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THE VANISHING OF POINCARÉ SERIES

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1. Introduction

Every holomorphic modular form of weight $k>2$ is a sum of Poincaré series; see, for example, Chapter 5 of (**5**). In particular, every cusp form of even weight $k \geqq 4$ for the full modular group $\Gamma(1)$ is a linear combination over the complex field $\boldsymbol{C}$ of the Poincaré series

$$G_k(z, m) = \tfrac{1}{2}\sum_T (cz+d)^{-k} \exp\{2\pi i m T(z)\}. \tag{1.1}$$

Here m is any positive integer, $z \in \boldsymbol{H} = \{z \in \boldsymbol{C}: \operatorname{Im} z > 0\}$ and

$$T(z) = \frac{az+b}{cz+d}.$$

The summation is over all matrices

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with different second rows in the (homogeneous) modular group, i.e. in $SL(2, \boldsymbol{Z})$. The factor $\frac{1}{2}$ is introducted for convenience.

When $k = 4, 6, 8, 10$ and 14, the space C_k of cusp forms has dimension zero, so that, in each of these five cases, $G_k(z, m)$ vanishes identically for all positive integers m. We write

$$\mu_k = \dim C_k. \tag{1.2}$$

Then $\mu_k > 0$ for $k = 12$ and all even $k \geqq 16$. In fact (**5**, Theorem 6.1.2), for $k \geqq 4$,

$$\mu_k = \begin{cases} \left[\dfrac{k}{12}\right] & \text{if} \quad k \not\equiv 2 \pmod{12}, \\ \left[\dfrac{k}{12}\right] - 1 & \text{if} \quad k \equiv 2 \pmod{12}. \end{cases} \tag{1.3}$$

It follows that $\mu_k = O(k)$ for large k.

Moreover, when $\mu_k \geqq 1$, the series $G_k(z, m)$ $(1 \leqq m \leqq \mu_k)$ span the space C_k, so that, in particular,

$$G_k(z, m) \not\equiv 0 \quad \text{for} \quad 1 \leqq m \leqq \mu_k. \tag{1.4}$$

See, for example, Theorem 6.2.1 of (**5**).

The object of the present paper is to consider whether (1.4) can be extended to values of m greater than μ_k, when k is large.

It may be noted that, since, as shown in §6.3 of (**5**),

$$G_{12}(z, m)=c_{12}m^{11}\tau(m)\Delta(z), \tag{1.5}$$

where Δ is the discriminant function, $\tau(m)$ is Ramanujan's coefficient, and c_{12} is a positive constant, more information is available in this particular case. For Lehmer has shown that $\tau(m)\neq 0$ for

$$1\leq m\leqq 113\ 740\ 230\ 287\ 998$$

and, accordingly, $G_{12}(z, m)\not\equiv 0$ for these values of m; see §6.6 of (**5**). Formulae similar to (1.5) hold also for $G_k(z, m)$ when $k=16, 18, 20, 22$ and 26.

We prove the following theorem.

Theorem 1. *There exist positive constants k_0 and B, where $B>4\log 2$, such that, for all even $k\geqq k_0$ and all positive integers*

$$m\leqq k^2\exp\{-B\log k/\log\log k\}, \tag{1.6}$$

the Poincaré series $G_k(z, m)$ does not vanish identically.

Note that the theorem remains true for arbitrary $\varepsilon>0$ and suitably large k_0 if the right-hand side of (1.6) is replaced by $k^{2-\varepsilon}$.

The proof uses sharp estimates for the magnitude of Kloosterman sums. If cruder approximations are used, it is easy to replace $k^{2-\varepsilon}$ by the weaker $k^{3/2}$. The method can also be applied to Poincaré series belonging to congruence subgroups of the modular group.

Some additional results concerning the vanishing or non-vanishing of particular subsets of Poincaré series are obtained in §6.

Throughout the paper $A_1, A_2, \ldots, B_1, B_2, \ldots$ denote positive absolute constants; in particular, these parameters do not depend upon the weight k.

2. Preliminaries

It is convenient to follow Petersson (**3**) and introduce the function

$$g_k(z, m)=m^{k-1}G_k(z, m). \tag{2.1}$$

We shall always assume that m and k are positive integers, the latter being even. Write

$$g_k(z, m)=\sum_{r=1}^{\infty} c_k(r, m)e^{2\pi irz}, \tag{2.2}$$

where $z\in \boldsymbol{H}$. Then it is known that

$$c_k(r, m)=(rm)^{(k-1)/2}\left\{\delta_{r,m}+2\pi(-1)^{k/2}\sum_{q=1}^{\infty}\frac{S(r, m; q)}{q}J_{k-1}\left(\frac{4\pi\sqrt{(rm)}}{q}\right)\right\}, \tag{2.3}$$

where $\delta_{r,m}$ is the Kronecker delta, J_{k-1} is the Bessel function of the first kind, and $S(r, m; q)$ is the Kloosterman sum

$$S(r, m; q)=\sum_{\substack{h=1\\(h,q)=1}}^{q}\exp\left\{\frac{2\pi i}{q}(rh+mh')\right\}; \tag{2.4}$$

here $hh'\equiv 1(\text{mod } q)$. See (**4**) or Theorem 5.3.2 of (**5**); (2.3) is a particular case of a result first proved by Petersson in 1931.

Since $S(r, m; q) = S(m, r; q)$, it is clear that

$$c_k(r, m) = c_k(m, r). \tag{2.5}$$

Moreover, since, by (6.2.7) in (**5**),

$$(g_k(z, r), g_k(z, m)) = \omega_k c_k(r, m), \tag{2.6}$$

where the left-hand side denotes the Petersson inner product and ω_k is positive, it follows that $c_k(r, m)$ is real and that

$$c_k(m, m) \geqq 0.$$

Further, we deduce, as Petersson did (**3**), that $g_k(z, m)$, and therefore $G_k(z, m)$ vanishes identically if and only if $c_k(m, m) = 0$. For this reason we use as our starting point the formula

$$c_k(m, m) = m^{k-1}\left\{1 + 2\pi(-1)^{k/2} \sum_{q=1}^{\infty} \frac{S(m, m; q)}{q} J_{k-1}\left(\frac{4\pi m}{q}\right)\right\}. \tag{2.7}$$

3. Estimates of certain arithmetical functions

Kloosterman sums have certain additive properties from which corresponding properties of the coefficients $c_k(r, m)$ can be deduced. For example, from the formula

$$S(rp^\rho, mp^\mu; q) = S(r, mp^{\rho+\mu}; q) + pS(rp^{\rho-1}, mp^{\mu-1}; q/p), \tag{3.1}$$

which holds for any prime p and positive integers q, r, m, ρ, μ satisfying

$$p \mid q, \quad p \nmid r, \quad p \nmid m,$$

we easily deduce that, under the same conditions,

$$c_k(rp^\rho, mp^\mu) = c_k(r, mp^{\rho+\mu}) + p^{k-1} c_k(rp^{\rho-1}, mp^{\mu-1}). \tag{3.2}$$

We use this result in §6.

Further, if $(q_1, q_2) = 1$, then

$$S(u, v; q_1 q_2) = S(u, v\bar{q}_2^2; q_1) S(u, v\bar{q}_1^2; q_2), \tag{3.3}$$

where $q_1\bar{q}_1 \equiv 1 (\text{mod } q_2)$ and $q_2\bar{q}_2 \equiv 1 (\text{mod } q_1)$. The estimation of $S(r, m; q)$ therefore reduces to the estimation of $S(r, m; p^n)$, where p is a prime and n is a positive integer.

From Lemma 8 of Estermann's paper (**1**), and from the proof of his Lemma 9, we deduce that

$$|S(r, m; p^n)| \leqq 2^\alpha p^{(n+h)/2} \quad (p \text{ odd}), \tag{3.4}$$

where $p^h = (r, m, p^n)$ and

$$\alpha = 0 \ (h = n), \quad \alpha = 1 \ (h < n).$$

When $p = 2$, we have, similarly,

$$|S(r, m; 2^n)| \leqq 2^{3\alpha/2} . 2^{(n+h)/2}, \tag{3.5}$$

where $2^h = (r, m, 2^n)$. This result improves slightly Lemma 3 of (**1**) (where $\frac{3}{2}\alpha$ is replaced by $\frac{5}{2}\alpha$) and is easily deduced by a more efficient application of Lemma 2 of that paper.

From (3.3, 4, 5) we easily deduce

Lemma 3.1. *Let $d=(q, r, m)$. Then*

$$|S(r, m; q)| \leqq 2^{\omega(q/d)} q^{1/2} d^{1/2}, \tag{3.6}$$

where

$$\omega(n) = \sum_{p|n} \omega(p) \quad (p \text{ prime}),$$

and where $\omega(2)=\frac{3}{2}$ and $\omega(p)=1 (p>2)$.

It is easy to deduce from the Prime Number Theorem that there exists a positive constant $B_1 > \log 2$ such that

$$2^{\omega(n)} \leqq \exp\left(\frac{B_1 \log x}{\operatorname{loglog} 2x}\right) =: M(x) \quad \text{for} \quad n \leqq x, \tag{3.7}$$

where $x \geqq 2$.

Now let, for any real λ,

$$\sigma_\lambda(n) = \sum_{\substack{d \mid n \\ d>0}} d^\lambda. \tag{3.8}$$

We shall require upper bounds for $\sigma_0(n)$ and $\sigma_{-\frac{1}{2}}(n)$. It is easily deduced from the Prime Number Theorem that

$$\sigma_0(m) \leqq M(m) \quad \text{for} \quad m \geqq 2, \tag{3.9}$$

and that, for some $B_2 > 2$,

$$\sigma_{-1/2}(m) \leqq \exp\left\{\frac{B_2(\log m)^{1/2}}{\operatorname{loglog} 2m}\right\} \quad (m \geqq 2). \tag{3.10}$$

4. Bessel functions

Write

$$\nu = k-1. \tag{4.1}$$

We shall assume that $k \geqq 16$ so that $\nu \geqq 15$; as fractional powers of ν will occur it is convenient to write

$$\sigma = \nu^{-1/6}. \tag{4.2}$$

We obtain upper bounds for $J_\nu(\nu x)$ valid for all $\nu \geqq 15$ and all $x \geqq 0$. These involve various positive constants (independent of ν and x), which we denote by $A_1, A_2, \ldots$.

Lemma 4.1. *For all $x \geqq 0$ and $\nu \geqq 15$,*

$$|J_\nu(\nu x)| \leqq (2\pi\nu)^{-1/2}(\tfrac{1}{2}ex)^\nu.$$

Proof. The upper bound is, in fact, valid for $\nu \geqq 1$. We start from Poisson's integral

$$J_\nu(\nu x) = \frac{2(\frac{1}{2}\nu x)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_0^{\pi/2} \cos(\nu x \cos\theta) \sin^{2\nu}\theta \, d\theta;$$

see formula (5) on p. 48 of (**7**). Hence

$$|J_\nu(\nu x)| \leqq \frac{2(\frac{1}{2}\nu x)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\int_0^{\pi/2}\sin^{2\nu}\theta\, d\theta = \frac{(\frac{1}{2}\nu x)^\nu}{\nu!},$$

from which the required result follows, since, for $\nu \geqq 1$,

$$\nu! \geqq \sqrt{(2\pi)}\,\nu^{\nu+1/2}e^{-\nu}.$$

Now put

$$x_\nu = (1-\sigma^4)^{1/2}, \quad y_\nu = (1+\sigma^4)^{1/2}, \tag{4.3}$$

and write

$$x = \operatorname{sech}\alpha \quad (0 < x \leqq 1), \quad x = \sec\beta \quad (x \geqq 1). \tag{4.4}$$

Here $\alpha > 0$ and $0 \leqq \beta \leqq \frac{1}{2}\pi$. We also write

$$z = \nu(\alpha - \tanh\alpha) \quad (0 < x \leqq 1), \quad z = \nu(\tan\beta - \beta) \quad (x \geqq 1). \tag{4.5}$$

Then, since $\nu \geqq 15$, it follows easily that

$$z \geqq \tfrac{1}{3}\nu\tanh^3\alpha \geqq \tfrac{1}{3} \quad (0 < x \leqq x_\nu), \quad 0 \leqq z < \tfrac{2}{5} \quad (x_\nu \leqq x \leqq 1) \tag{4.6}$$

and

$$0 \leqq z \leqq \tfrac{1}{3} \quad (1 \leqq x \leqq y_\nu), \quad z > \tfrac{3}{10} \quad (y_\nu \leqq x). \tag{4.7}$$

The parameters α and β were introduced by Langer (**2**), but we have written ν for his ρ and replaced $|\xi|$ by z.

Lemma 4.2. *For* $0 < x \leqq 1$ *and* $\nu \geqq 15$,

$$|J_\nu(\nu x)| \leqq \begin{cases} A_1\sigma^3(1-x^2)^{-1/4}\exp\{-\frac{1}{3}\nu(1-x^2)^{3/2}\} & (0 < x \leqq x_\nu), \quad (4.8)\\ A_1\sigma^2 & (x_\nu \leqq x \leqq 1). \quad (4.9)\end{cases}$$

Proof. For $x \leqq x_\nu$ we use formula (64) on p. 59 of (**2**). This express $J_\nu(\nu x)$ as an asymptotic series

$$J_\nu(\nu\operatorname{sech}\alpha) = \frac{e^{-z}}{(2\pi\nu\tanh\alpha)^{1/2}}\left\{1 + O\left(\frac{1}{z}\right)\right\}$$

for large z; note that here "large" means "bounded away from zero". From this and (4.6) we deduce (4.8).

For $x_\nu \leqq x \leqq 1$, z is "small" by (4.6). We use formula (68) on p. 61 of (**2**), which shows that

$$J_\nu(\nu\operatorname{sech}\alpha) = \frac{1}{\pi}\left(\frac{\alpha - \tanh\alpha}{\tanh\alpha}\right)^{1/2}K_{1/3}(z) + O(\nu^{-4/3}).$$

By Basset's formula on p. 172 of (**7**) for the modified Bessel function of the third kind,

$$K_{1/3}(z) = \frac{2^{1/3}\Gamma(\frac{5}{6})}{z^{1/3}\sqrt{\pi}}\int_0^\infty \frac{\cos zu\, du}{(u^2+1)^{5/6}},$$

so that

$$|K_{1/3}(z)| \leqq \frac{2^{1/3}\Gamma(\frac{5}{6})}{z^{1/3}\sqrt{\pi}}\int_0^\infty \frac{du}{(u^2+1)^{5/6}} = \frac{\Gamma(\frac{1}{3})}{(4z)^{1/3}}.$$

From this (4.9) follows, since $\tanh\alpha \leqq \sigma^2$.

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Lemma 4.3. *For $x \geqq 1$ and $\nu \geqq 15$*

$$|J_\nu(\nu x)| \leqq \begin{cases} A_2\sigma^2 \quad (1 \leqq x \leqq y_\nu), & (4.10) \\ A_2\sigma^3(x^2-1)^{-1/4} \quad (y_\nu \leqq x). & (4.11) \end{cases}$$

Proof. For the range $1 \leqq x \leqq y_\nu$ we note from (4.7) that z is small and use formula (66)(a) on p. 60 of (**2**). This gives

$$J_\nu(\nu \sec \beta) = \left(\frac{\tan \beta - \beta}{3 \tan \beta}\right)^{1/2} \{J_{1/3}(z) + J_{-1/3}(z)\} + O(\nu^{-4/3}).$$

Since $J_\mu(z) = O(z^\mu)$ for small z, (4.10) follows.

For $x \geqq y_\nu$ we use formula (63) on p. 58 of (**2**), which shows that

$$J_\nu(\nu \sec \beta) = \frac{2}{(\pi\nu \tan \beta)^{1/2}}\left\{\cos\left(z - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right)\right\},$$

since, by (4.7), z is large. From this (4.11) follows.

We now write x_0 for the positive root of the equation

$$x \exp\{\tfrac{1}{3}(1-x^2)^{3/2}\} = 2e^{-1}, \tag{4.12}$$

so that $x_0 = 0{\cdot}629$ approximately. We use Lemma 4.1 in the range $0 \leqq x \leqq x_0$ and Lemma 4.2 for $x_0 \leqq x \leqq 1$.

It is convenient to define functions f, F, g and G as described below. We define f on $[0, 1]$ as follows:

$$f(x) = \begin{cases} A_3\sigma^3(\tfrac{1}{2}ex)^\nu & (0 \leqq x \leqq x_0), \\ A_4\sigma^3(1-x^2)^{-1/4} \exp\{-\tfrac{1}{3}\nu(1-x^2)^{3/2}\} & (x_0 \leqq x \leqq x_\nu), \\ \sigma^2 & (x_\nu \leqq x \leqq 1). \end{cases} \tag{4.13}$$

The positive constants A_3 and A_4 are chosen to make f continuous on $[0, 1]$; note that they are independent of ν. Moreover $f(x)$, and therefore $x^{1/2}f(x)$, increase for $0 \leqq x \leqq 1$ and

$$f(x) \leqq F(x) \quad (0 \leqq x < 1), \tag{4.14}$$

where $F(x) = f(x)$ for $0 \leqq x \leqq x_\nu$ and $F(x)$ is defined for $x_\nu \leqq x \leqq 1$ by the second formula in (4.13).

We define g on $[0, \infty[$ by

$$g(x) = \left\{\begin{matrix} \sigma^2 x^{1/2} & (1 \leqq x \leqq y_\nu) \\ \dfrac{\sigma^3 x^{1/2}}{(x^2-1)^{1/4}} & (y_\nu \leqq x) \end{matrix}\right\}. \tag{4.15}$$

Then g is continuous and $g(x)$ increases for $1 \leqq x \leqq y_\nu$ and decreases for $y_\nu \leqq x$. Moreover

$$g(x) \leqq G(x) \qquad (x > 1), \tag{4.16}$$

where

$$G(x) = \frac{\sigma^3 x^{1/2}}{(x^2-1)^{1/4}} \quad (x > 1).$$

From Lemmas 4.1, 4.2 and 4.3 we deduce

Lemma 4.4. *A positive constant A_5 exists such that*

$$|J_\nu(\nu x)| \leqq A_5 f(x) \quad \textit{for} \quad 0 \leqq x \leqq 1 \tag{4.17}$$

and

$$|x^{1/2} J_\nu(\nu x)| \leqq A_5 g(x) \quad \textit{for} \quad x \geqq 1. \tag{4.18}$$

5. Proof of Theorem 1

We consider the sum

$$S_m = \sum_{q=1}^{\infty} \frac{S(m, m; q)}{q} J_{k-1}\left(\frac{4\pi m}{q}\right). \tag{5.1}$$

To estimate S_m we use Lemma 3.1 for values of

$$q < \frac{4\pi m}{\nu} = Q, \tag{5.2}$$

and the estimate

$$|S(m, m; q)| \leqq q, \tag{5.3}$$

when $q \geqq Q$. We also put

$$d = (q, m), \quad q = rd. \tag{5.4}$$

Then

$$|S_m| \leqq S'_m + S''_m, \tag{5.5}$$

where

$$S'_m = \sum_{d \mid m} \sum_{r < Q/d} 2^{\omega(r)} r^{-1/2} \left| J_\nu\left(\frac{4\pi m}{rd}\right) \right| \tag{5.6}$$

and

$$S''_m = \sum_{q \geqq Q} \left| J_\nu\left(\frac{4\pi m}{q}\right) \right|. \tag{5.7}$$

In view of what is known about $G_{12}(z, m)$, and since $G_{14}(z, m)$ vanishes identically for $m > 0$, we may assume that $k \geqq 16$, so that $\nu \geqq 15$. We may also assume that

$$m > \mu_k.$$

It is then easily checked that

$$m > \frac{4\pi m}{15} \geqq Q \geqq \frac{4\pi(\mu_k + 1)}{k-1} \geqq \frac{8\pi}{25} > 1. \tag{5.8}$$

Accordingly, by (3.7) and (5.6),

$$S'_m \leqq \frac{M(m)}{Q^{1/2}} \sum_{\substack{d \mid m \\ d < Q}} d^{1/2} T_d, \tag{5.9}$$

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where, for $d \mid m$ and $d<Q$,

$$T_d = \sum_{1 \leqq r < Q/d} \left(\frac{Q}{rd}\right)^{1/2} \left| J_\nu\left(\frac{\nu Q}{rd}\right)\right| \leqq A_5 \sum_{1 \leqq r < Q/d} g\left(\frac{Q}{rd}\right), \tag{5.10}$$

by (4.18).

But, by the properties of the function g,

$$\begin{aligned}\sum_{1 \leqq r \leqq Q/d} g\left(\frac{Q}{rd}\right) &\leqq \int_1^{Q/d} g\left(\frac{Q}{ud}\right) du + 2g(y_\nu)\\ &= \frac{Q}{d}\int_1^{Q/d} x^{-2} g(x)\, dx + 2g(y_\nu)\\ &\leqq \frac{Q}{d}\int_1^{\infty} x^{-2} G(x)\, dx + 2\sigma^2 y_\nu^{1/2}\\ &\leqq \frac{Q\sigma^3}{d}\int_1^{\infty} \frac{x^{-3/2}\, dx}{(x^2-1)^{1/4}} + 3\sigma^2\\ &= \frac{2(2\pi)^{5/2}}{\Gamma^2(\frac{1}{4})} \frac{m\sigma^9}{d} + 3\sigma^2.\end{aligned}$$

Hence, by (5.9) and (5.10),

$$\begin{aligned}S_m' &\leqq A_6 M(m) \sum_{\substack{d \mid m\\ d<Q}} \left\{\left(\frac{m}{d}\right)^{1/2}\sigma^6 + \left(\frac{d}{m}\right)^{1/2}\sigma^{-1}\right\}\\ &\leqq A_6 M(m)\{\sigma^6 m^{1/2}\sigma_{-1/2}(m) + (4\pi)^{1/2}\sigma^2\sigma_0(m)\}.\end{aligned} \tag{5.11}$$

Also, by (5.7), (4.17) and (4.14),

$$\begin{aligned}S_m'' &= \sum_{q \geqq Q} \left| J_\nu\left(\frac{\nu Q}{q}\right)\right| \leqq A_5 \sum_{q \geqq Q} f(Q/q)\\ &\leqq A_5\left\{\int_Q^\infty f(Q/u)\, du + f(1)\right\}\\ &= A_5\left\{Q\int_0^1 x^{-2} f(x)\, dx + \sigma^2\right\}\\ &\leqq A_5\left\{Q\int_0^1 x^{-2} F(x)\, dx + \sigma^2\right\}.\end{aligned} \tag{5.12}$$

Now, by (4.13),

$$\int_0^1 x^{-2} F(x)\, dx = A_3 I_1 + A_4 I_2,$$

where

$$I_1 = \int_0^{x_0} \sigma^3 x^{-2} (\tfrac{1}{2}ex)^\nu\, dx$$

and

$$I_2 = \int_{x_0}^1 \sigma^3 x^{-2}(1-x^2)^{-1/4} \exp\{-\tfrac{1}{3}\nu(1-x^2)^{3/2}\}\, dx.$$

Hence

$$I_1 = \frac{\sigma^3(\frac{1}{2}ex_0)^\nu}{(\nu-1)x_0};$$

note that $\frac{1}{2}ex_0 < 0{\cdot}9$. Also

$$I_2 \leqq \left(\frac{\sigma}{x_0}\right)^3 \int_{x_0}^1 x(1-x^2)^{-1/4} \exp\{-\tfrac{1}{3}\nu(1-x^2)^{3/2}\}\, dx$$

$$\leqq \frac{\sigma^6}{x_0^3\sqrt{3}} \int_1^\infty t^{-1/2}e^{-t}\, dt = \left(\frac{\pi}{3}\right)^{1/2} \frac{\sigma^6}{x_0^3}.$$

Accordingly, by (5.7) and (5.12),

$$S''_m \leqq A_5\sigma^2 + A_7 m\sigma^{15}(\tfrac{1}{2}ex_0)^\nu + A_8 m\sigma^{12} \leqq A_9 m\sigma^{12}, \tag{5.13}$$

if $m \leqq \nu^2$.

From (5.5), (5.11), (5.13) and (3.9) we deduce that

$$|\, S_m \,| \leqq A_6 m^{1/2}\sigma^6 M(m)\sigma_{-\frac{1}{2}}(m) + A_{10}\sigma^2 M^2(m) + A_9 m\sigma^{12}, \tag{5.14}$$

provided that $m \leqq \nu^2$. We now restrict m further by taking it to satisfy (1.6) (recall that $k = \nu + 1$), where $B > 4B_1$. The second and third terms on the right of (5.14) are then clearly $o(1)$ for large k, while the first term is, by (3.10),

$$A_6 \frac{k}{\nu} \exp\left\{\frac{(4B_1 - B)\log k}{2 \operatorname{loglog} k}(1 + o(1))\right\}.$$

It follows that

$$|\, S_m \,| < \frac{1}{2\pi}$$

for sufficiently large k. Hence, by (2.7), $c_k(m, m) > 0$ and this completes the proof of Theorem 1.

6. Further results

We require the following lemma:

Lemma 6.1. (i) *If $c_k(m, m) = 0$, then $c_k(mn, mn) = 0$ whenever $(m, n) = 1$.*

(ii) *If $c_k(m, m) \neq 0$ and p is any prime not dividing m, then, for each integer $\mu \geqq 1$, either* (a) *$c_k(mp^\mu, mp^\mu) \neq 0$ or* (b) *$c_k(mp^{\mu-1}, mp^{\mu+1}) \neq 0$.*

Proof. By repeated applications of (3.2) we find, when $p \nmid m$ and $\mu \geqq 1$,

$$c_k(mp^\mu, mp^\mu) = \sum_{\lambda=0}^{\mu} p^{(k-1)\lambda} c_k(m, mp^{2(\mu-\lambda)}).$$

Similarly, when $\mu \geqq 2$,

$$c_k(mp^{\mu-1}, mp^{\mu+1}) = \sum_{\lambda=0}^{\mu-1} p^{(k-1)\lambda} c_k(m, mp^{2(\mu-\lambda)}).$$

Note that this is also trivially true when $\mu = 1$. From these two formulae we deduce that, when $\mu \geqq 1$,

$$c_k(mp^\mu, mp^\mu) = p^{(k-1)\mu} c_k(m, m) + c_k(mp^{\mu-1}, mp^{\mu+1}). \tag{6.1}$$

Thus, if $c_k(m, m) = 0$ and also $c_k(mp^{\mu-1}, mp^{\mu-1}) = 0$ for any $\mu \geqq 1$, then $g_k(z, mp^{\mu-1}) \equiv 0$ and so the right-hand side of (6.1) vanishes. It follows that $c_k(mp^\mu, mp^\mu) = 0$. By induction on $\mu(\mu = 1, 2, 3, \ldots)$, we deduce that

$$c_k(m, m) = 0 \Rightarrow c_k(mp^\mu, mp^\mu) = 0 \quad \text{for all} \quad \mu \geqq 0,$$

and part (i) of the lemma follows from this.

Moreover, if $c_k(m, m) \neq 0$, it follows from (6.1) that it is not possible for both $c_k(mp^\mu, mp^\mu)$ and $c_k(mp^{\mu-1}, mp^{\mu+1})$ to be zero. This gives part (ii).

From Lemma 6.1 and (2.5) we deduce

Theorem 2. (i) *If $g_k(z, m) \equiv 0$, then, for any positive integer n prime to m,*

$$g_k(z, mn) \equiv 0.$$

(ii) *If $g_k(z, m) \not\equiv 0$ and if p is a prime not dividing m and $\mu \geqq 1$, then* (a) $g_k(z, mp^\mu) \not\equiv 0$, *or* (b) $g_k(z, mp^{\mu-1}) \not\equiv 0$ *and* $g_k(z, mp^{\mu+1}) \not\equiv 0$.

As a corollary of part (ii) of the theorem we note that it follows that, if $g_k(z, m) \not\equiv 0$, then it is not possible for $g_k(z, mp^\mu)$ to be identically zero for two consecutive positive integers μ.

Theorem 2(i) can also be deduced from the relation

$$g_k(z, m) \mid T_n = g_k(z, n) \mid T_m = \sum_{d \mid (m, n)} d^{k-1} g_k(z, mn/d^2); \tag{6.2}$$

see equation (23) of (**3**), or Theorem 9.3.1 of (**5**). Here T_n and T_m are Hecke operators.

From (6.2) we also deduce

Theorem 3. *For any positive integer m,*

$$g_k(z, m) \equiv 0 \text{ if and only if } T_m \text{ annihilates } C_k.$$

Proof. That $g_k(z, m) \equiv 0$ implies that $C_k \mid T_m = 0$ follows immediately from the first equation in (6.2). Conversely, if $C_k \mid T_m = 0$ then

$$g_k(z, m) = g_k(z, 1) \mid T_m = 0.$$

In conclusion, we remark that the formula (3.1) is related to the formula

$$S(m, n; q) = \sum_{d \mid (m, n, q)} d S\left(1, \frac{mn}{d^2}; \frac{q}{d}\right) \tag{6.3}$$

quoted by Selberg (**6**). He mentions that the multiplicative properties of cusp form coefficients such as $\tau(n)$ can easily be deduced from (6.3), but gives no details. In this connexion it may be of interest to record that the formula (3.2) was used by the author in his Ph.D. thesis (1940) to prove multiplicative properties of this kind.

7. Postscript

It is possible that recent work of N. V. Kuznetsov on the order of magnitude of partial sums of the form

$$\sum_{0<c\leqq T} \frac{1}{c} S(m, n; c)$$

may enable Theorem 1 to be improved. This is referred to in the preprint entitled "Peterson's hypothesis for forms of zero weight and Linnik's hypothesis" (Akad. Nauk SSSR, Khabarovsk, 1977).

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