -(1+a)(2h_1 - h_2 - ah_3), \\
 e_{1,2}^{(2)} &= 0, & e_{1,3}^{(2)} &= 0, & e_{1,2,3}^{(2)} &= 0, & f_{2,1}^{(2)} &= 0, & f_{3,1}^{(2)} &= 0, & f_{3,2,1}^{(2)} &= 0.
 \end{aligned}$$

We now modify just two root vectors, taking

$$e_{1,1,2,3} := +(1+a)^{-1}e'_{1,1,2,3}, \quad f_{3,2,1,1} := -(1+a)^{-1}f'_{3,2,1,1}$$

(recall that  $a \neq -1$  by assumption); the above formulae have to then be modified accordingly.

In particular, one now checks that the even part of  $\mathfrak{g} := D(2, 1; a)$  is  $\mathfrak{g}_0 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ . Moreover, the three triples  $(e_{1,1,2,3}, f_{3,2,1,1}, (1+a)^{-1}(2h_1 - h_2 - ah_3))$ ,  $(e_2, f_2, h_2)$  and  $(e_3, f_3, h_3)$  are  $\mathfrak{sl}_2$ -triples inside  $\mathfrak{g}$ , each one being associated with a (positive) even root  $2\varepsilon_i$  ( $i = 1, 2, 3$ ).

### 3. Chevalley bases and Kostant superalgebras for $D(2, 1; a)$

In this section we introduce the first results we build upon to construct Chevalley supergroups of type  $D(2, 1; a)$ . As before,  $\mathbb{K}$  is an algebraically closed field of characteristic 0.

### 3.1. Chevalley bases and Chevalley Lie superalgebras

The subject of this subsection is an analogue, in the super setting, of a classical result due to Chevalley: the notion of Chevalley basis and, correspondingly, of Chevalley Lie superalgebra. For  $\mathfrak{g} := D(2, 1; a)$ , this notion is introduced exactly as in [6, Definition 3.3], up to changing  $\mathbb{Z}$  to  $\mathbb{Z}[a]$ , the latter being the unital subring of  $\mathbb{K}$  generated by  $a$  (see § 2.3).

**Definition 3.1.** We call a *Chevalley basis* of  $\mathfrak{g} := D(2, 1; a)$  any homogeneous  $\mathbb{K}$ -basis  $B = \{H_i\}_{i=1,2,3} \amalg \{X_\alpha\}_{\alpha \in \Delta}$  of  $\mathfrak{g}$  with the following properties.

- (a)  $\{H_1, H_2, H_3\}$  is a  $\mathbb{K}$ -basis of  $\mathfrak{h}$ ; moreover, with  $H_\alpha \in \mathfrak{h}$  as in § 2.3,

$$\mathfrak{h}_{\mathbb{Z}[a]} := \text{Span}_{\mathbb{Z}[a]}(H_1, H_2, H_3) = \text{Span}_{\mathbb{Z}[a]}(\{H_\alpha \mid \alpha \in \Delta\}).$$

- (b)  $[H_i, H_j] = 0$ ,  $[H_i, X_\alpha] = \alpha(H_i)X_\alpha$  for all  $i, j \in \{1, \dots, \ell\}$ ,  $\alpha \in \Delta$ .
- (c)  $[X_\alpha, X_{-\alpha}] = \sigma_\alpha H_\alpha$  for all  $\alpha \in \Delta$ , with  $H_\alpha$  as in (a), and  $\sigma_\alpha := -1$  if  $\alpha \in \Delta_1^-$  or  $\sigma_\alpha := 1$  otherwise.
- (d)  $[X_\alpha, X_\beta] = c_{\alpha, \beta} X_{\alpha+\beta}$  for all  $\alpha, \beta \in \Delta: \alpha + \beta \neq 0$ , with the conditions that
- (d.1) if  $(\alpha + \beta) \notin \Delta$ , then  $c_{\alpha, \beta} = 0$  and  $X_{\alpha+\beta} := 0$ ,
- (d.2) if  $(\alpha, \alpha) \neq 0$  or  $(\beta, \beta) \neq 0$ , and if  $\Sigma_\beta^\alpha := (\beta + \mathbb{Z}\alpha) \cap (\Delta \cup \{0\}) = \{\beta - r\alpha, \dots, \beta + q\alpha\}$  is the  $\alpha$ -string through  $\beta$ , then  $c_{\alpha, \beta} = \pm(r+1)$ ,
- (d.3) if  $(\alpha, \alpha) = 0 = (\beta, \beta)$ , then  $c_{\alpha, \beta} = \pm\beta(H_\alpha)$ .

**Definition 3.2.** If  $B$  is a Chevalley basis of  $\mathfrak{g}$ , we denote by  $\mathfrak{g}^{\mathbb{Z}[a]}$  the  $\mathbb{Z}[a]$ -span of  $B$ , and we call it the *Chevalley Lie superalgebra* (of  $\mathfrak{g}$ ).

**Remark 3.3.** The above notions are taken from [6]; in general, one should (and can) adapt them to the notion of ‘Lie superalgebra’ in the stronger sense of Definition 2.2 (involving the ‘2-operation’). However, in the present case, i.e. for  $\mathfrak{g} = D(2, 1; a)$ , no change is necessary. Also, all examples of Chevalley bases considered in [6, § 3.3] are Chevalley bases in the ‘stronger’ sense too.

### 3.2. Existence of Chevalley bases

We consider in  $\mathfrak{g}$  the generators  $h_i, e_i, f_i$  ( $i = 1, 2, 3$ ) and the root vectors  $e_{1,2}, e_{1,3}, e_{1,2,3}, e_{1,1,2,3}, f_{2,1}, f_{3,1}, f_{3,2,1}, f_{3,2,1,1}$  constructed in § 2.3. Looking at all brackets among those considered there, it is a routine matter to check that the set

$$B := \{H_i, e_i, f_i\}_{i=1,2,3} \cup \{e_{1,2}, e_{1,3}, e_{1,2,3}, e_{1,1,2,3}, f_{2,1}, f_{3,1}, f_{3,2,1}, f_{3,2,1,1}\},$$

with  $H_1 := h_1$ ,  $H_2 := (1+a)^{-1}(2h_1 - h_2 - ah_3)$ ,  $H_3 := h_3$ , is indeed a *Chevalley basis* for  $\mathfrak{g} = D(2, 1; a)$  (the proof is a bookkeeping matter).



To be precise, the root vectors in  $B$  are the following.

Even:

$$\begin{aligned} X_{+(2\alpha_1+\alpha_2+\alpha_3)} &= e_{1,1,2,3}, & X_{+\alpha_2} &= e_2, & X_{+\alpha_3} &= e_3 & (\text{positive}), \\ X_{-(2\alpha_1+\alpha_2+\alpha_3)} &= f_{3,2,1,1}, & X_{-\alpha_2} &= f_2, & X_{-\alpha_3} &= f_3 & (\text{negative}). \end{aligned}$$

Odd:

$$\begin{aligned} X_{+\alpha_1} &= e_1, & X_{+\alpha_1+\alpha_2} &= e_{1,2}, & X_{+\alpha_1+\alpha_3} &= e_{1,3}, & X_{+\alpha_1+\alpha_2+\alpha_3} &= e_{1,2,3} & (\text{positive}), \\ X_{-\alpha_1} &= f_1, & X_{-\alpha_1-\alpha_2} &= f_{2,1}, & X_{-\alpha_1-\alpha_3} &= e_{3,1}, & X_{-\alpha_1-\alpha_2-\alpha_3} &= f_{3,2,1} & (\text{negative}). \end{aligned}$$

On the other hand, the Cartan elements in  $B$  are just  $H_1, H_2$  and  $H_3$ , defined as above (see also [10]).

### 3.3. Kostant superalgebra

For any  $\mathbb{K}$ -algebra  $A$ , given  $n \in \mathbb{N}$  and  $y \in A$ , we define the  $n$ th binomial coefficients  $\binom{y}{n}$  and the  $n$ th divided power  $y^{(n)}$  by

$$\binom{y}{n} := \frac{y(y-1) \cdots (y-n+1)}{n!}, \quad y^{(n)} := \frac{y^n}{n!}.$$

Recall that  $\mathbb{Z}[a]$  is the unital subring of  $\mathbb{K}$  generated by  $a$ . We also need to consider the unital subring  $\mathbb{Z}_a$  of  $\mathbb{K}$  generated by the subset  $\{\binom{P(a)}{n} \mid P(a) \in \mathbb{Z}[a], n \in \mathbb{N}\}$ .

By a classical result on integer-valued polynomials (see [6, Lemma 4.1]), one shows that  $\mathbb{Z}_a$  is in fact also generated by  $\{\binom{a}{n} \mid n \in \mathbb{N}\}$ . Note also that  $\mathbb{Z}_a = \mathbb{Z}$  if and only if  $a \in \mathbb{Z}$ .

Fix in  $\mathfrak{g} := D(2, 1; a)$  a Chevalley basis  $B = \{H_1, H_2, H_3\} \amalg \{X_\alpha\}_{\alpha \in \Delta}$  as in Definition 3.1. Let  $U(\mathfrak{g})$  be the universal enveloping superalgebra of  $\mathfrak{g}$ .

In [6, § 4.1], the Kostant superalgebra  $K_{\mathbb{Z}}(\mathfrak{g})$  was defined as the subalgebra of  $U(\mathfrak{g})$  generated by divided powers of the root vectors attached to even roots, the root vectors attached to odd roots and binomial coefficients in the elements of  $\mathfrak{h}_{\mathbb{Z}}$ , the  $\mathbb{Z}$ -span of the elements of  $\{H_1, H_2, H_3\}$ .

If we try to perform the same construction *verbatim* for  $\mathfrak{g} = D(2, 1; a)$  for  $a \in \mathbb{K} \setminus \mathbb{Z}$ , we are soon forced to include among the generators all binomial coefficients of type  $\binom{H}{n}$ , with  $H \in \mathfrak{h}_{\mathbb{Z}[a]}$ , the  $\mathbb{Z}[a]$ -span of  $\{H_1, H_2, H_3\}$ . When commuting such binomial coefficients with divided powers, coefficients of type  $\binom{\alpha(H)}{n}$  show up, where  $\alpha$  is a root and  $H \in \mathfrak{h}_{\mathbb{Z}[a]}$ . By construction we have  $\alpha(H) \in \mathbb{Z}[a]$ ; hence,  $\binom{\alpha(H)}{n}$  belongs to the ring  $\mathbb{Z}_a$  defined above.

By the above remarks, for  $\mathfrak{g} := D(2, 1; a)$  we define the Kostant superalgebra  $K_{\mathbb{Z}_a}(\mathfrak{g})$  as we did in [6] (for the other classical Lie superalgebras), *but* with  $\mathbb{Z}_a$  as the ground ring; more precisely, we have the following.

**Definition 3.4.** We call the *Kostant superalgebra*, or Kostant’s  $\mathbb{Z}_a$ -form of  $U(\mathfrak{g})$ , the unital  $\mathbb{Z}_a$ -subsuperalgebra  $K_{\mathbb{Z}_a}(\mathfrak{g})$  of  $U(\mathfrak{g})$  generated by all the elements

$$X_\alpha^{(n)} := X_\alpha^n/n!, \quad X_\gamma, \quad \binom{H}{n} \quad \forall \alpha \in \Delta_0, n \in \mathbb{N}, \gamma \in \Delta_1, H \in \mathfrak{h}_{\mathbb{Z}[a]}.$$

The following analogue of [6, Corollary 4.2] holds (with the same proof), and is needed in the proof of the PBW-like theorem for  $K_{\mathbb{Z}_a}(\mathfrak{g})$ .

**Corollary 3.5.**

(a)  $\mathbb{H}_{\mathbb{Z}_a} := \{h \in U(\mathfrak{h}) \mid h(z'_1, z'_2, z'_3) \in \mathbb{Z}_a \ \forall z'_1, z'_2, z'_3 \in \mathbb{Z}_a\}$  is a free  $\mathbb{Z}_a$ -submodule of  $U(\mathfrak{h})$ , with basis

$$B_{U(\mathfrak{h})} := \left\{ \binom{H_1}{n_1} \binom{H_2}{n_2} \binom{H_3}{n_3} \mid n_1, n_2, n_3 \in \mathbb{N} \right\}.$$

(b) The  $\mathbb{Z}_a$ -subalgebra of  $U(\mathfrak{g})$  generated by all the elements  $\binom{H'-z'}{n}$  with

$$H' \in \mathfrak{h}_{\mathbb{Z}_a} := \sum_{i=1}^3 \mathbb{Z}_a H_i, \quad z' \in \mathbb{Z}_a, \quad n \in \mathbb{N},$$

is merely  $\mathbb{H}_{\mathbb{Z}_a}$ .

**3.4. Commutation rules and Kostant’s theorem**

In [6] Fioresi and Gavarini proved a ‘super PBW-like’ theorem for Kostant’s superalgebras, namely, the latter is a free  $\mathbb{Z}$ -module with  $\mathbb{Z}$ -basis the set of ordered monomials (with respect to any total order) whose factors are binomial coefficients in the  $H_i$ , or *odd* root vectors, or divided powers of *even* root vectors. This result follows from a direct analysis of commutation rules among the generators of Kostant’s superalgebra. One can perform the same proof for  $D(2, 1; a)$ , using a list of relevant ‘commutation rules’ (all proved by easy induction) whose main feature is that all coefficients belong to  $\mathbb{Z}_a$ . We split the list into two sections: first, relations involving only *even* generators (known by classical theory); second, relations also involving *odd* generators.

(1) Even generators only (that is,  $\binom{H_i}{m}$  and  $X_\alpha^{(n)}$  only,  $\alpha \in \Delta_0$ ):

$$\begin{aligned} \binom{H_i}{n} \binom{H_j}{m} &= \binom{H_j}{m} \binom{H_i}{n} && \forall i, j \in \{1, 2, 3\}, \forall n, m \in \mathbb{N}, \\ X_\alpha^{(n)} f(H) &= f(H - n\alpha(H)) X_\alpha^{(n)} && \forall \alpha \in \Delta_0, H \in \mathfrak{h}, n \in \mathbb{N}, f(T) \in \mathbb{K}[T], \\ X_\alpha^{(n)} X_\alpha^{(m)} &= \binom{n+m}{m} X_\alpha^{(n+m)} && \forall \alpha \in \Delta_0, \forall n, m \in \mathbb{N}, \\ X_\alpha^{(n)} X_\beta^{(m)} &= X_\beta^{(m)} X_\alpha^{(n)} + l.h.t && \forall \alpha, \beta \in \Delta_0, \forall n, m \in \mathbb{N}, \end{aligned}$$

where *l.h.t.* stands for a  $\mathbb{Z}_a$ -linear combination of monomials in the  $X_\delta^{(k)}$  and in the  $\binom{H_i}{c}$  whose ‘height’ (by definition, the sum of all ‘exponents’  $k$  occurring in such a monomial) is less than  $n + m$ . A *special case* is

$$X_\alpha^{(n)} X_{-\alpha}^{(m)} = \sum_{k=0}^{\min(m,n)} X_{-\alpha}^{(m-k)} \binom{H_\alpha - m - n + 2k}{k} X_\alpha^{(n-k)} \quad \forall \alpha \in \Delta_0, m, n \in \mathbb{N}.$$

(2) Odd and even generators (also involving the  $X_\gamma$ ,  $\gamma \in \Delta_1$ ):

$$\begin{aligned} X_\gamma f(H) &= f(H - \gamma(H))X_\gamma & \forall \gamma \in \Delta_1, h \in \mathfrak{h}, f(T) \in \mathbb{K}[T], \\ X_\gamma^n &= 0 & \forall \gamma \in \Delta_1, \forall n \geq 2, \\ X_{-\gamma}X_\gamma &= -X_\gamma X_{-\gamma} + H_\gamma & \forall \gamma \in \Delta_1, \\ X_\gamma X_\delta &= -X_\delta X_\gamma + c_{\gamma, \delta} X_{\gamma+\delta} & \forall \gamma, \delta \in \Delta_1, \gamma + \delta \neq 0, \end{aligned}$$

with  $c_{\gamma, \delta}$  as in Definition 3.1,

$$X_\alpha^{(n)} X_\gamma = X_\gamma X_\alpha^{(n)} + \sum_{k=1}^n \left( \prod_{s=1}^k \varepsilon_s \right) \binom{r+k}{k} X_{\gamma+k\alpha} X_\alpha^{(n-k)} \quad \forall n \in \mathbb{N}, \alpha \in \Delta_0, \gamma \in \Delta_1,$$

with  $\sigma_\gamma^\alpha = \{\gamma - r\alpha, \dots, \gamma, \dots, \gamma + q\alpha\}$ ,  $X_{\gamma+k\alpha} := 0$  if  $(\gamma + k\alpha) \notin \Delta$  and  $\varepsilon_s = \pm 1$  such that  $[X_\alpha, X_{\gamma+(s-1)\alpha}] = \varepsilon_s(r+s)X_{\gamma+s\alpha}$ .

Our super version of Kostant’s theorem for  $K_{\mathbb{Z}_a}(\mathfrak{g})$  is the following.

**Theorem 3.6.** *The Kostant superalgebra  $K_{\mathbb{Z}_a}(\mathfrak{g})$  is a free  $\mathbb{Z}_a$ -module. For any given total order  $\preceq$  of the set  $\Delta \cup \{1, 2, 3\}$ , a  $\mathbb{Z}_a$ -basis of  $K_{\mathbb{Z}_a}(\mathfrak{g})$  is the set  $\mathcal{B}$  of ordered ‘PBW-like monomials’, i.e. all products (without repetition) of factors of type  $X_\alpha^{(n_\alpha)}$ ,  $\binom{H_i}{n_i}$  and  $X_\gamma$ , with  $\alpha \in \Delta_0$ ,  $i \in \{1, 2, 3\}$ ,  $\gamma \in \Delta_1$  and  $n_\alpha, n_i \in \mathbb{N}$ , taken in the right order with respect to  $\preceq$ .*

This result is proved like the similar one in [6], making use of the commutation relations considered above. It also has a direct consequence, again proved similarly to that in [6]. To state it, we first consider  $\mathfrak{g}_1^{\mathbb{Z}[a]}$ , the odd part of  $\mathfrak{g}^{\mathbb{Z}[a]}$ : it has  $\{X_\gamma \mid \gamma \in \Delta_1\}$  as a  $\mathbb{Z}[a]$ -basis, by construction. Then let  $\bigwedge \mathfrak{g}_1^{\mathbb{Z}[a]}$  be the exterior  $\mathbb{Z}[a]$ -algebra over  $\mathfrak{g}_1^{\mathbb{Z}[a]}$ , and let  $\bigwedge \mathfrak{g}_1^{\mathbb{Z}_a}$  be its scalar extension to  $\mathbb{Z}_a$ . Also let  $K_{\mathbb{Z}}(\mathfrak{g}_0)$  be the classical Kostant algebra of  $\mathfrak{g}_0$  (over  $\mathbb{Z}$ ) and let  $K_{\mathbb{Z}_a}(\mathfrak{g}_0)$  be its scalar extension to  $\mathbb{Z}_a$ . The tensor factorization  $U(\mathfrak{g}) \cong U(\mathfrak{g}_0) \otimes_{\mathbb{K}} \bigwedge \mathfrak{g}_1$  (see [15]) then has the following ‘integral version’.

**Corollary 3.7.** *There exists an isomorphism of  $\mathbb{Z}_a$ -modules*

$$K_{\mathbb{Z}_a}(\mathfrak{g}) \cong K_{\mathbb{Z}_a}(\mathfrak{g}_0) \otimes_{\mathbb{Z}_a} \bigwedge \mathfrak{g}_1^{\mathbb{Z}_a}.$$

**Remarks 3.8.** (a) Following a classical pattern (and see [2, 3, 14] in the super context) we can define the *superalgebra of distributions*  $\text{Dist}(G)$  on any supergroup  $G$ . Then,  $\text{Dist}(G) = K_{\mathbb{Z}_a}(\mathfrak{g}) \otimes_{\mathbb{Z}_a} \mathbb{K}$  when  $\mathfrak{g} := \text{Lie}(G)$  is just  $D(2, 1; a)$ .

(b) All the above proves that the assumptions of [14, Theorem 2.8] hold for any supergroup  $G$  with tangent Lie superalgebra  $\mathfrak{g} = D(2, 1; a)$ . Thus, *all results in [14] do apply to such supergroups.*

#### 4. Chevalley supergroups of type $D(2, 1; a)$

We now present the construction of affine supergroups associated with the Lie superalgebra  $\mathfrak{g} = D(2, 1; a)$ . The method, inspired by Chevalley’s original one (dealing with

complex semisimple Lie algebras), follows closely the one presented in [6] for the other classical Lie superalgebras, including  $\mathfrak{g} = D(2, 1; a)$  when  $a \in \mathbb{Z}$ . However, the occurrence of the (possibly non-integral) parameter  $a$  demands that we revisit that construction and introduce some suitable, delicate, modifications.

#### 4.1. Admissible lattices

Let  $\mathbb{K}$  be an algebraically closed field of characteristic 0. If  $R$  is a unital subring of  $\mathbb{K}$ , and  $V$  is a finite-dimensional  $\mathbb{K}$ -vector space, any  $M \subseteq V$  is called an  $R$ -lattice (or  $R$ -form) of  $V$  if  $M = \text{Span}_R(B)$  for some  $\mathbb{K}$ -basis  $B$  of  $V$ . Let  $\mathfrak{g} = D(2, 1; a)$  be defined over  $\mathbb{K}$ , and fix the ring  $\mathbb{Z}_a$ , a Chevalley basis  $B$  of  $\mathfrak{g}$  and the Kostant algebra  $K_{\mathbb{Z}_a}(\mathfrak{g})$  as in § 3.

The following definition and results are just slight variations of those in [6, § 5.1].

**Definition 4.1.** Let  $V$  be a  $\mathfrak{g}$ -module, and let  $M$  be a corresponding  $\mathbb{Z}_a$ -lattice.

(a) We call  $V$  *rational* if

- (i)  $\mathfrak{h}_{\mathbb{Z}[a]} := \text{Span}_{\mathbb{Z}[a]}(H_1, H_2, H_3)$  acts diagonally on  $V$  with eigenvalues in  $\mathbb{Z}[a]$ ; in other words, one has that  $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ , with  $V_\mu := \{v \in V \mid h.v = \mu(h)v \ \forall h \in \mathfrak{h}\}$ , and  $\mu(H_i) \in \mathbb{Z}[a]$  for all  $i$  and all  $\mu$  such that  $V_\mu \neq \{0\}$ ,
- (ii)  $\mu(H_\alpha) \in \mathbb{Z}$  for all  $\alpha \in \Delta_0$  and  $\mu \in \mathfrak{h}^*$  such that  $V_\mu \neq \{0\}$ .

(b) We call  $M$  an *admissible (lattice)* if it is  $K_{\mathbb{Z}_a}(\mathfrak{g})$ -stable.

**Theorem 4.2.** Any rational, finite-dimensional, semisimple  $\mathfrak{g}$ -module  $V$  contains an admissible lattice  $M$ . Any such  $M$  is the (direct) sum of its weight components, i.e.  $M = \bigoplus_{\mu \in \mathfrak{h}^*} (M \cap V_\mu)$ .

**Theorem 4.3.** Let  $V$  be a rational, finite-dimensional  $\mathfrak{g}$ -module, let  $M$  be an admissible lattice of  $V$ , and let  $\mathfrak{g}_V = \{X \in \mathfrak{g} \mid X.M \subseteq M\}$ . If  $V$  is faithful, then

$$\mathfrak{g}_V = \mathfrak{h}_V \bigoplus (\bigoplus_{\alpha \in \Delta} \mathbb{Z}_a X_\alpha), \quad \mathfrak{h}_V := \{H \in \mathfrak{h} \mid \mu(H) \in \mathbb{Z}_a \ \forall \mu \in \Lambda\},$$

where  $\Lambda$  is the set of all weights of  $V$ . In particular,  $\mathfrak{g}_V$  is a  $\mathbb{Z}_a$ -lattice in  $\mathfrak{g}$ , independent of the choice of the admissible lattice  $M$  (but not of  $V$ ).

From now on, we retain the following notation:  $V$  is a rational, finite-dimensional  $\mathfrak{g}$ -module, and  $M$  is an admissible lattice of  $V$ . Also, we assume that  $\mathbb{k}$  is a commutative unital  $\mathbb{Z}_a$ -algebra.

With these assumptions, we set  $\mathfrak{g}_{\mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}_a} \mathfrak{g}_V$ ,  $V_{\mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}_a} M$ ,  $U_{\mathbb{k}}(\mathfrak{g}) := \mathbb{k} \otimes_{\mathbb{Z}_a} K_{\mathbb{Z}_a}(\mathfrak{g})$ ; so  $\mathfrak{g}_{\mathbb{k}}$  acts faithfully on  $V_{\mathbb{k}}$ , yielding an embedding of  $\mathfrak{g}_{\mathbb{k}}$  into  $\mathfrak{gl}(V_{\mathbb{k}})$ . For any  $A \in (\text{salg})_{\mathbb{k}}$ , the Lie superalgebra  $\mathfrak{g}_A := A \otimes_{\mathbb{k}} \mathfrak{g}_{\mathbb{k}}$  acts faithfully on  $V_{\mathbb{k}}(A) := A \otimes_{\mathbb{k}} V_{\mathbb{k}}$ , so it embeds into  $\mathfrak{gl}(V_{\mathbb{k}}(A))$ , etc.

**4.2. Additive one-parameter supersubgroups**

Let  $\alpha \in \Delta_0$ , let  $\beta \in \Delta_1$ , and let  $X_\alpha, X_\beta$  be the associated root vectors (in our fixed Chevalley basis of  $\mathfrak{g}$ ). Both  $X_\alpha$  and  $X_\beta$  act as nilpotent operators on  $V$ , hence on  $M$  and  $V_{\mathbb{k}}$ , so they are represented by nilpotent matrices in  $\mathfrak{gl}(V_{\mathbb{k}}(A))$ ; the same holds for all operators

$$tX_\alpha, \vartheta X_\beta \in \text{End}(V_{\mathbb{k}}(A)) \quad \forall t \in A_0, \vartheta \in A_1. \tag{4.1}$$

Of course we have that  $Y^{(n)} := Y^n/n! \in (K_{\mathbb{Z}}(\mathfrak{g}))(A)$  for any  $Y$  as in (4.1) and  $n \in \mathbb{N}$ ; moreover,  $Y^{(n)} = 0$  for  $n \gg 0$ , by nilpotency. Thus, the formal power series

$$\exp(Y) := \sum_{n=0}^{+\infty} Y^{(n)},$$

when computed for  $Y$  as in (4.1), gives a well-defined element in  $\text{GL}(V_{\mathbb{k}}(A))$ , expressed as a finite sum.

**Definition 4.4.** Let  $\alpha \in \Delta_0$ , let  $\beta \in \Delta_1$  and let  $X_\alpha, X_\beta$  be as above. We define the supergroup functors  $x_\alpha$  and  $x_\beta$  from (salg) to (groups) as

$$x_\alpha(A) := \{x_\alpha(t) := \exp(tX_\alpha) \mid t \in A_0\} = \left\{ \left( 1 + tX_\alpha + \frac{t}{2!}X_\alpha^2 + \dots \right) \mid t \in A_0 \right\},$$

$$x_\beta(A) := \{x_\beta(\vartheta) := \exp(\vartheta X_\beta) \mid \vartheta \in A_1\} = \{(1 + \vartheta X_\beta) \mid \vartheta \in A_1\}.$$

For later convenience we also write that  $x_\zeta(t) := 1$  when  $\zeta$  belongs to the  $\mathbb{Z}$ -span of  $\Delta$  but  $\zeta \notin \Delta$ .

As in [6, Proposition 5.8 (a)], one sees that these supergroup functors are in fact representable; hence, they are both *affine supergroups*, namely,  $x_\alpha$  is represented by  $\mathbb{k}[x]$  and  $x_\beta$  by  $\mathbb{k}[\xi]$ . We refer to both  $x_\alpha$  and  $x_\beta$  as *additive one-parameter (super)subgroups*.

**4.3. Multiplicative one-parameter supersubgroups of classical type**

For any  $\alpha \in \Delta_0 (\subseteq \mathfrak{h}^*)$ , let  $H_\alpha \in \mathfrak{h}_{\mathbb{Z}}^0$  be the corresponding coroot (see §2.3). Then consider  $\mathfrak{h}_{\mathbb{Z}}^0 := \text{Span}_{\mathbb{Z}}(\{H_\alpha \mid \alpha \in \Delta_0\})$ ; clearly, this is a  $\mathbb{Z}$ -form of  $\mathfrak{h}$ , and by definition we have that  $\mathfrak{h}_{\mathbb{Z}}^0 \subseteq \mathfrak{h}_{\mathbb{Z}[a]} := \text{Span}_{\mathbb{Z}[a]}(H_1, H_2, H_3)$ . Let  $V = \bigoplus_{\mu} V_{\mu}$  be the splitting of  $V$  into weight spaces; as  $V$  is rational, we have  $\mu(H_\alpha) \in \mathbb{Z}$  for all  $\alpha \in \Delta_0$  and  $\mu \in \mathfrak{h}^* : V_{\mu} \neq \{0\}$ . Now, for any  $A \in (\text{salg})$ ,  $\alpha \in \Delta_0$  and  $t \in U(A_0)$  (the group of invertible elements in  $A_0$ ) set  $h_\alpha(t).v := t^{\mu(H_\alpha)}v$  for all  $v \in V_{\mu}$ ,  $\mu \in \mathfrak{h}^*$ ; this defines another operator (also locally expressed by exponentials)

$$h_\alpha(t) \in \text{GL}(V_{\mathbb{k}}(A)) \quad \forall t \in U(A_0), \alpha \in \Delta_0. \tag{4.2}$$

More generally, if  $\{H_{\alpha_i}\}_{i=1,2,3}$  is any basis of  $\Delta_0$  and  $H = \sum_{i=1}^3 z_i H_{\alpha_i}$  (with  $z_1, z_2, z_3 \in \mathbb{Z}$ ), then we define

$$h_H(t) := \prod_{i=1}^3 (h_{\alpha_i}(t))^{z_i} \quad \text{for } \alpha \in \Delta_0.$$

**Definition 4.5.** Let  $H \in \mathfrak{h}_{\mathbb{Z}}^0$  as above. We define the supergroup functor  $h_H$  (also writing that  $h_{\alpha} := h_{H_{\alpha}}$  for any  $\alpha \in \Delta_0$ ) from (salg) to (groups) as  $A \mapsto h_H(A) := \{t^H := h_H(t) \mid t \in U(A_0)\}$ . As in [6, Proposition 5.8 (b)], one sees that these functors are representable, so they are affine supergroups; moreover, they are also closed subgroups of the diagonal subgroup of  $GL(V_{\mathbb{k}}(A))$ .

**4.4. Multiplicative one-parameter supersubgroups of  $a$ -type**

In order to attach a suitable ‘multiplicative one-parameter supersubgroup’ to *any* element in  $\mathfrak{h}_{\mathbb{Z}[a]} := \text{Span}_{\mathbb{Z}[a]}(H_1, H_2, H_3)$  (not only in  $\mathfrak{h}_{\mathbb{Z}}^0$ ) we need to adapt our previous construction.

Consider a Cartan element  $H_i$  in our fixed Chevalley basis: we want to define a suitable, representable supergroup functor associated with it, to be called  $h_i^{[a]}$ . Given any  $A \in$  (salg), and  $t \in U(A_0)$ , we look for an operator like  $h_i^{[a]}(t) := t^{H_i} \in GL(V_{\mathbb{k}}(A))$ . This should be given by

$$h_i^{[a]}(t) := t^{H_i} = (1 + (t - 1))^{H_i} = \sum_{n=0}^{+\infty} (t - 1)^n \binom{H_i}{n}.$$

Let  $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_{\mu}$  be the splitting of  $V$  into weight spaces; definitions imply that

$$h_i^{[a]}(t)|_{V_{\mu}} = \sum_{n=0}^{+\infty} (t - 1)^n \binom{\mu(H_i)}{n} \text{id}_{V_{\mu}} = t^{\mu(H_i)} \text{id}_{V_{\mu}}$$

on weight spaces, which makes sense (and then globally yields a well-defined operator on all of  $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_{\mu}$ ) as soon as

$$t^{\mu(H_i)} := \sum_{n=0}^{+\infty} (t - 1)^n \binom{\mu(H_i)}{n}$$

is a well-defined element of  $A$ . Now,  $V$  is rational and  $M$  is admissible, so  $\mu(H_i) \in \mathbb{Z}[a]$ . Thus, a necessary condition we may require is that

$$t^{z(a)} := \sum_{n=0}^{+\infty} (t - 1)^n \binom{z(a)}{n} \in A \quad \forall z(a) \in \mathbb{Z}[a], \tag{4.3}$$

which in the end is equivalent to having

$$t^{\pm a^k} := \sum_{n=0}^{+\infty} (t - 1)^n \binom{\pm a^k}{n} \in A \quad \forall k \in \mathbb{N}. \tag{4.4}$$

Both (4.3) and (4.4) must be read as conditions defining a suitable subset of  $A_0$ , namely, that of all elements  $t \in A_0$  for which the condition does hold. Now we fix details, to give a well-defined meaning to the expressions in (4.3) and (4.4) and to the just-sketched construction.

Consider the polynomial  $\mathbb{Z}_a$ -superalgebra  $\mathbb{Z}_a[\ell]$ , where  $\ell$  is an *even* indeterminate; this is also a (super)bialgebra, with  $\Delta(\ell) = \ell \otimes \ell$ ,  $\epsilon(\ell) = 1$ . Let  $\mathbb{Z}_a[[\ell - 1]]$  be the  $(\ell - 1)$ -adic completion of  $\mathbb{Z}_a[\ell]$ ; this also contains an inverse of  $\ell$ , namely,  $\ell^{-1} = \sum_{n=0}^{+\infty} (-1)^n (\ell - 1)^n$ , so  $\mathbb{Z}_a[\ell, \ell^{-1}] \subseteq \mathbb{Z}_a[[\ell - 1]]$ . The coproduct of  $\mathbb{Z}_a[\ell]$  extends to  $\mathbb{Z}_a[[\ell - 1]]$ , making it into a *formal* bialgebra (the coproduct taking values into the  $(\ell - 1)$ -adic completion of the algebraic tensor square of  $\mathbb{Z}_a[[\ell - 1]]$ ), which, indeed, is a *formal Hopf algebra*. In the latter, both  $\ell$  and  $\ell^{-1}$  are group-like elements, so  $\mathbb{Z}_a[\ell, \ell^{-1}]$  is a (non-formal) Hopf subalgebra of  $\mathbb{Z}_a[[\ell - 1]]$ . Consider the elements

$$\ell^{\pm a^k} := \sum_{n=0}^{+\infty} (\ell - 1)^n \binom{\pm a^k}{n} \in \mathbb{Z}_a[[\ell - 1]] \quad \forall k \in \mathbb{N}. \tag{4.5}$$

A key property of these elements is the following.

**Lemma 4.6.** *All the  $\ell^{\pm a^k}$  ( $k \in \mathbb{N}$ ) are group-like elements of  $\mathbb{Z}_a[[\ell - 1]]$ .*

**Proof.** When  $\ell$  belongs to  $\mathbb{C}$  and  $a \in \mathbb{C} \setminus \{0, -1\}$ , the power series

$$\sum_{n=0}^{+\infty} (\ell - 1)^n \binom{\pm a^k}{n}$$

represents the Taylor expansion of the analytic function  $\phi_a: \ell \mapsto \ell^{\pm a^k}$  in a neighbourhood of  $\ell_0 := 1$ . Now, the function  $\phi_a$  is multiplicative, i.e.  $\phi_a(\ell_1 \ell_2) = \phi_a(\ell_1) \phi_a(\ell_2)$ ; this identity for all complex values of  $\ell_1$  and  $\ell_2$  in a neighbourhood of 1 implies (passing through a Taylor expansion) a similar identity at the level of power series. In turn, the latter identity implies an identity among *formal power series* (i.e. still holding when the complex numbers  $\ell_1, \ell_2$  are replaced with indeterminates). This can be recast to say that the formal power series

$$\sum_{n=0}^{+\infty} (\ell - 1)^n \binom{\pm a^k}{n}$$

is a group-like element in  $\mathbb{Z}_a[[\ell - 1]]$ . As this holds for *any*  $a \in \mathbb{C}$ , it must hold for a formal parameter  $a$ , i.e. if the complex value of  $a$  is replaced by an indeterminate. Then, the formal symbol  $a$  can be replaced by any genuine value in  $\mathbb{K} \setminus \{0, -1\}$ , and the formal series in  $\mathbb{Z}_a[[\ell - 1]]$  will still be group-like. □

**4.7. The affine (super)groups  $P_a$ .** Let  $\mathcal{L}_a := \mathbb{Z}_a[\{\ell^{\pm a^k}\}_{k \in \mathbb{N}}]$  be the  $\mathbb{Z}_a$ -subalgebra of  $\mathbb{Z}_a[[\ell - 1]]$  generated by all the  $\ell^{\pm a^k}$ : as these generators are group-like,  $\mathbb{Z}_a[\{\ell^{\pm a^k}\}_{k \in \mathbb{N}}]$  is in fact a Hopf sub(super)algebra of  $\mathbb{Z}_a[[\ell - 1]]$ . In particular, it is a (totally even) Hopf algebra over  $\mathbb{Z}_a$ . If  $\mathbb{k}$  is now any (unital, commutative)  $\mathbb{Z}_a$ -algebra, as above, then

$$\mathcal{L}_{a,\mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}_a} \mathbb{Z}_a[\{\ell^{\pm a^k}\}_{k \in \mathbb{N}}] =: \mathbb{k}[\{\ell^{\pm a^k}\}_{k \in \mathbb{N}}] \quad (\subseteq \mathbb{k} \otimes_{\mathbb{Z}_a} \mathbb{Z}_a[[\ell - 1]])$$

belongs to  $(\text{salg})_{\mathbb{k}} = (\text{salg})$ , and, in addition, it is a (commutative) Hopf algebra over  $\mathbb{k}$ .

Let  $P_a := \text{Spec}(\mathcal{L}_{a,\mathbb{k}})$  be the affine scheme associated with  $\mathcal{L}_{a,\mathbb{k}}$ : as  $\mathcal{L}_{a,\mathbb{k}}$  is a Hopf algebra,  $P_a$  is indeed an affine group-scheme, which we can also see as an (affine)

supergroup. As usual, we identify  $P_a := \text{Spec}(\mathcal{L}_{a,\mathbb{k}})$  with its functors of points, namely,  $P_a = \text{Hom}_{(\text{salg})_{\mathbb{k}}}(\mathcal{L}_{a,\mathbb{k}}, \cdot)$ . Similarly, we identify  $U := \text{Spec}(\mathbb{k}[\ell, \ell^{-1}])$  with its functor of points, so for any  $A \in (\text{salg})_{\mathbb{k}}$  we have that  $U(A) = \overline{\text{Hom}}_{(\text{salg})_{\mathbb{k}}}(\mathbb{k}[\ell, \ell^{-1}], A)$  is the set of units of  $A_0$ . The natural embedding of  $\mathbb{k}[\ell, \ell^{-1}]$  into  $\mathcal{L}_{a,\mathbb{k}} = \mathbb{k}[\{\ell^{\pm a^k}\}_{k \in \mathbb{N}}]$  then yields a (super)group morphism  $\pi_a: P_a \rightarrow U$ .

Given  $A \in (\text{salg})_{\mathbb{k}}$ , any  $\varphi \in P_a(A) = \text{Hom}_{(\text{salg})_{\mathbb{k}}}(\mathcal{L}_{a,\mathbb{k}}, A)$  (an algebra morphism from  $\mathcal{L}_{a,\mathbb{k}}$  to  $A$ ) is uniquely determined by a double sequence in  $A$ , namely,  $\underline{t} := (t_k^+, t_k^-)_{k \in \mathbb{N}}$ , with  $t_k^{\pm} := \varphi(\ell^{\pm a^k})$ ; as  $t_k^{\mp} = (t_k^{\pm})^{-1}$ , just one half of this double sequence is actually enough to determine  $\varphi$ . Similarly, any  $\phi \in U(A)$  is uniquely determined by  $t := \phi(\ell)$ . Via these identifications, the morphism  $\pi_a: P_a \rightarrow U$  is described by  $\pi_a((t_k^+, t_k^-)_{k \in \mathbb{N}}) = t_0^+$ .

Note that when  $a \in \mathbb{Z}$  one has that  $P_a = U$ , by the very definitions, and  $\pi_a$  is the identity.

**Remark 4.8.** In some cases,  $\varphi \in P_a(A)$  is uniquely determined by its image  $\pi_a(\varphi)$  in  $U(A)$ ; under mild assumptions, this may happen for all elements of  $P_a(A)$ , so that  $\pi_a$  then turns out to be injective and  $P_a(A)$  identifies with a subgroup of  $U(A)$ . Indeed, given  $\varphi \in P_a(A)$  and  $t := \pi_a(\varphi) \in U(A)$ , let  $\hat{A}$  be the  $(t - 1)$ -adic completion of  $A$ , and let  $A \xrightarrow{j_t} \hat{A}$  be the natural morphism from  $A$  to  $\hat{A}$ . There then exists a unique morphism  $\varphi': \mathbb{k}[\ell - 1] \rightarrow A$  given by  $\varphi'(\ell - 1) = t - 1$ ; in turn, this uniquely yields  $\hat{\varphi}: \mathbb{k}[[\ell - 1]] \rightarrow \hat{A}$  such that  $\hat{\varphi}(\ell - 1) = t - 1$ . Finally, the restriction of  $\hat{\varphi}$  to  $\mathcal{L}_{a,\mathbb{k}}$  necessarily coincides with the composition  $j_t \circ \varphi: \mathcal{L}_{a,\mathbb{k}} \xrightarrow{j_t} A \xrightarrow{j_t} \hat{A}$ . Thus,  $t = \pi_a(\varphi)$  uniquely determines  $j_t \circ \varphi$ ; in particular, if  $\varphi, \psi \in P_a(A)$  and  $\pi_a(\varphi) = t = \pi_a(\psi)$ , then  $j_t \circ \varphi = j_t \circ \psi$ . Thus, if, in addition,  $A \xrightarrow{j_t} \hat{A}$  is injective, then  $\varphi = \psi$ , i.e.  $t := \pi_a(\varphi)$  is enough to determine  $\varphi$ .

**Notation 4.9.** In the following, we identify any  $\varphi \in P_a(A)$  by its corresponding double sequence  $\underline{t}$  (as above) and any  $\phi \in U(A)$  by  $t := \phi(\ell)$ ; thus, we write  $\underline{t}$  for  $\varphi$  and  $t$  for  $\phi$ .

Given  $A \in (\text{salg})_{\mathbb{k}}$  and  $\underline{t} \in P_a(A)$ , for any  $z(a) = \sum_k z_k a^k \in \mathbb{Z}[a]$  we use the notation

$$\underline{t}^{z(a)} := \prod_k (t_k^+)^{z_k},$$

which is just

$$\varphi(\ell^{z(a)}) = \varphi\left(\prod_k (\ell^{a^k})^{z_k}\right) \quad \text{if } \underline{t} = \varphi, \quad \text{with } \ell^{z(a)} := \prod_k (\ell^{a^k})^{z_k}.$$

**Remarks 4.10.** (a) By (4.5) it is easy to see, using notation of §2, that for any  $A \in (\text{salg})$  one has that  $P_a(A) \supseteq (1 + \mathfrak{N}(A_0)) \supseteq (1 + A_1^2)$ .

(b) It is clear that  $P_a(\mathbb{C}) = \mathbb{C}^* = U(\mathbb{C})$ , with  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ . Now, assume that  $\mathbb{K} = \mathbb{C}$  and  $A = \mathbb{C}[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$ , where the  $x_i$  and the  $\xi_j$ , respectively, are even and odd indeterminates. Letting  $(\xi_1, \dots, \xi_n)$  be the ideal generated by the  $\xi_i$ , one has that  $P_a(\mathbb{C}[x_1, \dots, x_m, \xi_1, \dots, \xi_n]) = \mathbb{C}^* + (\xi_1, \dots, \xi_n)^2$ ; in particular,

$$P_a(\mathbb{C}[x_1, \dots, x_m, \xi_1, \dots, \xi_n]) = U(\mathbb{C}[x_1, \dots, x_m, \xi_1, \dots, \xi_n]_0).$$

By the same argument, one also has that  $P_a(A) = U(A)$  (that is,  $\pi_a$  is the identity) for all those  $A \in (\text{salg})_{\mathbb{C}}$  such that  $U(A) = \mathbb{C}^* + \mathfrak{N}(A_0)$ , where  $\mathfrak{N}(A_0)$  is the nilradical of  $A_0$ .



Now let  $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$  and  $M$  be as above. For any  $H \in \mathfrak{h}_{\mathbb{Z}[a]} := \text{Span}_{\mathbb{Z}[a]}(H_1, H_2, H_3)$ ,  $A \in (\text{salg})$  and  $\underline{t} \in P_a(A)$ , the formula

$$h_H^{[a]}(\underline{t}) := \underline{t}^H = \sum_{n=0}^{+\infty} (t-1)^n \binom{H}{n}, \quad \text{with } t := t_0^+,$$

yields a well-defined element of  $\text{GL}(V_{\mathbb{k}}(A))$ , whose action is  $h_H^{[a]}(\underline{t}).v := \underline{t}^{\mu(H)}v$  for all  $v \in V_\mu$ ,  $\mu \in \mathfrak{h}^* : V_\mu \neq \{0\}$  (this makes sense, since  $\mu(H) \in \mathbb{Z}[a]$ ). In particular, we write that

$$h_i^{[a]}(\underline{t}) := h_{H_i}^{[a]}(\underline{t}) \quad \text{for } i = 1, 2, 3;$$

thus, if

$$H = \sum_{i=1}^3 z_i(a)H_i \in \mathfrak{h}_{\mathbb{Z}[a]} := \text{Span}_{\mathbb{Z}[a]}(H_1, H_2, H_3), \quad \text{with } z_i(a) \in \mathbb{Z}[a] \ \forall i,$$

one has that  $h_H^{[a]}(\underline{t}) = h_1^{[a]}(\underline{t})h_2^{[a]}(\underline{t})h_3^{[a]}(\underline{t})$  for all  $\underline{t} \in P_a(A)$ .

**Definition 4.11.** Let  $H \in \mathfrak{h}_{\mathbb{Z}[a]}$  as above. Consider the morphism  $\hat{h}_H^{[a]} : P_a \rightarrow \text{GL}(V_{\mathbb{k}})$  given on objects by

$$P_a(A) \xrightarrow{\hat{h}_H^{[a]}(A)} \text{GL}(V_{\mathbb{k}}(A)), \quad \underline{t} \mapsto h_H^{[a]}(\underline{t}).$$

We define the supergroup functor  $h_H^{[a]}$  from  $(\text{salg})$  to  $(\text{groups})$  as being the image of  $\hat{h}_H^{[a]}$ ; in particular, it is given on objects by

$$A \mapsto h_H^{[a]}(A) := \{h_H^{[a]}(\underline{t}) := \underline{t}^H \mid \underline{t} \in P_a(A)\},$$

while its definition on arrows should (hopefully) be clear.

**Remark 4.12 (representability of the functors  $h_H^{[a]}$ ).** By construction, the kernel  $\mathcal{K}_a := \text{Ker}(\hat{h}_H^{[a]})$  of  $\hat{h}_H^{[a]}$  (notation as above) is the closed subgroup of  $P_a$  defined by the ideal  $I(\mathcal{K}_a)$  given as follows. If  $\mu \in \mathfrak{h}^*$  and  $\mu(H) = \sum_k z_{\mu,k} a^k \in \mathbb{Z}[a]$  for some  $z_k \in \mathbb{Z}$ , then

$$I(\mathcal{K}_a) = \left( \left\{ \prod_k (\ell^{a^k})^{z_{\mu,k}} - 1 \mid \mu \in \mathfrak{h}^* : V_\mu \neq \{0\} \right\} \right).$$

As the functor  $P_a$  is represented by  $\mathcal{L}_{a,\mathbb{k}}$ , it follows that its quotient  $h_H^{[a]} \cong P_a/\mathcal{K}_a$  is represented by the Hopf algebra

$$\mathcal{L}_{a,\mathbb{k}}^{\text{co-}I(\mathcal{K}_a)} := \{f \in \mathcal{L}_{a,\mathbb{k}} \mid (\Delta(f) - f \otimes 1) \in \mathcal{L}_{a,\mathbb{k}} \otimes I(\mathcal{K}_a)\}$$

of right  $I(\mathcal{K}_a)$ -coinvariants of  $\mathcal{L}_{a,\mathbb{k}}$ . In particular, the supergroup functor  $h_H^{[a]}$  is representable; hence it is itself an affine supergroup.

In the following, we call the  $h_H^{[a]}$  multiplicative one-parameter (super)subgroups of  $a$ -type.

**4.5. Construction of Chevalley supergroups**

In order to define our Chevalley supergroups, we first need the definition of a suitable algebraic group  $\mathbf{G}_0$  associated with  $\mathfrak{g} = D(2, 1; a)$  and  $V$ .

First, for each  $A \in (\text{salg})$  consider the subgroup  $G'_0(A) := \langle h_H(A), x_\alpha(A) \mid H \in \mathfrak{h}_{\mathbb{Z}}^0, \alpha \in \Delta_0 \rangle$  generated in  $\text{GL}(V_{\mathbb{k}}(A))$  by the one-parameter supersubgroups  $h_H(A)$  and  $x_\alpha(A)$ ,  $H \in \mathfrak{h}_{\mathbb{Z}}^0$ ,  $\alpha \in \Delta_0$ . Overall, this yields a group functor  $G'_0$  defined on  $(\text{salg})$ , which clearly factors through  $(\text{alg}) = (\text{alg})_{\mathbb{k}}$ , the category of unital commutative  $\mathbb{k}$ -algebras. This  $G'_0$  is a presheaf (see [6, Appendix, Definition A.5]); hence, we can define the functor  $\mathbf{G}'_0$  as the sheafification of  $G'_0$ : on local algebras (in  $(\text{alg})$ ) the functor  $\mathbf{G}'_0$  coincides with the functor of points of the classical Chevalley group-scheme associated with the semisimple Lie algebra  $\mathfrak{g}_0$  (isomorphic to  $\mathfrak{sl}(2)^{\oplus 3}$ ) and the  $\mathfrak{g}_0$ -module  $V$ . In particular,  $\mathbf{G}'_0$  is representable. Inside  $G'_0$ , the  $h_H$  generate the subgroup  $T'(A_0) := \langle h_H(A) \mid H \in \mathfrak{h}_{\mathbb{Z}}^0 \rangle$ , yielding another supergroup functor, which also factors through  $(\text{alg}) = (\text{alg})_{\mathbb{k}}$ ; its sheafification  $\mathbf{T}'$  coincides with  $T'$  itself, and is a maximal torus in  $\mathbf{G}'_0$ .

Second, consider the subgroup

$$T(A) := \langle \{h_H^{[a]}(A) \mid H \in \mathfrak{h}_{\mathbb{Z}[a]}^0\} \cup T'(A) \rangle \text{ of } \text{GL}(V_{\mathbb{k}}(A)),$$

which for various  $A$  in  $(\text{salg})$  yields another (sub)group functor  $T: (\text{salg}) \rightarrow (\text{groups})$  (also factoring through  $(\text{alg})$ ): as above, we can also consider the sheafification  $\mathbf{T}$  of  $T$ .

For later use, note that  $\mathbf{T}' \leq \mathbf{G}'_0 \leq \text{GL}(V_{\mathbb{k}})$  and  $\mathbf{T}' \leq \mathbf{T} \leq \text{GL}(V_{\mathbb{k}})$ , with  $\mathbf{G}'_0 \cap \mathbf{T} = \mathbf{T}'$ .

**4.13. Description of  $\mathbf{T}^{[a]}$ .** We devote a little more of the paper to describing  $T$  and  $\mathbf{T}$ ; indeed, we show that  $T$  is representable, so that  $\mathbf{T} = T$ .

We write  $T^{[a]} := \langle h_1^{[a]}, h_2^{[a]}, h_3^{[a]} \rangle$  for the (supersub)group functor generated by  $h_1^{[a]}$ ,  $h_2^{[a]}$  and  $h_3^{[a]}$ , so  $T = \langle T^{[a]} \cup T' \rangle$ . It is clear by construction that  $T^{[a]}$  is a direct product

$$T^{[a]} \cong h_1^{[a]} \times h_2^{[a]} \times h_3^{[a]},$$

just like  $T' \cong U \times U \times U$ , and both of these groups commute with each other inside  $\text{GL}(V_{\mathbb{k}})$ ; also, it is clear that the morphism  $\pi_a: P_a \rightarrow U$  uniquely induces a similar morphism  $\pi_a: \mathcal{H}^{[a]} \rightarrow T'$ . It follows that  $T = \langle T^{[a]} \cup T' \rangle$  can be seen as the fibred product  $T \times_{T'} T'$  of  $T^{[a]}$  and  $T'$  with respect to the pair of morphisms

$$T^{[a]} \xrightarrow{\pi_a} T' \xleftarrow{\text{id}_{T'}} T'.$$

We now realize this fibred product in concrete terms.

Take the direct product structure on  $T^{[a]} \times T'$ . Recall that  $T$  is representable; hence, it coincides with its sheafification  $\mathbf{T}$ . Similarly, we see by construction that  $T^{[a]}$  is also representable, so for its sheafification  $\mathbf{T}^{[a]}$  we have that  $\mathbf{T}^{[a]} = T^{[a]}$ . It follows that  $T^{[a]} \times T' = \mathbf{T}^{[a]} \times \mathbf{T}'$  is also representable. The fibred product  $T^{[a]} \times_{T'} T'$  is a quotient of the above direct product. Indeed, let

$$K(A_0) := \{(x, y) \mid x \in T^{[a]}(A_0), y \in T'(A_0), \pi_a(x) = y^{-1}\}$$

for any  $A \in (\text{salg})$ , so  $A \mapsto K(A_0)$  defines, on  $(\text{salg})$ , through  $(\text{alg})$ , a (normal) subgroup functor  $K$  of  $T^{[a]} \times T'$ . We then have a (functor) isomorphism  $T \cong (T^{[a]} \times T')/K$ . In addition, let  $\mathbf{K}$  denote the sheafification of the functor  $K$ .

We see now that  $\mathbf{K}$  as a subgroup of  $T^{[a]} \times T'$  is *closed*. To begin with, recall that, by construction, the group multiplication provides the isomorphisms

$$h_1^{[a]} \times h_2^{[a]} \times h_3^{[a]} \cong T^{[a]} \quad \text{and} \quad h_{H_{2\varepsilon_1}} \times h_{H_{2\varepsilon_2}} \times h_{H_{2\varepsilon_3}} \cong T'.$$

Thus, we can write the  $A$ -points (for  $A \in (\text{salg})$ ) of  $T^{[a]} \times T'$  as pairs of triples of the form  $((\underline{t}_1, \underline{t}_2, \underline{t}_3), (\tau_1, \tau_2, \tau_3))$ , with  $\underline{t}_1, \underline{t}_2, \underline{t}_3 \in P_a(A)$  and  $\tau_1, \tau_2, \tau_3 \in U(A)$  (with a slight abuse of notation: for all  $i$  and  $j$  we are identifying  $h_i^{[a]}(\underline{t}_i)$  with  $\underline{t}_i$  and  $h_{H_{2\varepsilon_j}}(\tau_j)$  with  $\tau_j$ ).

Recall the identities  $H_{2\varepsilon_1} = H_2$ ,  $H_{2\varepsilon_3} = H_3$ ,  $H_{2\varepsilon_2} = 2H_1 - (1 + a)H_2 - aH_3$ , which hold inside  $\mathfrak{h}$ ; in turn, these yield, for every  $\underline{t} \in P_a(A)$  with  $\vartheta := \pi_a(\underline{t}) \in U(A)$ , the formal identities

$$\begin{aligned} h_{H_{2\varepsilon_1}}(\vartheta) &= \vartheta^{H_{2\varepsilon_1}} = \underline{t}^{H_2} = h_{H_2}^{[a]}(\underline{t}) = h_2^{[a]}(\underline{t}), \\ h_{H_{2\varepsilon_3}}(\vartheta) &= \vartheta^{H_{2\varepsilon_3}} = \underline{t}^{H_3} = h_{H_3}^{[a]}(\underline{t}) = h_3^{[a]}(\underline{t}), \\ h_{H_{2\varepsilon_2}}(\vartheta) &= \vartheta^{H_{2\varepsilon_2}} = \underline{t}^{2H_1 - (1+a)H_2 - aH_3} = h_{2H_1}^{[a]}(\underline{t}) \cdot h_{-(1+a)H_2}^{[a]}(\underline{t}) \cdot h_{-aH_3}^{[a]}(\underline{t}), \end{aligned}$$

which in shorter notation read

$$(\vartheta, 1, 1) = \pi_a(1, \underline{t}, 1), \quad (1, 1, \vartheta) = \pi_a(1, 1, \underline{t}), \quad (1, \vartheta, 1) = \pi_a(\underline{t}^2, \underline{t}^{-(1+a)}, \underline{t}^{-a}). \quad (4.6)$$

Now, consider the condition  $\pi_a(x) = y^{-1}$  for a pair  $(x, y)$  to belong to  $K(A)$ , rewritten as  $y\pi_a(x) = 1$ : we want to read it for any pair  $(x, y) := ((\underline{t}_1, \underline{t}_2, \underline{t}_3), (\tau_1, \tau_2, \tau_3))$  as above. By the previous analysis, it corresponds to the three equations that, in turn, correspond to the three conditions in (4.6). Recall that the group  $T^{[a]} \times T'$  is represented by the algebra  $\mathcal{O}(T^{[a]} \times T') \cong \mathcal{O}(T^{[a]}) \otimes \mathcal{O}(T')$ : the left-hand tensor factor is a quotient of

$$\mathcal{O}(P_a^{\times 3}) \cong \mathcal{O}(P_a)^{\otimes 3} \cong \bigotimes_{i=1}^3 \mathbb{k}[\{\ell_i^{\pm a^k}\}_{k \in \mathbb{N}}],$$

while the right-hand one is

$$\mathcal{O}(T') \cong \mathcal{O}(U^{\times 3}) \cong \bigotimes_{i=1}^3 \mathbb{k}[z_i^{\pm 1}]$$

(see §4.7). The first two conditions in (4.6) then correspond to the equations  $z_1 \ell_2 = 1$ ,  $z_3 \ell_3 = 1$  in  $\mathcal{O}(T^{[a]} \times T')$ .

The third condition can be handled as follows. For any  $\underline{t} \in P_a(A)$ , let  $\hat{\underline{t}} := (\underline{t}_1, \underline{t}_2, \underline{t}_3) \in P_a(A)^{\times 3}$ , with  $\underline{t}_1 =: p_1(\hat{\underline{t}}) = \underline{t}$ , where  $p_1$  is the projection of  $P_a(A)^{\times 3}$  onto its leftmost factor, and consider the (functorial) ‘diagonalization map’

$$P_a \xrightarrow{\Delta_3} P_a(A)^{\times 3}, \quad \underline{t} \mapsto (\underline{t}, \underline{t}, \underline{t}).$$

Then,  $\Delta_3 \circ p_1$  is a (group functor) morphism, and

$$\pi_a(\underline{t}^2, \underline{t}^{-(1+a)}, \underline{t}^{-a}) = (\hat{\ell}_1^2, \hat{\ell}_2^{-(1+a)}, \hat{\ell}_3^{-a})(\hat{\underline{t}}),$$

where

$$\hat{\ell}_i := \ell_i \circ \Delta_3 \circ p_1 \in \mathcal{O}(P_a^{\times 3}) \cong \mathbb{k}[\{\ell_1^{\pm a^k}, \ell_2^{\pm a^k}, \ell_3^{\pm a^k}\}_{k \in \mathbb{N}}].$$

The outcome is that the third condition in (4.6) corresponds to  $z_2 \hat{\ell}_1^2 \hat{\ell}_2^{-1} \hat{\ell}_2^{-a} \hat{\ell}_3^{-a} = 1$ . To sum up, we get that  $K$  is the closed subgroup of  $T^{[a]} \times T'$  defined by the ideal  $I(K) = (z_1 \ell_2 - 1, z_3 \ell_3 - 1, z_2 \hat{\ell}_1^2 \hat{\ell}_2^{-1} \hat{\ell}_2^{-a} \hat{\ell}_3^{-a} - 1)$ .

Finally, as  $T^{[a]} \times T'$  is representable (hence, it is an affine group scheme) and its subgroup  $K$  is closed, we argue that  $\mathbf{K} = K$ , that the latter is also representable and hence that the quotient  $T \cong (T^{[a]} \times T')/\mathbf{K}$  is representable too; in particular,  $\mathbf{T} = T$  as well.

We can now introduce the algebraic group  $\mathbf{G}_0$  we were looking for.

**Definition 4.14.** For every  $A \in (\text{salg})$ , we let  $G_0(A)$  be the subgroup  $G_0(A) := \langle G'_0(A) \cup T(A) \rangle$  of  $\text{GL}(V_{\mathbb{k}}(A))$ . We denote by  $G_0: (\text{salg}) \rightarrow (\text{groups})$  the supergroup functor that is the full subfunctor of  $\text{GL}(V_{\mathbb{k}}(A))$  given on objects by  $A \mapsto G_0(A)$ ; we denote by  $\mathbf{G}_0: (\text{salg}) \rightarrow (\text{groups})$  the sheafification functor of  $G_0$ . Note that both  $G_0$  and  $\mathbf{G}_0$  factor through  $(\text{alg})$ .

**Proposition 4.15.** *The supergroup functor  $\mathbf{G}_0$  is representable; hence (as it factors through  $(\text{alg})$ ), it is an affine group.*

**Proof.** We argue much as we did above (see § 4.13) to describe  $T$ , so we only sketch the proof here.

The groups  $G'_0$  and  $T$  are subgroups of  $\text{GL}(V_{\mathbb{k}})$ , and their mutual intersection is  $G'_0 \cap T = T'$ ; thus,  $G_0$  is a fibred product of  $G'_0$  and  $T$  over  $T'$ , which we can describe in down-to-earth terms.

Fix  $A \in (\text{salg})$ . Inside  $\text{GL}(V_{\mathbb{k}})(A_0)$ , the subgroup  $T(A_0)$  acts on  $G'_0(A_0)$  by adjoint action, so  $G_0(A_0)$ , generated by  $T(A_0)$  and  $G'_0(A_0)$ , is a quotient of the semi-direct product  $T(A_0) \ltimes G'_0(A_0)$ . Indeed, let  $J(A_0) := \{(j^{-1}, j) \mid j \in T(A_0) \cap G'_0(A_0) = T'(A_0)\}$ , so  $A \mapsto J(A_0)$  defines, on  $(\text{salg})$ , through  $(\text{alg})$ , a normal subgroup functor of  $T \ltimes G'_0$ . We then have a (functor) isomorphism  $G_0 \cong (T \ltimes G'_0)/J$ . Thus,  $G_0 \cong (T \ltimes G'_0)/J$  as group functors; hence (forgetting the group structure), we also have  $G_0 \cong (T \ltimes G'_0)/J$  as set-valued functors. Taking sheafifications, as both  $\mathbf{T}$  and  $\mathbf{G}'_0$  are representable, we infer that  $\mathbf{T} \ltimes \mathbf{G}'_0$  is also representable. In addition, if  $\mathbf{J}$  is the sheafification of the functor  $J$ , we see that  $\mathbf{J}$  as a subgroup of  $\mathbf{T} \ltimes \mathbf{G}'_0$  is closed.

To see all this in detail, we revisit our construction. We started with a representation of  $A \otimes_{\mathbb{k}} U_{\mathbb{k}}(\mathfrak{g})$  on  $V_{\mathbb{k}}(A) := A \otimes_{\mathbb{k}} V_{\mathbb{k}}$  (see § 4.1). By construction,  $\mathbf{G}'_0$  is just the algebraic group associated with  $\mathfrak{g}_0$  and  $V$  (as a  $\mathfrak{g}_0$ -module) by the classical Chevalley construction. Indeed,  $\mathfrak{g}_0 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ , where the three summands are given by  $\mathfrak{sl}_2$ -triples associated with positive even roots  $2\varepsilon_i$  (see § 2.3), and  $\mathbf{G}'_0 \cong \mathbf{H}_1 \times \mathbf{H}_2 \times \mathbf{H}_3$ , where  $\mathbf{H}_i \in \{\mathbf{SL}_2, \mathbb{P}\mathbf{SL}_2\}$  for each  $i$ . Each  $\mathbf{H}_i$  is represented by  $\mathcal{O}(\mathbf{SL}_2) = \mathbb{k}[a, b, c, d]/(ad - bc - 1)$  if  $\mathbf{H}_i \cong \mathbf{SL}_2$ ,

and by the unital subalgebra of  $\mathcal{O}(\mathbf{SL}_2)$  generated by all products of any two elements in the set  $\{a, b, c, d\}$  if  $\mathbf{H}_i \cong \mathbb{P}\mathbf{SL}_2$ . The torus  $T = \mathbf{T} \cong U^{\times 3}$  is then embedded in  $\times_{i=1}^3 \mathbf{H}_i$  via

$$(\tau_1, \tau_2, \tau_3) \mapsto \left( \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1^{-1} \end{pmatrix}, \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2^{-1} \end{pmatrix}, \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3^{-1} \end{pmatrix} \right);$$

hence, it is the closed subgroup of  $\mathbf{G}'_0$  defined by the ideal  $I(T) = (b_1, b_2, b_3, c_1, c_2, c_3)$ .

Now, the description of the subgroup  $J$  of  $\mathbf{T} \times \mathbf{G}'$  is along the same lines as for the description of the subgroup  $\mathbf{T}'$  of  $T^{[a]} \times T'$  in §4.7. Then, by the same arguments one eventually finds that  $J = \mathbf{J}$  is closed in  $\mathbf{T} \times \mathbf{G}'_0$ , as claimed.

A similar analysis also works when some  $\mathbf{H}_i$  (possibly all of them) are isomorphic to  $\mathbb{P}\mathbf{SL}_2$ .

Finally, as  $\mathbf{T} \times \mathbf{G}'_0$  is representable (so it is an affine group scheme), its quotient by the closed normal subgroup  $\mathbf{J}$  is also representable; hence, the same holds for the isomorphic functor  $\mathbf{G}_0$ . □

We can now, eventually, define our Chevalley supergroups.

**Definition 4.16.** Let  $\mathfrak{g}$  and  $V$  be as above. We call a *Chevalley supergroup functor*, associated with  $\mathfrak{g}$  and  $V$ , the functor  $G: (\text{salg}) \rightarrow (\text{grps})$  given by the following.

- If  $A \in \text{Ob}((\text{salg}))$ , we let  $G(A)$  be the subgroup of  $\text{GL}(V_{\mathbb{k}}(A))$  generated by  $G_0(A)$  and the one-parameter subgroups  $x_{\beta}(A)$ , with  $\beta \in \Delta_1$ , that is,

$$G(A) := \langle G_0(A), x_{\beta}(A) \mid \beta \in \Delta_1 \rangle = \langle T(A), x_{\delta}(A) \mid \delta \in \Delta \rangle,$$

where the second identity follows from the previous description of  $G_0$ .

- If  $\phi \in \text{Hom}_{(\text{salg})}(A, B)$ , then  $\text{End}_{\mathbb{k}}(\phi): \text{End}_{\mathbb{k}}(V_{\mathbb{k}}(A)) \rightarrow \text{End}_{\mathbb{k}}(V_{\mathbb{k}}(B))$  (given on matrix entries by  $\phi$  itself) respects the sum and the associative product of matrices; then,  $\text{End}_{\mathbb{k}}(\phi)$  clearly restricts to a group morphism  $\text{GL}(V_{\mathbb{k}}(A)) \rightarrow \text{GL}(V_{\mathbb{k}}(B))$ . The latter maps the generators of  $G(A)$  to those of  $G(B)$ , and hence restricts to a group morphism  $G(\phi): G(A) \rightarrow G(B)$ .

We call a *Chevalley supergroup*, associated with  $\mathfrak{g} = D(2, 1; a)$  and  $V$ , the sheafification  $\mathbf{G}$  of  $G$  (see [6, Appendix]). Thus,  $\mathbf{G}: (\text{salg}) \rightarrow (\text{grps})$  is a sheaf functor such that  $\mathbf{G}(A) = G(A)$  if  $A \in (\text{salg})$  is *local*. To stress the dependence on  $V$ , we also write  $G_V$  for  $G$  and  $\mathbf{G}_V$  for  $\mathbf{G}$ .

**Remarks 4.17.** (a) For  $a \in \mathbb{Z}$ , a construction of Chevalley supergroups of type  $D(2, 1; a)$  was given in [6]; it coincides with the present one, because  $P_a(A) = U(A_0)$  if  $a \in \mathbb{Z}$ , for  $A \in (\text{salg})$ .

(b) An alternative definition of Chevalley supergroups can be given letting the subgroup (functor)  $\text{Up}: A \mapsto \text{Up}(A_0)$  play the role of  $P_a$ , where  $\text{Up}(A_0) := (1 + \mathfrak{N}(A_0))$  is the subgroup of  $U(A_0)$  of all *unipotent* elements of  $A_0$  (and  $\mathfrak{N}(A_0)$  is the nilradical of  $A_0$ ). All our arguments and results from now on are also valid. Nevertheless, using the subgroup functor  $\text{Up}$  one does not recover the construction of [6] for  $a \in \mathbb{Z}$ , which instead is the case with  $P_a$  (see (a) above).

4.6. Chevalley supergroups as affine supergroups

Our definition of the Chevalley supergroup  $\mathbf{G}$  does not imply (at first sight) that  $\mathbf{G}$  is representable, so that it is indeed an affine supergroup scheme. In this section we prove this fact.

**Definition 4.18.** For any  $A \in (\text{salg})$ , we define the subsets of  $G(A)$

$$\begin{aligned}
 G_1(A) &:= \left\{ \prod_{i=1}^n x_{\gamma_i}(\vartheta_i) \mid n \in \mathbb{N}, \gamma_i \in \Delta_1, \vartheta_i \in A_1 \right\}, \\
 G_0^\pm(A) &:= \left\{ \prod_{i=1}^n x_{\alpha_i}(t_i) \mid n \in \mathbb{N}, \alpha_i \in \Delta_0^\pm, t_i \in A_0 \right\}, \\
 G_1^\pm(A) &:= \left\{ \prod_{i=1}^n x_{\gamma_i}(\vartheta_i) \mid n \in \mathbb{N}, \gamma_i \in \Delta_1^\pm, \vartheta_i \in A_1 \right\}, \\
 G^\pm(A) &:= \left\{ \prod_{i=1}^n x_{\beta_i}(t_i) \mid n \in \mathbb{N}, \beta_i \in \Delta^\pm, t_i \in A_0 \cup A_1 \right\} \\
 &= \langle G_0^\pm(A), G_1^\pm(A) \rangle.
 \end{aligned}$$

Moreover, fixing any total order  $\preceq$  on  $\Delta_1^\pm$ , and letting  $N_\pm = |\Delta_1^\pm|$ , we set

$$G_1^{\pm, <}(A) := \left\{ \prod_{i=1}^{N_\pm} x_{\gamma_i}(\vartheta_i) \mid \gamma_1 \prec \dots \prec \gamma_{N_\pm} \in \Delta_1^\pm, \vartheta_1, \dots, \vartheta_{N_\pm} \in A_1 \right\},$$

and for any total order  $\preceq$  on  $\Delta_1$ , and letting  $N := |\Delta| = N_+ + N_-$ , we set

$$G_1^{<}(A) := \left\{ \prod_{i=1}^N x_{\gamma_i}(\vartheta_i) \mid \gamma_1 \prec \dots \prec \gamma_N \in \Delta_1, \vartheta_1, \dots, \vartheta_N \in A_1 \right\}.$$

Incidentally, note that  $N_\pm = 4$  and  $N = 8$  for  $\mathfrak{g} = D(2, 1; a)$ .

Similar notation denotes the sheafifications  $\mathbf{G}_1, \mathbf{G}^\pm, \mathbf{G}_0^\pm, \mathbf{G}_1^\pm$ , etc.

We begin by studying commutation relations among generators of Chevalley groups. As a matter of notation, when  $\Gamma$  is any group and  $g, h \in \Gamma$  we denote by  $(g, h) := ghg^{-1}h^{-1}$  their commutator.

**Lemma 4.19.** (a) Let  $\alpha \in \Delta_0, \gamma \in \Delta_1, A \in (\text{salg})$  and  $t \in A_0, \vartheta \in A_1$ . There then exist  $c_s \in \mathbb{Z}$  such that

$$(x_\gamma(\vartheta), x_\alpha(t)) = \prod_{s>0} x_{\gamma+s\alpha}(c_s t^s \vartheta) \in G_1(A)$$

(the product being finite). Indeed, with  $\varepsilon_k = \pm 1$  and  $r$  as in § 3.4 (2),

$$(1 + \vartheta X_\gamma, x_\alpha(t)) = \prod_{s>0} \left( 1 + \prod_{k=1}^s \varepsilon_k \cdot \binom{s+r}{r} \cdot t^s \vartheta X_{\gamma+s\alpha} \right),$$

where the factors in the product are taken in any order (as they do commute).

(b) Let  $\gamma, \delta \in \Delta_1$ ,  $A \in (\text{salg})$ ,  $\vartheta, \eta \in A_1$ . Then (using the notation of Definition 3.1),

$$(x_\gamma(\vartheta), x_\delta(\eta)) = x_{\gamma+\delta}(-c_{\gamma,\delta}\vartheta\eta) = (1 - c_{\gamma,\delta}\vartheta\eta X_{\gamma+\delta}) \in G_0(A)$$

if  $\delta \neq -\gamma$ ; otherwise, for  $\delta = -\gamma$ , we have that

$$(x_\gamma(\vartheta), x_{-\gamma}(\eta)) = (1 - \vartheta\eta H_\gamma) = h_{H_\gamma}^{[a]}(1 - \vartheta\eta) \in G_0(A).$$

(c) Let  $\beta \in \Delta$ ,  $A \in (\text{salg})$ ,  $\mathbf{u} \in A_0 \cup A_1$ . Then,

$$\begin{aligned} h_{H_0}(t)x_\beta(\mathbf{u})h_{H_0}(t)^{-1} &= x_\beta(t^{\beta(H_0)}\mathbf{u}) \in G_{p(\beta)}(A) \quad \forall H_0 \in \mathfrak{h}_{\mathbb{Z}}^0, t \in U(A_0), \\ h_H^{[a]}(\underline{t})x_\beta(\mathbf{u})h_H^{[a]}(\underline{t})^{-1} &= x_\beta(\underline{t}^{\beta(H)}\mathbf{u}) \in G_{p(\beta)}(A) \quad \forall H \in \mathfrak{h}_{\mathbb{Z}[a]}, \underline{t} \in P_a(A_0), \end{aligned}$$

where  $p(\beta) := s$ , by definition, if and only if  $\beta \in \Delta_s$ .

**Proof.** As for [6, Lemma 5.13], these results follow directly from the classical ones and simple calculations, using the relations in § 3.4 and the identity  $(\vartheta\eta)^2 = -\vartheta^2\eta^2 = 0$ . In addition, for (b) in the present case we have also to take into account that  $(1 - \vartheta\eta) \in (1 + \mathfrak{N}(A_0)) \subseteq P_a(A)$  for all  $\vartheta, \eta \in A_1$ , so  $h_{H_\gamma}^{[a]}(1 - \vartheta\eta)$  is well defined and equal to  $(1 - \vartheta\eta H_\gamma)$ .  $\square$

Now, with our definition of Chevalley groups at hand and the commutation rules among their generators available (as given in Lemma 4.19), one can reproduce what was done in [6, § 5.3]. Just some minimal changes are in order, due to a couple of facts: first, the presence among the generators of the multiplicative one-parameter supersubgroups of  $a$ -type, which are handled like the classical ones using Lemma 4.19; second, several shortcuts and simplifications are possible, as the structure of  $\mathfrak{g} = D(2, 1; a)$  is much simpler than that of the general classical Lie superalgebras. Thus, in the following we simply list the results we get (essentially, the main steps in the line of arguing of [6]), as the proofs can be easily recovered from [6, § 5]. The first result is the following.

**Theorem 4.20.** For any  $A \in (\text{salg})$ , there exist the set-theoretic factorizations

- (a)  $G(A) = G_0(A)G_1(A), \quad G(A) = G_1(A)G_0(A),$
- (b)  $G^\pm(A) = G_0^\pm(A)G_1^\pm(A), \quad G^\pm(A) = G_1^\pm(A)G_0^\pm(A),$
- (c)  $G(A) = G_0(A)G_1^<(A), \quad G(A) = G_1^<(A)G_0(A).$

Item (a) above is a group-theoretical counterpart for  $G$  of the splitting  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  (a superanalogue of the Cartan decomposition for reductive groups), while (b) is a similar result for  $G^+$  and  $G^-$ . Item (c) improves (a): as in [6, Theorem 5.17], it follows from Theorem 4.20, simply reordering the factors in  $G_1(A)$  by means of the commutation rules in Lemma 4.19.

With a more careful analysis, one can improve the previous results to get the following key one.

**Theorem 4.21.** *The group product yields the isomorphisms of set valued functors*

$$G_0 \times G_1^{-, <} \times G_1^{+, <} \xrightarrow{\cong} G, \quad \mathbf{G}_0 \times \mathbf{G}_1^{-, <} \times \mathbf{G}_1^{+, <} \xrightarrow{\cong} \mathbf{G},$$

as well as those obtained by permuting the  $(-)$ -factor and the  $(+)$ -factor and/or moving the  $(0)$ -factor to the right. All of these induce similar functor isomorphisms with the left-hand side obtained by permuting the factors above, such as  $G_1^{+, <} \times G_0 \times G_1^{-, <} \xrightarrow{\cong} G$ ,  $\mathbf{G}_1^{-, <} \times \mathbf{G}_0 \times \mathbf{G}_1^{+, <} \xrightarrow{\cong} G$ , etc.

The same technique used to prove Theorem 4.21 also yields the following.

**Proposition 4.22.** *The functors  $G_1^{\pm, <} : (\text{salg}) \rightarrow (\text{sets})$  are representable: they are the functor of points of the superscheme  $\mathbb{A}_k^{0|N_{\pm}}$ , with  $N_{\pm} = |\Delta_1^{\pm}|$ . In particular, we have  $G_1^{\pm, <} = \mathbf{G}_1^{\pm, <}$ .*

Indeed, this holds since natural transformations  $\Psi^{\pm} : \mathbb{A}_k^{0|N_{\pm}} \rightarrow G_1^{\pm, <}$  exist, by definition, given on objects by

$$\begin{aligned} \Psi^{\pm}(A) : \mathbb{A}_k^{0|N_{\pm}}(A) &\rightarrow G_1^{\pm, <}(A) \\ (\vartheta_1, \dots, \vartheta_{N_{\pm}}) &\mapsto \prod_{i=1}^{N_{\pm}} x_{\gamma_i}(\vartheta_i), \end{aligned}$$

and obvious on morphisms. One then proves that these  $\Psi^{\pm}$  are isomorphisms of (set-valued) functors.

Finally, we can prove that the Chevalley supergroups are affine.

**Theorem 4.23.** *Every Chevalley supergroup  $\mathbf{G}$  is an affine supergroup.*

Indeed, this is a direct consequence of the last two results, as they imply that the group functor  $\mathbf{G}$  is isomorphic (as a set-valued functor) to the direct product of three representable group functors; hence, it is also representable, which entails that it is an affine supergroup.

Another immediate consequence is the following, which improves, for Chevalley supergroups, a more general result proved by Masuoka (see [12, Theorem 4.5]) in the algebraic setting.

**Proposition 4.24.** *For any Chevalley supergroup  $\mathbf{G}$ , there exist isomorphisms*

$$\begin{aligned} \mathcal{O}(\mathbf{G}) &\cong \mathcal{O}(\mathbf{G}_0) \otimes \mathcal{O}(\mathbf{G}_1^{-, <}) \otimes \mathcal{O}(\mathbf{G}_1^{+, <}) \\ &\cong \mathcal{O}(\mathbf{G}_0) \otimes \mathbb{k}[\xi_1, \dots, \xi_{N_-}] \otimes \mathbb{k}[\chi_1, \dots, \chi_{N_+}] \end{aligned}$$

of commutative superalgebras, where  $N_{\pm} = |\Delta_1^{\pm}|$ , the subalgebra  $\mathcal{O}(\mathbf{G}_0)$  is totally even, and  $\xi_1, \dots, \xi_{N_-}$  and  $\chi_1, \dots, \chi_{N_+}$  are odd elements.

Finally, one has the following result for any  $A \in (\text{salg})$  that is the central extension of the commutative algebra  $A_0$  by the  $A_0$ -module  $A_1$ .



**Proposition 4.25.** *Let  $G$  be a Chevalley supergroup functor, and let  $\mathbf{G}$  be its associated Chevalley supergroup. Assume that  $A \in (\text{salg})$  is such that  $A_1^2 = \{0\}$ , and let  $N_{\pm} = |\Delta_1^{\pm}|$ .*

*Then,  $G_1^+(A)$ ,  $G_1^-(A)$  and  $G_1(A)$  are normal subgroups of  $G(A)$ , and we have that*

$$G_1^{\pm}(A) = G_1^{\pm, <}(A) \cong \mathbb{A}_k^{0|N_{\pm}}(A), \quad G_1(A) = G_1^-(A) \cdot G_1^+(A) = G_1^+(A) \cdot G_1^-(A),$$

$$G_1(A) \cong G_1^-(A) \times G_1^+(A) \cong G_1^+(A) \times G_1^-(A) \cong \mathbb{A}_k^{0|N^-}(A) \times \mathbb{A}_k^{0|N^+}(A)$$

(where ‘ $\cong$ ’ means isomorphic as groups), the group structure on  $\mathbb{A}_k^{0|N_{\pm}}(A)$  being the obvious one. In particular,

$$G(A) \cong G_0(A) \ltimes G_1(A) \cong G_0(A_0) \ltimes (\mathbb{A}_k^{0|N^-}(A) \times \mathbb{A}_k^{0|N^+}(A)),$$

the semi-direct product of the classical group  $G_0(A_0)$  with the totally odd affine superspace  $\mathbb{A}_k^{0|N^-}(A) \times \mathbb{A}_k^{0|N^+}(A)$ .

Similar results hold with the symbol ‘ $\mathbf{G}$ ’ replacing ‘ $G$ ’ everywhere.

## 5. Independence of the construction, and Lie’s third theorem

In this section, we discuss the dependence on  $V$  of the Chevalley supergroups  $\mathbf{G}_V$ , and a superanalogue of Lie’s third theorem for  $\mathbf{G}_V$  and its converse. Here, again, we follow [6, §§ 5.4 and 5.5].

### 5.1. Independence of the Chevalley and Kostant superalgebras

The construction of Chevalley supergroups depends on the finite-dimensional, rational  $\mathfrak{g}$ -representation  $V$  fixed from the very beginning, and on an admissible  $\mathbb{Z}$ -lattice  $M$  in  $V$ . We now show how it depends on  $V$ ; as a consequence, one finds that it is in fact *independent of  $M$* . Once again, we stick to statements, as the proofs follow the same arguments as in [6, §§ 5.4 and 5.5].

Let  $\mathbf{G}'$  and  $\mathbf{G}$  be two Chevalley supergroups obtained by the same  $\mathfrak{g}$ , possibly with a different choice of the representation. We denote by  $X_{\alpha}$  and by  $X'_{\alpha}$ , respectively, the elements of the Chevalley basis in  $\mathfrak{g}$  identified (as usual) with their images under the two representations of  $\mathfrak{g}$ .

Our first result is technical, yet important.

**Lemma 5.1.** *Let  $\phi: \mathbf{G} \rightarrow \mathbf{G}'$  be a morphism of Chevalley supergroups such that for all local superalgebras  $A$  we have that  $\phi_A(\mathbf{G}_0(A)) = \mathbf{G}'_0(A)$  and  $\phi_A(1 + \vartheta X_{\beta}) = 1 + \vartheta X'_{\beta}$  for all  $\beta \in \Delta_1$ ,  $\vartheta \in A_1$ . Then,  $\text{Ker}(\phi_A) \subseteq \mathbf{T}$ , where  $\mathbf{T}$  is the maximal torus in the group  $\mathbf{G}_0 \subseteq \mathbf{G}$  (see § 4.5).*

Let  $L_V$  be the lattice spanned by the weights in the  $\mathfrak{g}$ -representation  $V$ . The relation between Chevalley supergroups attached to different weight lattices is the same as in the classical setting.

**Theorem 5.2.** *Let  $\mathbf{G}$  and  $\mathbf{G}'$  be two Chevalley supergroups constructed using two representations  $V$  and  $V'$  of the same  $\mathfrak{g}$  over the same field  $\mathbb{K}$  (as in § 4.1). If  $L_V \supseteq L_{V'}$ , then there exists a unique morphism  $\phi: \mathbf{G} \rightarrow \mathbf{G}'$  such that  $\phi_A(1 + \vartheta X_\alpha) = 1 + \vartheta X'_\alpha$ , and  $\text{Ker}(\phi_A) \subseteq Z(\mathbf{G}(A))$ , for every local algebra  $A$ . Moreover,  $\phi$  is an isomorphism if and only if  $L_V = L_{V'}$ .*

As a direct consequence, we have the following ‘independence result’.

**Corollary 5.3.** *Every Chevalley supergroup  $\mathbf{G}_V$  is independent, up to isomorphism, of the choice of an admissible lattice  $M$  of  $V$  considered in the very construction of  $\mathbf{G}_V$  itself.*

## 5.2. Lie’s third theorem

In the present context, the analogue of ‘Lie’s third theorem’ concerns the question of whether the tangent Lie superalgebra of our supergroups  $\mathbf{G}$  is  $\mathfrak{g} = D(2, 1; a)$ . We now answer said question.

Let  $\mathbb{k}$  now be a field (note that this assumption makes the discussion simpler, but it may be dropped, if one acts as in [8, § 4.6]). Let  $\mathbf{G}_V$  be a Chevalley supergroup scheme over  $\mathbb{k}$ , built out of  $\mathfrak{g} = D(2, 1; a)$  and a rational  $\mathfrak{g}$ -module  $V$  as in § 4.5. In § 4.1, we have constructed the Lie superalgebra  $\mathfrak{g}_{\mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}_a} \mathfrak{g}_V$  over  $\mathbb{k}$  starting from the  $\mathbb{Z}_a$ -lattice  $\mathfrak{g}_V$ . We now show that the affine supergroup  $\mathbf{G}_V$  has  $\mathfrak{g}_{\mathbb{k}}$  as its tangent Lie superalgebra.

It is well known that one can associate a Lie superalgebra to a supergroup scheme, in a functorial way. Let us quickly remind ourselves (we refer the reader to [4] for further details).

Let  $A \in (\text{salg})$  and let  $A[\epsilon] := A[x]/(x^2)$  be the *superalgebra* of dual numbers, in which  $\epsilon := x \bmod (x^2)$  is taken to be *even*. We have that  $A[\epsilon] = A \oplus \epsilon A$ , and there exist two natural morphisms  $i: A \rightarrow A[\epsilon]$ ,  $a \xrightarrow{i} a$ , and  $p: A[\epsilon] \rightarrow A$ ,  $(a + \epsilon a') \xrightarrow{p} a$ , such that  $p \circ i = \text{id}_A$ .

**Definition 5.4.** For each supergroup scheme  $G$ , consider  $G(p): G(A(\epsilon)) \rightarrow G(A)$ . There then exists a supergroup functor  $\text{Lie}(G): (\text{salg}) \rightarrow (\text{sets})$  given on objects by  $\text{Lie}(G)(A) := \text{Ker}(G(p))$ .

One shows that the functor  $\text{Lie}(G)$  is represented by a super vector space, which can be identified with (the functor of points of) the tangent (super)space at the identity of  $G$ . Then, by an abuse of notation one denotes by the same symbol both the functor and its representing super vector space. One also proves that the functor  $\text{Lie}(G)$  takes values in the category  $(\text{Lie-alg})$  of Lie  $\mathbb{k}$ -algebras (this is equivalent to saying that the super vector space representing it is a Lie  $\mathbb{k}$ -superalgebra).

We now compare with  $\mathfrak{g} = D(2, 1; a)$  the Lie superalgebra  $\text{Lie}(\mathbf{G}_V)$  of any of our Chevalley supergroups  $\mathbf{G}_V$ ; the outcome is (as in [6, Theorem 5.35], with the same proof) the following.

**Theorem 5.5.** *If  $\mathbf{G}_V$  is a Chevalley supergroup built upon  $\mathfrak{g}$  and  $V$ , then  $\text{Lie}(\mathbf{G}_V) = \mathfrak{g}$  as functors with values in  $(\text{Lie-alg})$ .*

**Remark 5.6.** In the proof of Theorem 5.5, one also uses the fact that the (classical) group  $P_a: (\text{alg}) \rightarrow (\text{groups})$  has the functor  $\text{Lie}(P_a): (\text{alg}) \rightarrow (\text{Lie-alg})$ ,  $A \mapsto A$ , as tangent Lie algebra. The same also holds for the group  $\text{Up}: (\text{alg}) \rightarrow (\text{groups})$ , which is relevant for Remark 4.17 (b).

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