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ON THE CONFORMAL DEFORMATION OF RIEMANNIAN STRUCTURES

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In this paper, we study a nonlinear partial differential equation on a compact manifold;

$$\Delta u + ru + Hu^a = 0, \quad u > 0,$$

where $a > 1$ is a constant, r is a positive constant, and H is a prescribed smooth function.

Kazdan and Warner showed that if $\lambda_1(g) < 0$ and $\overline{H} < 0$, where $\overline{H}$ is the mean of H, then there is a constant $0 < r_0(H) \leqslant \infty$ such that one can solve this equation for $0 < r < r_0(H)$, but not for $r > r_0(H)$. They also proved that if $r_0(H) = \infty$, then $H(x) \leqslant 0 (\not\equiv 0)$ for all $x \in M$. They conjectured that this necessary condition might be sufficient.

I show that this conjecture is right; that is, if $H(x) \leqslant 0 (\not\equiv 0)$ for all $x \in M$, then $r_0(H) = \infty$.

1. INTRODUCTION

In this paper, we consider the problem of describing the set of scalar curvature functions associated with Riemannian metrics on a given connected, but not necessarily orientable, compact manifold of dimension greater than or equal to 3.

We shall call metrics g and g_1 pointwise conformal if $g_1 = p(x)g$ for some positive function $p \in C^\infty(M)$. Now if a given metric g on M, where $\dim M = n \geqslant 3$, has scalar curvature $k \in C^\infty(M)$ and we seek $K \in C^\infty(M)$ as the scalar curvature of the metric $g_1 = u^{4/(n-2)}g$ pointwise conformal to g, then $u(> 0)$ must satisfy

$$\frac{4(n-1)}{n-2}\Delta u - ku + Ku^{(n+2)/(n-2)} = 0, \tag{1.1}$$

where Δ is the Laplacian in the g metric.

In carrying out analysis of (1.1), the sign of the lowest eigenvalue $\lambda_1(g)$ of the linear part of (1.1), in other words,

$$L\phi = -\frac{4(n-1)}{n-2}\Delta\phi + k\phi = \lambda_1(g)\phi, \tag{1.2}$$

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plays a prominent part because the sign of $\lambda_1(g)$ is a conformal invariant. In this paper our results are proved in the case of $\lambda_1(g) < 0$. For basic existence theorems, we use the method of upper and lower solutions ([**2**, p.370–371] or [**5**, Lemma 2.6]).

2. Main results

Let M be a compact connected n-dimensional manifold, which is not necessarily orientable and possesses a given Riemannian structure g. We denote the volume element of this metric by dV, the gradient by ∇, and the mean value of a function f on M is written $\overline{f}$, that is,

$$\overline{f} = \frac{1}{\mathrm{vol}(M)} \int_M f\,dV.$$

We let $H_{s,p}(M)$ denote the Sobolev space of functions on M whose derivatives through order s are in $L_p(M)$. The norm on $H_{s,p}(M)$ will be denoted by $\| \ \|_{s,p}$. The usual $L_2(M)$ inner product will be written $\langle \, , \rangle$.

Lemma 1. *Assume $K < 0$. Then K is the scalar curvature of some metric pointwise conformal to the given metric g if and only if $\lambda_1(g) < 0$.*

Proof: See Theorem 4.1 in [**5**]. ∎

The above Lemma 1 shows that if $\lambda_1(g) < 0$, then one can always pointwise conformally deform g to a metric of constant negative scalar curvature $k = -c$, where $c > 0$ is a constant. Thus (1.1) reads

$$\frac{4(n-1)}{n-2}\Delta u + cu = -Ku^{(n+2)/(n-2)}, \quad u > 0. \tag{2.1}$$

In order to understand (2.1), one must first free it from geometric considerations and consider the equation

$$-Lu = \Delta u + ru = -Hu^a, \quad u > 0, \tag{2.2}$$

where $a > 1$ and $r > 0$ are constants, and $H \in C^\infty(M)$. Throughout this paper, we shall assume that all data (M, metric g, and curvature K, *et cetera*) are smooth merely for convenience.

Kazdan and Warner showed that if $\lambda_1(g) < 0$ and $\overline{H} < 0$, then there is a constant $0 < r_0(H) \leqslant \infty$ such that one can solve (2.2) for $0 < r_0 < r_0(H)$, but not for $r > r_0(H)$ (see Proposition 4.8 in [**5**]). They also showed that if $r_0(H) = \infty$ then $H(x) \leqslant 0$ for all $x \in M$. In fact, they proved that if $H(x_0) > 0$ for some $x_0 \in M$, then $r_0(H) < \infty$ (see Proposition 4.10 in [**5**]). Since $\lambda_1(g) < 0$, Theorem 2.11 in [**5**] implies that $H \not\equiv 0$. Kazdan and Warner [**5**] conjectured that this necessary condition might be sufficient, such as in Theorem 10.5(a) of [**4**]. Now we shall prove that this necessary condition is also a sufficient condition, that is, if $H(x) \leqslant 0 (\not\equiv 0)$ for all $x \in M$, then $r_0(H) = \infty$.

Lemma 2. (Existence of lower solutions.) *Let $H \in L_p(M)$ with $p > \dim M$. If $\lambda_1 < 0$, then given any positive continuous function u on M, there is a function $u_- \in H_{2,p}(M)$ with $0 < u_- < u$ satisfying $Lu_- \leqslant Hu_-^a$, that is, $\Delta u_- + ru_- + Hu_-^a \geqslant 0$.*

Proof: See Lemma 2.8 in [5], substituting $-r$ for h, where r is a positive constant. ∎

We consider the differential operator

$$Lv = -\Delta v - \alpha H v, \tag{2.3}$$

where α is a positive constant and $H \leqslant 0 \, (\not\equiv 0)$. For each $\alpha > 0$, if $\lambda_1(\alpha)$ is the lowest eigenvalue of (2.3), then

$$\begin{aligned}\lambda_1(\alpha) &= \min_{v \not\equiv 0} \frac{\|v\|_2^2 + \langle v, -\alpha H v\rangle}{\|v\|_2^2}, \quad v \in H_{1,2}(M) \\ &= \min\left(\|v\|_2^2 + \langle v, -\alpha H v\rangle\right), \quad \|v\|_2 = 1, \, v \in H_{1,2}(M).\end{aligned}$$

Note that the eigenfunction is never zero (see Remark 2.4 in [5]). Let $\phi_\alpha > 0$ be the corresponding eigenfunction of (2.3) with $\|\phi_\alpha\|_2 = 1$, that is,

$$\Delta \phi_\alpha + \alpha H \phi_\alpha = -\lambda_1(\alpha)\phi_\alpha. \tag{2.4}$$

By integrating (2.4) over M, we can see that $\lambda_1(\alpha) > 0$. Now in order to investigate the behaviour of $\lambda_1(\alpha)$ as $\alpha \to \infty$, we shall prove the following key lemma.

Lemma 3. *Let M be a connected compact manifold without boundary. Let L be as in (2.3) and $\lambda_1(\alpha)$ be the corresponding eigenvalue of L for $\alpha > 0$. If $H \leqslant 0 \, (\not\equiv 0)$, then $\lambda_1(\alpha) \to \infty$ as $\alpha \to \infty$.*

Proof: For each $\alpha > 0$,

$$\Delta \phi_\alpha + \alpha H \phi_\alpha = -\lambda_1(\alpha)\phi_\alpha,$$

where $\phi_\alpha > 0$ is the corresponding eigenfunction with $\|\phi_\alpha\|_2 = 1$. To prove our conclusion we have several steps.

Step 1. $\{\lambda_1(\alpha)\}_{\alpha \in N}$ is a strictly increasing sequence. Let $\alpha_1 < \alpha_2$. Since $\Delta \phi_{\alpha_1} + \alpha_1 H \phi_{\alpha_1} = -\lambda_1(\alpha_1)\phi_{\alpha_1}$,

$$\int_M \Delta \phi_{\alpha_1} \phi_{\alpha_2} dV + \alpha_1 \int_M H \phi_{\alpha_1} \phi_{\alpha_2} dV = -\lambda_1(\alpha_1) \int_M \phi_{\alpha_1} \phi_{\alpha_2} dV.$$

But the fact that $\partial M = \phi$ implies that

$$\int_M \Delta \phi_{\alpha_1} \phi_{\alpha_2} dV = \int_M \phi_{\alpha_1} \Delta \phi_{\alpha_2} \, dV$$

and also ϕ_{α_2} satisfies

$$\Delta\phi_{\alpha_2} + \alpha_2 H\phi_{\alpha_2} = -\lambda_1(\alpha_2)\phi_{\alpha_2},$$

so we find that

$$(\alpha_1 - \alpha_2)\int_M H\phi_{\alpha_1}\phi_{\alpha_2}dV = \{\lambda_1(\alpha_2) - \lambda_1(\alpha_1)\}\int_M \phi_{\alpha_1}\phi_{\alpha_2}dV. \tag{2.5}$$

Since $\phi_{\alpha_1}, \phi_{\alpha_2} > 0$ and $H \leqslant 0 (\not\equiv 0)$ on M and $\alpha_1 < \alpha_2$, $\lambda_1(\alpha_1) < \lambda_1(\alpha_2)$. Hence $\{\lambda_1(\alpha)\}_{\alpha\in N}$ is a strictly increasing sequence. From (2.5) we find that $|\lambda_1(\alpha_2) - \lambda_1(\alpha_1)| \leqslant \|H\|_\infty |\alpha_1 - \alpha_2|$. This means that $\lambda_1(\alpha)$ is continuous with respect to α.

Suppose $\{\lambda_1(\alpha)\}_{\alpha\in N}$ is bounded. Then there exists λ_0 such that $\lambda_1(\alpha) < \lambda_0$ and $\lambda_1(\alpha) \to \lambda_0$ as $\alpha \to \infty$.

STEP 2. If $\lambda_1(\alpha) \to \lambda_0$ as $\alpha \to \infty$, then $\alpha \int (-H)\phi_\alpha^2 dV \to 0$ and $\alpha \int (-H)\phi_\alpha dV \to 0$ as $\alpha \to \infty$.

The variational characterisation of $\lambda_1(\alpha)$ implies that

$$\begin{aligned}\lambda_1(\alpha+\ell) &= \|\nabla\phi_{\alpha+\ell}\|_2^2 + (\alpha+\ell)\int(-H)\phi_{\alpha+\ell}^2 dV \\ &= \|\nabla\phi_{\alpha+\ell}\|_2^2 + \alpha\int(-H)\phi_{\alpha+\ell}^2 dV + \ell\int(-H)\phi_{\alpha+\ell}^2 dV \\ &\geqslant \lambda_1(\alpha) + \ell\int(-H)\phi_{\alpha+\ell}^2 dV.\end{aligned}$$

Hence for all $\ell > 0$,

$$\lambda_1(\alpha+\ell) - \lambda_1(\alpha) \geqslant \ell\int(-H)\phi_{\alpha+\ell}^2 dV.$$

Since $\lambda_1(\alpha) \to \lambda_0$ as $\alpha \to \infty$, for all $\varepsilon > 0$ there exists $\alpha > 0$ such that

$$|\lambda_1(\alpha) - \lambda_0| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}(\alpha+\ell)\int(-H)\phi_{\alpha+\ell}^2 dV &= \frac{\alpha+\ell}{\ell}\,\ell\int(-H)\phi_{\alpha+\ell}^2 dV \\ &\leqslant \frac{\alpha+\ell}{\ell}\{\lambda_1(\alpha+\ell) - \lambda_1(\alpha)\} \\ &\leqslant \frac{\alpha+\ell}{\ell}|\lambda_0 - \lambda_1(\alpha)| < \varepsilon\end{aligned}$$

for sufficiently large $\ell > 0$. Hence $\alpha \int (-H)\phi_\alpha^2 dV \to 0$ as $\alpha \to \infty$. By the Hölder's inequality,

$$\begin{aligned}\left|\alpha \int (-H)\phi_\alpha dV\right| &\leqslant \alpha \int |H|\, \phi_\alpha dV \\ &\leqslant \alpha\left(\int H^2\phi_\alpha^2 dV\right)\left(\int l^2 dV\right) \\ &= \mathrm{vol}(M)\cdot \|H\|_\infty\, \alpha \int (-H)\phi_\alpha^2 dV, \quad (\text{note } H \in C^\infty(M))\end{aligned}$$

so the second assertion in Step 2 follows easily.

STEP 3. Since $\int \phi_\alpha^2 dV = 1$ and $\Delta\phi_\alpha + \alpha H\phi_\alpha = -\lambda_1(\alpha)\phi_\alpha$,

$$\int |\nabla\phi_\alpha|^2\, dV = \alpha \int H\phi_\alpha^2 dV + \lambda_1(\alpha).$$

But $\left|\alpha \int_M H\phi_\alpha^2 dV\right| \to 0$ as $\alpha \to \infty$ and $\lambda_1(\alpha) \to \lambda_0$, hence $\{\int_M |\nabla\phi_\alpha|^2\, dV\}_{\alpha\in N}$ is bounded. Therefore, $\{\phi_\alpha\}_{\alpha\in N}$ is bounded in $H_{1,2}(M)$. By Kondrakov Theorem ([1], Theorem 2.34), $\{\phi_\alpha\}_{\alpha\in N}$ is compact in $L_2(M)$. Thus there exists $\phi_0 \in L_2(M)$ such that $\phi_{n_\alpha} \to \phi_0$ strongly, where $\{\phi_{n_\alpha}\}$ is a subsequence of $\{\phi_\alpha\}_{\alpha\in N}$. We may assume that $\phi_\alpha \to \phi_0$ in $L_2(M)$. Since $\int_M \phi_\alpha^2 dV = 1$ and $\phi_\alpha > 0$ on M, $\int_M \phi_0^2 dV = 1$ and $\phi_0 \geqslant 0\ (\not\equiv 0)$. (See [1], Proposition 3.43.) Note that $\int_M \phi_0 dV > 0$. But for each α,

$$\int_M \Delta\phi_\alpha dV + \alpha \int_M H\phi_\alpha dV = -\lambda_1(\alpha) \int_M \phi_\alpha dV. \tag{2.6}$$

Since $\lambda_1(\alpha) \to \lambda_0$ and

$$\begin{aligned}\left|\int_M \phi_\alpha dV - \int_M \phi_0 dV\right| &\leqslant \int_M |\phi_\alpha - \phi_0|\, dV \\ &\leqslant \text{ constant } \times \|\phi_\alpha - \phi_0\|_2^2 \to 0 \text{ as } \alpha \to \infty,\end{aligned}$$

the right side of (2.6) converges to $-\lambda_0 \int_M \phi_0 dV \neq 0$. But $\int_M \Delta\phi_\alpha dV = 0$ and $\left|\alpha \int_M H\phi_\alpha dV\right| \to 0$ as $\alpha \to \infty$, so the left side of (2.6) converges to 0 as $\alpha \to \infty$. Hence we have a contradiction. Thus $\{\lambda_1(\alpha)\}_{\alpha\in N}$ is not bounded, that is, $\lambda_1(\alpha) \to \infty$ as $\alpha \to \infty$. ∎

Using the previous key Lemma 3, we can prove the following main theorem, that is, the necessary condition $H(x) \leqslant 0\,(\not\equiv 0)$ for $r_0(H) = \infty$ is also sufficient.

THEOREM. (Existence of upper solutions). *If $H(x) \leqslant 0\,(\not\equiv 0)$ for all $x \in M$, then (2.2) has a solution for any positive constant r, so $r_0(H) = \infty$.*

PROOF: If we show that $Lu_+ \geqslant Hu_+^a$ for some positive function $u_+ > 0$ and any positive constant $r > 0$, that is,

$$\Delta u_+ + ru_+ + Hu_+^a \leqslant 0,$$

then Lemma 2 implies that there exists a solution of (2.2), so $r_0(H) = \infty$. Let r be any positive constant. If we put $u_+ = e^{\psi}$, then $\Delta u_+ = e^{\psi}\left(\Delta\psi + |\nabla\psi|^2\right)$. Hence

$$\Delta u_+ + r u_+ + H u_+^a \leqslant 0$$

if and only if

$$\Delta\psi + |\nabla\psi|^2 + r + H e^{c\psi} \leqslant 0$$

for some function ψ and $c = a - 1 > 0$.

If $Lv = -\Delta v - \alpha H v$, then by Lemma 3 the first eigenvalue $\lambda_1(\alpha)$ of L converges to ∞ as $\alpha \to \infty$ and $\lambda_1(\alpha)$ is continuous with respect to α. Hence there is a constant $\alpha > 0$ such that $\lambda_1(\alpha) = r$. Let ϕ be the corresponding eigenfunction, that is,

$$\Delta\phi + \alpha H\phi = -\lambda_1(\alpha)\phi = -r\phi, \quad \phi > 0.$$

Put $\phi = e^{\widetilde{\psi}}$. Then

$$\Delta\widetilde{\psi} + \left|\nabla\widetilde{\psi}\right|^2 + r + \alpha H = 0.$$

Define $\psi = \widetilde{\psi} + \lambda$ for some positive constant λ. Therefore,

$$\begin{aligned} &\Delta\psi + |\nabla\psi|^2 + r + H e^{c\psi} \\ &\quad = \Delta\widetilde{\psi} + \left|\nabla\widetilde{\psi}\right|^2 + r + H e^{c\widetilde{\psi} + c\lambda} \\ &\quad = -\alpha H + H e^{c\widetilde{\psi} + c\lambda} \\ &\quad = H\left(e^{c\widetilde{\psi} + c\lambda} - \alpha\right) \leqslant 0 \end{aligned}$$

for sufficiently large λ, since $H \leqslant 0 \, (\not\equiv 0)$. This completes our theorem. ∎

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