ANZIAM J. 59(2017), 103–114
doi:10.1017/S144618111700027X

FIFTH-ORDER EVOLUTION EQUATION OF GRAVITY–CAPILLARY WAVES

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(Received 11 May, 2016; accepted 29 November, 2016; first published online 9 August 2017)

Abstract

We extend the evolution equation for weak nonlinear gravity–capillary waves by including fifth-order nonlinear terms. Stability properties of a uniform Stokes gravity–capillary wave train is studied using the evolution equation obtained here. The region of stability in the perturbed wave-number plane determined by the fifth-order evolution equation is compared with that determined by third- and fourth-order evolution equations. We find that if the wave number of longitudinal perturbations exceeds a certain critical value, a uniform gravity–capillary wave train becomes unstable. This critical value increases as the wave steepness increases.

2010 *Mathematics subject classification*: primary 76B07; secondary 76B15, 76B45.

Keywords and phrases: evolution equation, gravity–capillary wave, modulational instability, wind effect.

1. Introduction

There has been considerable development in the study of the evolution of weakly nonlinear surface gravity–capillary waves. Many authors [2, 5, 8, 15, 21, 22] have contributed towards the understanding of the evolution of surface gravity–capillary waves. Djordjevic and Redekopp [5] studied the motion of a two-dimensional gravity–capillary wave packet on water of finite depth. They obtained evolution equations in the form of a coupled system of partial differential equations correct up to third order in wave steepness. They showed that the Stokes gravity–capillary wave train for several distinct wave bands becomes unstable due to modulational perturbations. They also found the existence of a resonant interaction between a gravity–capillary wave and a long gravity wave.

Dysthe [6] derived an evolution equation for a weakly nonlinear surface gravity wave packet which is correct up to fourth order in wave steepness. He showed that a band of waves remains stable up to fourth order. Dysthe also pointed out that

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the results of stability analysis of Stokes waves made from fourth-order evolution equations agree very well with the exact numerical computational results of Longuet-Higgins [13, 14], for waves of steepness up to 0.25. However, if the stability analysis is made from a third-order evolution equation, then the results differ from the exact results of Longuet-Higgins [13, 14], when wave steepness exceeds the value 0.15. Considering the importance of fourth-order nonlinear terms in the evolution equation, Hogan [8] derived an evolution equation for a weakly nonlinear deep water gravity–capillary wave packet, which is correct up to fourth order in the wave steepness. The evolution equation of Hogan [8, equation (2.20)] was derived under the assumption that the wave packet was of narrow bandwidth for which the order of the bandwidth, $|\Delta\vec{k}|/|\vec{k}|$, was same as the order of the smallness of the wave steepness $|\vec{k}|a$. Here $|\vec{k}|$ and a denote the characteristic wave-number and characteristic amplitude, respectively, and $|\Delta\vec{k}|$ is the modulational wave number.

Zhang and Melville [21] investigated the stability of weakly nonlinear gravity–capillary waves in resonant triad and quartet interactions by numerically solving an eigenvalue system. They found that the triad and quartet instabilities which are separated in the wave-number plane for infinitesimal waves may merge for weakly nonlinear waves. Later on, Zhang and Melville [22] presented a modified numerical scheme for investigating the three-dimensional instability properties of gravity–capillary waves in deep water. Evolution of a weakly nonlinear gravity–capillary wave packet was also studied by Debsarma and Das [2] in the presence of a thin thermocline in deep water. They obtained evolution equations in the form of a coupled system of two equations which are correct up to fourth order of wave steepness.

It is interesting to see the effect of the fifth-order nonlinear interaction terms on the modulational instability properties of uniform Stokes wave trains. In this paper, we have extended the analysis of Hogan [8] to include the fifth-order nonlinear terms. Equation (2.20) of Hogan [8, Section 2] describes the evolution of a weakly nonlinear gravity–capillary wave packet for which the bandwidth is of order $O(\epsilon)$, ϵ being the order of smallness of wave steepness, and the equation is correct up to $O(\epsilon^4)$. In this paper, we consider the evolution of a weakly nonlinear gravity–capillary wave packet which follows the narrow bandwidth constraint of Hogan [8]. The evolution equation derived here is correct up to $O(\epsilon^5)$, and it is employed here to study the stability properties of uniform Stokes wave trains (see the work by Djordjevic and Redekopp [5]). We have plotted stable–unstable regions (Figures 1 to 3) and growth rates of instability curves for uniform Stokes wave trains (Figures 4 to 7).

For the derivation of the evolution equation, we have used the Zakharov integral equation [20]. Crawford et al. [1] gave an expression of the Zakharov kernel for surface gravity waves. Later on, Hogan et al. [9] derived the corrected form of the Zakharov kernel for gravity–capillary waves. However, these kernels fail to conserve the Hamiltonian structure [11] of the original water wave equations. Krasitskii [11] first obtained an expression for the interaction coefficient that enjoys a number of symmetry conditions and leads to the Hamiltonian form of the evolution equation.

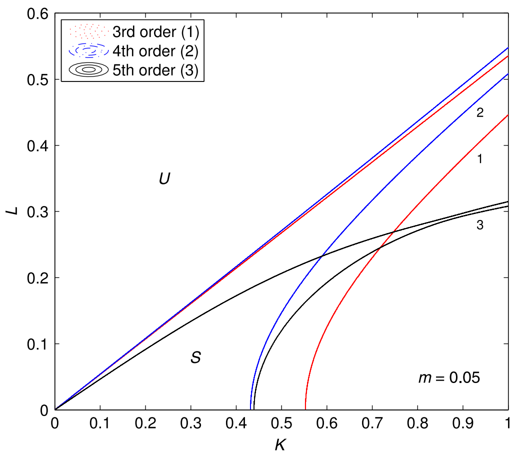

FIGURE 1. Stable–unstable region in the perturbed wave-number plane; S denotes stable region and U denotes unstable region; $m = 0.05$.

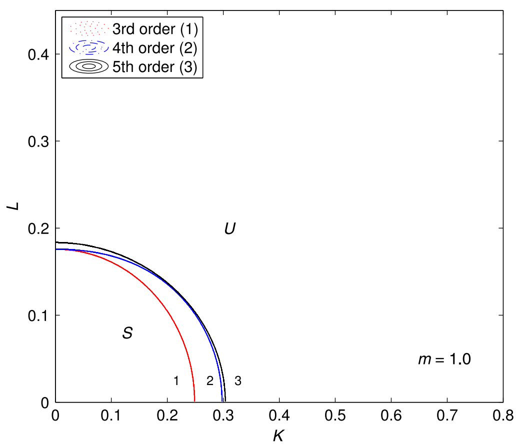

FIGURE 2. Stable–unstable region in the perturbed wave-number plane; S denotes stable region and U denotes unstable region; $m = 1.0$.

Krasitskii and Kalmykov [12] performed an investigation in order to compare the consequences obtained from the Hamiltonian and non-Hamiltonian forms of the Zakharov equation. They found that as far as modulational instability of a uniform Stokes wave train and the long time evolution of a discrete wave system are concerned, the differences in solutions are revealed only for sufficiently large nonlinearity of the wave system. Gramstad and Trulsen [7] derived the fourth-order modified Schrödinger equation for surface gravity wave packets which conserves the momentum and, hence, is in Hamiltonian form. In the current paper, we have compared our result with that obtained by the Hamiltonian evolution equation for the case of gravity waves.

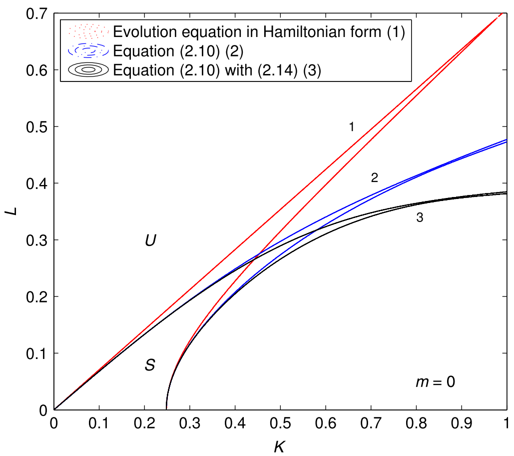

FIGURE 3. Stable-unstable region in the perturbed wave-number plane; S denotes stable region and U denotes unstable region; $m = 0$.

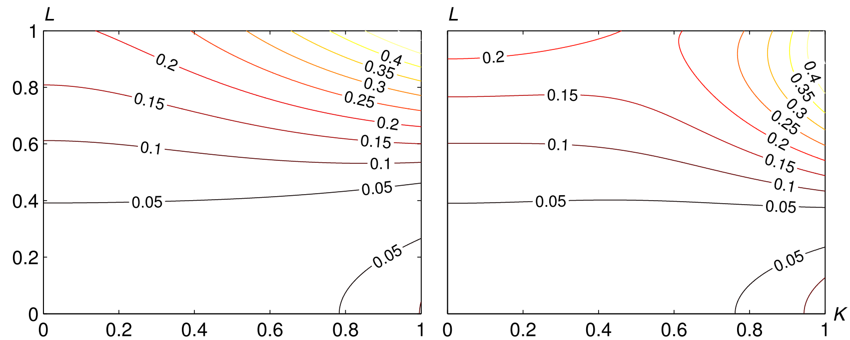

FIGURE 4. Contour plot of growth rate of instability in the (K, L)-plane; $m = 0.05$, $a_0 = 0.15$. Left: at fifth order; right: at fifth order of nonlinearity including higher-order linear dispersive terms.

Ocean water wave packets are not always of narrow bandwidth. Keeping this in mind, Trulsen and Dysthe [18] considered the evolution of a broad band surface gravity wave packet for which the bandwidth is of $O(\epsilon^{1/2})$. The evolution equation derived by them is correct up to $O(\epsilon^{7/2})$. When the stability of Stokes waves is studied using this evolution equation, the stable/unstable regions are in better agreement with the exact regions obtained by McLean et al. [16] and McLean [17] as compared to that for a narrow bandwidth wave packet. Debsarma and Das [3] modified the evolution equation of Trulsen and Dysthe [18] by including $O(\epsilon^4)$ nonlinear terms. A further relaxation in bandwidth was considered by Debsarma and Das [4]. They studied evolution of a gravity wave packet whose order of bandwidth is of $O(\epsilon^{1/3})$. Stable–unstable regions obtained by Debsarma and Das [3, 4] fit very nicely with those obtained by McLean et al. [16]. Encouraged by these results, here we have derived

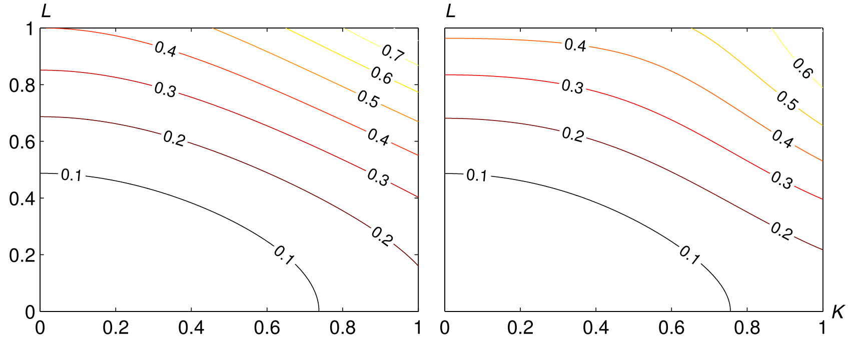

FIGURE 5. Contour plot of growth rate of instability in the (K, L)-plane; $m = 1.0$, $a_0 = 0.15$. Left: at fifth order; right: at fifth order of nonlinearity including higher-order linear dispersive terms.

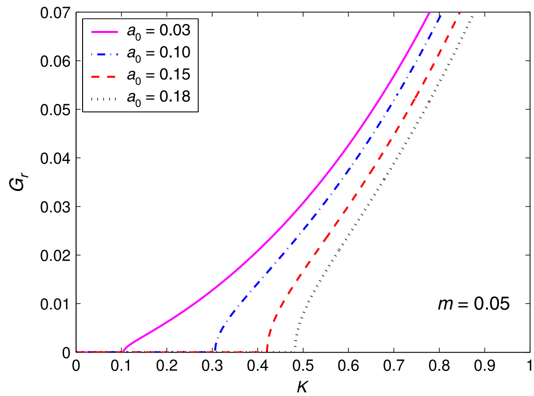

FIGURE 6. Growth rate of instability G_r against perturbation wave-number K, $m = 0.05$.

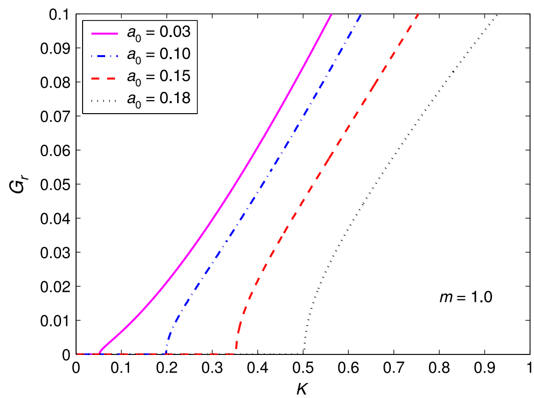

FIGURE 7. Growth rate of instability G_r against perturbation wave-number K, $m = 1.0$.

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a fifth-order evolution equation for surface gravity–capillary waves. Trulsen et al. [19] carried out stability analysis with an evolution equation for weakly nonlinear gravity waves, in which a complete expression for linear dispersive terms was used. In this paper, we have also considered the effect of higher-order linear dispersive terms on the stability properties of a uniform Stokes wave train.

2. Derivation of evolution equation

To derive the evolution equation we start with Zakharov's [20] integral equation

$$i\frac{\partial B(\vec{k},t)}{\partial t} = \iiint_{-\infty}^{\infty} T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)B^*(\vec{k}_1,t)B(\vec{k}_2,t)B(\vec{k}_3,t)\times\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3) \times \exp\{i\{\omega(\vec{k})+\omega(\vec{k}_1)-\omega(\vec{k}_2)-\omega(\vec{k}_3)\}t\}\,d\vec{k}_1\,d\vec{k}_2\,d\vec{k}_3, \tag{2.1}$$

where $\delta()$ is the Dirac delta function and $B(\vec{k},t)$ is related to the free surface elevation $\zeta(\vec{x},t)$ by

$$\zeta(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{|\vec{k}|}{2\omega(\vec{k})}\right]^{1/2}[B(\vec{k},t)\exp\{i(\vec{k}\cdot\vec{x}-\omega(\vec{k})t)\} + \text{ c.c.}]\,d\vec{k}. \tag{2.2}$$

Here $\vec{k}=(k,l)$ is the wave number vector, $\vec{x}=(x,y)$ is horizontal variable vector, c.c. denotes the complex conjugate of the previous term and ω is the wave frequency corresponding to the wave number vector $\vec{k}$. The linear dispersion relation for the gravity–capillary waves is given by $\omega=\omega(|\vec{k}|)=[g|\vec{k}|+\gamma|\vec{k}|^3]^{1/2}$, where g is the gravitational acceleration, and $\gamma = T^{(s)}/\rho$ with $T^{(s)}$ being the surface tension and ρ, the density of water. In equation (2.1), we have used the interaction coefficient $T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)$, known as the Zakharov kernel, given by Hogan et al. [9].

We consider the propagation of a broad band gravity–capillary wave packet having central wave number $\vec{k}_0=(k_0,0)$, and consider the vectors $\vec{k}$, $\vec{k}_j$, $j=1,2,3$, as $\vec{k}=\vec{k}_0+\vec{\chi}$, $\vec{k}_j=\vec{k}_0+\vec{\chi}_j$, $\vec{\chi}=(\chi,\lambda)$, and $\vec{\chi}_j=(\chi_j,\lambda_j)$.

We assume that the spectral widths $|\vec{\chi}|/|\vec{k}_0|$, $|\vec{\chi}_j|/|\vec{k}_0|$ are of $O(\epsilon)$, where ϵ is the order of smallness of the wave steepness. We also assume that the wave packet is sufficiently narrow, so that the triad solutions $\vec{k}_0=\vec{k}_1+\vec{k}_2$ and $\omega(|\vec{k}_0|)=\omega(|\vec{k}_1|)+\omega(|\vec{k}_2|)$ are not satisfied for some $\vec{k}_0$. Introducing a new variable

$$A(\vec{\chi},t) = B(\vec{k},t)\exp[-i\{\omega(\vec{k})-\omega(\vec{k}_0)\}t], \tag{2.3}$$

we rewrite equation (2.1) as

$$i\frac{\partial A(\vec{\chi},t)}{\partial t} - \{\omega(\vec{k})-\omega(\vec{k}_0)\}A(\vec{\chi},t) = \iiint_{-\infty}^{\infty} T(\vec{k}_0+\vec{\chi},\vec{k}_0+\vec{\chi}_1,\vec{k}_0+\vec{\chi}_2,\vec{k}_0+\vec{\chi}_3) \times A^*(\vec{\chi}_1,t)A(\vec{\chi}_2,t)A(\vec{\chi}_3,t)\times\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2-\vec{\chi}_3)\,d\vec{\chi}_1\,d\vec{\chi}_2\,d\vec{\chi}_3.$$

Inserting (2.3) into equation (2.2) and keeping terms up to second order in spectral width

$$\zeta(\vec{x},t) = \frac{1}{2}[a(\vec{x},t)\exp[i\{\vec{k}_0\cdot\vec{x} - \omega(k_0)t\}] + \text{c.c.}], \quad \text{where}$$

$$a(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2|\vec{k}_0|}{\omega(|\vec{k}_0|)}\right]^{1/2} f(\chi,\lambda)A(\vec{\chi},t)\exp(i\vec{\chi}\cdot\vec{x})\,d\vec{\chi},$$

$$f(\chi,\lambda) = 1 + \frac{1-m}{4k_0(1+m)}\cdot\chi - \frac{3+18m-5m^2}{32k_0^2(1+m)^2}\cdot\chi^2 + \frac{1-m}{8k_0^2(1+m)}\cdot\lambda^2, \quad m = \gamma k^2/g.$$

To derive the evolution equation we have followed the same procedure as in Debsarma and Das [3]. Finally, we obtain the desired evolution equation for a narrow band gravity–capillary wave packet correct up to fifth order in wave steepness as

$$\begin{aligned}
i\frac{\partial a}{\partial t} &+ i\epsilon\beta_0\frac{\partial a}{\partial x} + \epsilon^2\Big(\beta_1\frac{\partial^2 a}{\partial x^2} + \beta_2\frac{\partial^2 a}{\partial y^2}\Big) + i\epsilon^3\Big(\beta_3\frac{\partial^3 a}{\partial x^3} + \beta_4\frac{\partial^3 a}{\partial x\partial y^2}\Big)\\
&+ \epsilon^4\Big(\beta_5\frac{\partial^4 a}{\partial x^4} + \beta_6\frac{\partial^2 a}{\partial x^2\partial y^2} + \beta_7\frac{\partial^4 a}{\partial y^4}\Big)\\
&= \Lambda_1 a^2 a^* + i\epsilon\Lambda_2 a a^*\frac{\partial a}{\partial x} + i\epsilon\Lambda_3 a^2\frac{\partial a^*}{\partial x} + \epsilon\Lambda_4 a\frac{\partial}{\partial x}H[aa^*]\\
&+ \epsilon^2\Big(\mu_1 a a^*\frac{\partial^2 a}{\partial x^2} + \mu_2 a a^*\frac{\partial^2 a}{\partial y^2}\Big) + \epsilon^2\Big\{\mu_3 a^*\Big(\frac{\partial a}{\partial x}\Big)^2 + \mu_4 a^*\Big(\frac{\partial a}{\partial y}\Big)^2\Big\}\\
&+ \epsilon^2\Big(\mu_5 a\frac{\partial a}{\partial x}\frac{\partial a^*}{\partial x} + \mu_6\frac{\partial a}{\partial y}\frac{\partial a^*}{\partial y}\Big) + \epsilon^2\Big(\mu_7 a^2\frac{\partial^2 a^*}{\partial x^2} + \mu_8 a^2\frac{\partial^2 a^*}{\partial y^2}\Big)\\
&+ i\epsilon^2\mu_9 a\frac{\partial}{\partial x}H\Big[a^*\frac{\partial a}{\partial x}\Big] + i\epsilon^2\mu_{10} a\frac{\partial}{\partial y}H\Big[a^*\frac{\partial a}{\partial y}\Big] + i\epsilon^2\mu_{11} a\frac{\partial}{\partial x}H\Big[a\frac{\partial a^*}{\partial x}\Big]\\
&+ i\epsilon^2\mu_{12}\frac{\partial a}{\partial x}\frac{\partial}{\partial x}H[aa^*] + i\epsilon^2\mu_{13}\frac{\partial a}{\partial y}\frac{\partial}{\partial y}H[aa^*] + \epsilon^2\mu_{14} a\frac{\partial^3}{\partial x^3}P[aa^*]. \qquad (2.4)
\end{aligned}$$

The coefficients β_i, Λ_i, μ_i appearing in equation (2.4) are given in the Appendix A. In equation (2.4), H denotes the two-dimensional Hilbert transform operator [10] and P is another integral operator, defined as

$$H\Psi(x,y) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{(\xi - x)\Psi(\xi,\eta)}{[(\xi-x)^2 + (\eta-y)^2]^{3/2}}\,d\xi\,d\eta, \qquad (2.5)$$

$$P\Psi(x,y) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{(\xi - x)\Psi(\xi,\eta)}{(\xi-x)^2 + (\eta-y)^2}\,d\xi\,d\eta.$$

Equation (2.4) is written here in dimensionless form with the dimensionless variables

$$a' = k_0 a, \quad x' = k_0 x, \quad y' = k_0 x, \quad t' = \omega(k_0)t,$$

and later the primes are dropped for simplicity.

The coefficients β_0, β_1, β_2, β_3, β_4, Λ_1, Λ_2, Λ_3, Λ_4 in equation (2.4) are in agreement with the corresponding coefficients of the evolution equation of Hogan

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[8, equation (2.20)]. Also, by setting $m = 0$ in the coefficients of the evolution equation (2.4), we can recover the evolution equation of Debsarma and Das [3, equation (2.15)]. This provides a check on our computation.

The coefficients $\Lambda_1, \Lambda_3, \mu_i, i = 1, \dots, 8$, appearing in equation (2.4) contain a factor $(1 - 2m)$ in the denominator. As a result, the evolution equation (2.5) does not remain valid when $m = 1/2$. This is the case for a second harmonic resonance, when the wave travels at the same speed as one of its harmonics. McGoldrick [15] made a detailed study of this case in the context of Wilton's ripples.

Note that all the fourth- and fifth-order nonlinear terms in equation (2.4) emerge as a result of the narrow spectral width assumption, and none of these terms is of fourth or fifth order in the wave steepness itself. To examine the effect of higher-order linear dispersive terms on the stability properties of a uniform wave train, we have to include the following terms in the left-hand side of the evolution equation (2.4):

$$i\epsilon^5\Big(\beta_8\frac{\partial^5 a}{\partial x^5} + \beta_9\frac{\partial^5 a}{\partial x^3\partial y^2} + \beta_{10}\frac{\partial^5 a}{\partial x\partial y^4}\Big) + \epsilon^6\Big(\beta_{11}\frac{\partial^6 a}{\partial x^6} + \beta_{12}\frac{\partial^6 a}{\partial x^4\partial y^2} + \beta_{13}\frac{\partial^6 a}{\partial x^2\partial y^4} + \beta_{14}\frac{\partial^6 a}{\partial y^6}\Big). \tag{2.6}$$

We show the effect of the terms in (2.6) in the next section.

3. Stability analysis

The uniform Stokes wave solution of equation (2.4) is given by

$$a = a_0 \exp(-it\Delta\omega) \equiv \zeta^{(0)}, \tag{3.1}$$

where a_0 is a real constant and the nonlinear frequency shift

$$\Delta\omega = -\Lambda_1 a_0^2.$$

For the purpose of stability analysis of the uniform wave solution (3.1), we introduce the following infinitesimal perturbation to this solution:

$$a = \zeta^{(0)}[1 + a'(\vec{x}, t)]. \tag{3.2}$$

We now substitute the perturbed solution (3.2) into the evolution equation (2.4) and then linearize with respect to a'. Taking $a' = a'_r + ia'_i$, with a'_r, a'_i being real, we separate the real and imaginary parts. Thereby, we get a system of two coupled linear equations in a'_r and a'_i. Finally, assuming the space-time dependence of a'_r and a'_i to be in the form $\exp[i(Kx + Ly - \Omega t)]$, where (K, L) is the perturbation wavenumber and Ω denotes frequency, we obtain

$$i(\Omega + E)a'_i - Fa'_r = 0, \tag{3.3}$$

$$Ga'_i + i(\Omega + H)a'_r = 0, \tag{3.4}$$

where

$$E = \beta_0 K - (\beta_3 K^3 + \beta_4 KL^2) - (\Lambda_2 - \Lambda_3)Ka_0^2 + \frac{(\mu_9 K^3 + \mu_{10} KL^2 - \mu_{11} K^3)a_0^2}{\sqrt{K^2 + L^2}},$$

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$$F = (\beta_1 K^2 + \beta_2 L^2) - (\beta_5 K^4 + \beta_6 K^2 L^2 + \beta_7 L^4) + 2\Lambda_1 a_0^2 - \frac{2\Lambda_4 K^2 a_0^2}{\sqrt{K^2 + L^2}}$$

$$- \{(\mu_1 + \mu_7)K^2 + (\mu_2 + \mu_8)L^2\} a_0^2 + \frac{2\mu_{14} K^4 a_0^2}{(K^2 + L^2)},$$

$$G = (\beta_1 K^2 + \beta_2 L^2) - (\beta_5 K^4 + \beta_6 K^2 L^2 + \beta_7 L^4) - \{(\mu_1 - \mu_7)K^2 + (\mu_2 - \mu_8)L^2)\} a_0^2,$$

$$H = \beta_0 K - (\beta_3 K^3 + \beta_4 K L^2) - (\Lambda_2 + \Lambda_3) a_0^2 K + \frac{(\mu_9 K^3 + \mu_{10} K L^2 + \mu_{11} K^3) a_0^2}{\sqrt{K^2 + L^2}}.$$

Thus, the uniform Stokes wave given by (3.1) becomes unstable when the following condition is satisfied:

$$(E - H)^2 + 4FG < O. \tag{3.5}$$

Since $(E - H)^2 > 0$, condition (3.5) holds only when $FG < 0$. At the fourth order of nonlinearity, this implies that instability is impossible when m lies within the range $2/\sqrt{3} - 1 < m < 1/2$. In Figures 1 and 2, we have shown the stable and unstable regions in the perturbed wave number plane, that is, the (K, L)-plane for two different values of m, namely, $m = 0.05$ and $m = 1.0$. In these figures, we can also observe how the region of stability gets modified as we incorporate higher-order nonlinear terms in the evolution equation. We find that for small values of m ($m < 1/2$), the region of stability decreases as we include higher-order nonlinear terms in the evolution equation. However, for $m > 1/2$, the region of stability slightly increases as a result of including higher-order nonlinear terms.

In Figure 3 we have shown stable–unstable regions as obtained from condition (3.5), and also that obtained from the fourth-order evolution equation in Hamiltonian form for the case of surface gravity waves. When the instability condition (3.5) is satisfied, the growth rate of instability of the uniform wave train (3.1) is given by

$$G_r = \tfrac{1}{2}[-(E - H)^2 - 4FG]^{1/2}. \tag{3.6}$$

In Figures 4 and 5, we have shown contour plots of the growth rate of instability, taking $m = 0.05$ and $m = 1.0$, respectively. In these figures, we observe the growth rates of instability curves as obtained from the fifth-order evolution equation, and also when higher-order linear dispersive terms are taken in the fifth-order evolution equation. We observe that the effect of higher-order linear dispersive terms is to increase the growth rate of instability very slightly for $m < 1/2$ and to reduce the same slightly for $m > 1/2$.

We can recover the results of the stability analysis for unidirectional perturbations by setting $L = 0$ in equations (3.3)–(3.6). We have plotted the growth rates of instability curves against perturbation wave-number K in Figures 6 and 7 for unidirectional perturbations. In these figures we find that a uniform gravity–capillary wave train becomes unstable when the wave number of the longitudinal perturbations exceeds a certain critical value. This critical value increases with the increase in wave steepness. For $m = 3, 7$ and for $a_0 = 0.05$, Zhang and Melville's [21] full calculation found the values of the perturbation wave-number corresponding to the onset of stability of gravity–capillary waves to be $K = 0.042$ and 0.037, respectively. The corresponding values of K obtained by Hogan [8] are $K = 0.040$ and 0.034, respectively. Using condition (3.5), we have found the corresponding values to be $K = 0.0411$ and 0.036. These values are closer to the estimates of Zhang and Melville [21] than the estimates obtained from the fourth-order evolution equation of Hogan [8].

4. Conclusions

The evolution equation of a weakly nonlinear water wave packet can be derived in a number of ways. If one starts with the Zakharov equation, the Krasitskii kernel [11] should be the first choice for studying matters related to the space-time evolution of weakly nonlinear waves, especially for phenomena that rely on the conservative and Hamiltonian form. Here, starting with the Zakharov integral equation, we have derived an evolution equation for a weakly nonlinear gravity–capillary wave packet, under the assumption that the wave packet is sufficiently narrow. The evolution equation derived here is correct up to fifth order of nonlinearity. Thus, this equation is a modification of the evolution equation obtained by Hogan [8], which is correct up to fourth order of nonlinearity. Using this evolution equation, we have studied stability properties of a uniform gravity–capillary wave train. We find that the region of stability as computed from the fifth-order evolution equation is smaller than that from the fourth-order evolution equation for the case when $m < 1/2$. However, we see the opposite effect when $m > 1/2$. For longitudinal perturbations, we observe that a uniform gravity–capillary wave train becomes unstable when the perturbed wave-number exceeds a certain critical value, which increases as the wave steepness increases.

Appendix A. Coefficients of the evolution equation

$$\beta_0 = \frac{1+3m}{2(1+m)}, \quad \beta_1 = -\frac{1-6m-3m^2}{8(1+m)^2}, \quad \beta_2 = \frac{1+3m}{4(1+m)},$$

$$\beta_3 = -\frac{1+5m-5m^2-m^3}{16(1+m)^3}, \quad \beta_4 = \frac{3+2m+3m^2}{8(1+m)^2},$$

$$\beta_5 = \frac{5+20m+70m^2-28m^3-3m^4}{128(1+m)^4}, \quad \beta_6 = -\frac{15+35m+5m^2+9m^3}{32(1+m)^3},$$

$$\beta_7 = \frac{3+2m+3m^2}{32(1+m)^2}, \quad \beta_8 = \frac{7+3m+54m^2+210m^3-45m^4-3m^5}{256(1+m)^5},$$

$$\beta_9 = \frac{35+120m+150m^2+15m^4}{64(1+m)^4}, \quad \beta_{10} = \frac{21+45m+15m^2+15m^3}{64(1+m)^3},$$

$$\beta_{11} = \frac{21+118m+271m^2+132m^3+1295m^4-42m^5+21m^6}{1024(1+m)^6}, \quad \beta_{12} = \frac{315+1405m}{512(1+m)^5},$$

$$\beta_{13} = \frac{189+624m+690m^2+120m^3+105m^4}{256(1+m)^4}, \quad \beta_{14} = \frac{7+15m+5m^2+5m^3}{128(1+m)^3},$$

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$$\Lambda_1 = \frac{8 + m + 2m^2}{16(1+m)(1-2m)}, \quad \Lambda_2 = -\frac{24 + 93m + 15m^2 + 96m^3 - 60m^4 - 48m^5}{16(1+m)^2(1-2m)^2(1+4m)},$$

$$\Lambda_3 = -\frac{(1-m)(8+m+2m^2)}{32(1+m)^2(1-2m)}, \quad \Lambda_4 = \frac{1}{2}.$$

$$\mu_1 = -(40 + 219m + 1724m^2 + 3899m^3 + 4398m^4 + 1888m^5 + 6928m^6 - 4992m^7 - 3328m^8)/64(1+m)^3(1-2m)^3(1+4m)^2,$$

$$\mu_2 = \frac{35m + 149m^2 + 48m^3 + 32m^4 - 64m^5}{32(1+m)^2(1-2m)^2(1+4m)},$$

$$\mu_3 = -(36 + 343m + 1616m^2 + 5319m^3 + 6638m^4 - 6440m^5 - 1728m^6 - 896m^7 + 512m^8)/64(1+m)^3(1-2m)^3(1+4m)^2,$$

$$\mu_4 = -\frac{20 + 91m + 113m^2 + 300m^3 + 136m^4 + 32m^5}{32(1+m)^2(1-2m)^2(1+4m)},$$

$$\mu_5 = -\frac{4 - 13m - 58m^2 + 131m^3 - 124m^4 + 1268m^5 + 592m^6}{32(1+m)^3(1-2m)^2(1+4m)},$$

$$\mu_6 = \frac{4 + 29m + 25m^2 + 18m^3}{16(1+m)^2(1-2m)}, \quad \mu_7 = \frac{16 + 33m + 108m^2 + 337m^3 + 138m^4}{128(1+m)^3(1-2m)},$$

$$\mu_8 = \frac{16 + 51m + 51m^2 + 34m^3}{64(1+m)^2(1-2m)}, \quad \mu_9 = -\frac{1}{2}, \quad \mu_{10} = -\frac{1}{2},$$

$$\mu_{11} = -\frac{1-m}{4(1+m)}, \quad \mu_{12} = -\frac{1}{2}, \quad \mu_{13} = -\frac{1}{2}, \quad \mu_{14} = -\frac{(1+3m)^2}{8(1+m)}.$$

References

[1] D. R. Crawford, B. M. Lake, P. G. Saffman and H. C. Yuen, "Stability of weakly nonlinear deep-water waves in two and three dimensions", *J. Fluid Mech.* **105** (1981) 177–191; doi:10.1017/S0022112081003169.

[2] S. Debsarma and K. P. Das, "Fourth order nonlinear evolution equations for gravity-capillary waves in the presence of a thin thermocline in deep water", *ANZIAM J.* **43** (2002) 513–524; doi:10.1017/S1446181100012116.

[3] S. Debsarma and K. P. Das, "A higher order nonlinear evolution equation for broader bandwidth gravity waves in deep water", *Phys. Fluids* **17** (2005), 1–8; doi:10.1063/1.2046714.

[4] S. Debsarma and K. P. Das, "A higher order nonlinear evolution equation for much broader bandwidth gravity waves in deep water", *Int. J. Appl. Mech. Eng.* **12** (2007) 557–563.

[5] V. D. Djordjevic and L. G. Redekopp, "On two-dimensional packets of capillary-gravity waves", *J. Fluid Mech.* **79** (1977) 703–714; doi:10.1017/S0022112077000408.

[6] K. B. Dysthe, "Note on a modification to the nonlinear Schrodinger equation for application to deep water waves", *Proc. R. Soc. Lond. Ser. A* **369** (1979) 105–114; doi:10.1098/rspa.1979.0154.

[7] O. Gramstad and K. Trulsen, "Hamiltonian form of the modified nonlinear Schrodinger equation for gravity waves on arbitrary depth", *J. Fluid Mech.* **670** (2011) 404–426; doi:10.1017/S0022112010005355.

[8] S. J. Hogan, "Fourth order evolution equation for deep gravity-capillary waves", *Proc. R. Soc. Lond. A* **402** (1985) 359–372; doi:10.1098/rspa.1985.0122.

[9] S. J. Hogan, I. Gruman and M. Stiassnie, "On the changes in phase speed of one train of water waves in the presence of another", *J. Fluid Mech.* **192** (1988) 97–114; doi:10.1017/S0022112088001806.

[10] F. W. King, *Hilbert transforms* (Cambridge University Press, Cambridge, 2009).

[11] V. P. Krasitskii, "On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves", *J. Fluid Mech.* **272** (1994) 1–20; doi:10.1017/S0022112094004350.

[12] V. P. Krasitskii and V. A. Kalmykov, "Four-wave reduced equations for surface gravity waves", *Izv. Atmos. Phys.* **29** (1993) 222–228.

[13] M. S. Longuet-Higgins, "The instability of gravity waves of finite amplitude in deep water, I. Superharmonics", *Proc. R. Soc. Lond. A* **360** (1978a) 471–488; doi:10.1098/rspa.1978.0080.

[14] M. S. Longuet-Higgins, "The instability of gravity waves of finite amplitude in deep water, II. Subharmonics", *Proc. R. Soc. Lond. A* **360** (1978b) 489–505; doi:10.1098/rspa.1978.0081.

[15] L. F. McGoldrick, "On Wilton's ripple: a special case of resonant interactions", *J. Fluid Mech.* **42** (1970) 193–200; doi:10.1017/S0022112070001179.

[16] J. W. Mclean, Y. C. Ma, D. U. Martin, P. G. Saffman and H. C. Yuen, "Three dimensional instability of finite amplitude water waves", *Phys. Rev. Lett.* **46** (1981) 817–820; doi:10.1103/Phys.Rev.Lett.46.817.

[17] J. W. McLean, "Instabilities of finite amplitude water waves", *J. Fluid Mech.* **114** (1982) 315–330; doi:10.1017/S0022112082000172.

[18] K. Trulsen and K. B. Dysthe, "A modified nonlinear Schrodinger equation for border bandwidth gravity waves on deep water", *Wave Motion* **24** (1996) 281–289; doi:10.1016/S0165-2125(96)00020-0.

[19] K. Trulsen, I. Kliakhandler, K. B. Dysthe and M. G. Velarde, "On weakly nonlinear modulation of waves on deep water", *Phys. Fluids* **12** (2000) 2432–2437; doi:10.1063/1.1287856.

[20] V. E. Zakharov, "Stability of periodic waves of finite amplitude on the surface of a deep fluid", *J. Appl. Mech. Tech. Phys.* **2** (1968) 190–194; doi:10.1007/BF00913182.

[21] J. Zhang and W. K. Melville, "On the stability of weakly-nonlinear gravity-capillary waves", *Wave Motion* **8** (1986) 439–454; doi:10.1016/0165-2125(86)90029-6.

[22] J. Zhang and W. K. Melville, "3-dimensional instabilities of nonlinear gravity capillary waves", *J. Fluid Mech.* **174** (1987) 187–208; doi:10.1017/S0022112087000090.