DIFFERENTIABLE MANIFOLDS WITH AN AREA MEASURE

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1. Introduction. In this section we fix some notations and give a definition of an area measure on a differentiable manifold, where throughout the paper the word differentiable implies differentiability of class C^∞. Let M denote a differentiable manifold of dimension n and call a set of m linearly independent vectors $\{e_1, \ldots, e_m\}$ at a point of M an m-frame of M. The set E' of all such m-frames can be given the structure of a differentiable fibre bundle over M and we denote the projection of E' onto M by π'.

E' is acted on by the group L_m^+ of non-singular $m \times m$ matrices with positive determinants according to the rule

$$e = \{e_\alpha\} \to e_\beta\, \psi_\alpha{}^\beta = e\psi \qquad (\alpha, \beta = 1, \ldots, m)$$

where e denotes the frame of vectors e_α and ψ is a matrix of components $\psi_\alpha{}^\beta$ contained in L_m^+. The repeated index β implies summation, and this convention is used throughout the paper. The quotient space of E' by this action of L_m^+ is the space E of oriented m-planes of M. It is a differentiable fibre bundle over M and we denote the projection of E onto M by π.

Let ϕ: $M \to \tilde{M}$ be a local differentiable mapping between differentiable manifolds M, $\tilde{M}$. The derived mapping on tangent vectors is denoted by ϕ_*, and the dual mapping on differential forms by ϕ^*. ϕ^* extends to exterior differential forms. The mapping ϕ_* induces a mapping $E' \to \tilde{E}'$ and we shall denote this by the same symbol ϕ_*. We can now give some definitions.

An *area measure* of dimension m $(1 \leqslant m \leqslant n - 1)$ on M is a positive differentiable function L on E' such that $L(e\psi) = \det \psi L(e)$ for all $\psi \in L_m^+$.

A *local equivalence* of two area measures L, $\tilde{L}$ on manifolds M, $\tilde{M}$ is a local diffeomorphism ϕ of a neighbourhood U of M onto a neighbourhood of $\tilde{M}$, such that $\tilde{L}(\phi_*\, e) = L(e)$ for all e in $\pi'^{-1}(U)$.

An *automorphism* of an area measure is a diffeomorphism of M onto itself which is a local equivalence on all neighbourhoods.

We now explain some known ideas in the subject and summarize the results obtained in the present paper. We use the word intrinsic to mean "invariant under all local equivalences of the area measure."

In §3 we impose a regularity condition on the area measure; with this condition imposed the extreme cases $m = 1$, $m = n - 1$ are known as the geometries of Finsler and E. Cartan respectively **(3, 4)**. The area measure for a general m (also known as an areal space) has been studied by E. T. Davies, R. Debever,

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H. Iwamoto, and A. Kawaguchi and his pupils. In the geometries of Finsler and E. Cartan a very important step was the introduction of an intrinsic metric tensor or, equivalently, the introduction of an intrinsic positive definite scalar product in the vector bundle $T' = \pi^{-1}(T(M))$ where $T(M)$ is the tangent bundle to M. This metric tensor was extended to the regular area measure of dimension m by H. Iwamoto **(8)** but the formula given is very complicated. In the present paper we show the existence of intrinsic positive definite scalar products in the differentiable case by using the theory of fibre bundles and some results of R. S. Palais **(13)**. Then, by integrating over the fibres of E we obtain a Riemannian metric on M intrinsically associated with the regular area measure. One immediate consequence of the existence of this metric is that the group of automorphisms of a regular area measure is a Lie group, and the metric can also be used to extend some known theorems of Finsler geometry to a regular area measure of dimension m.

2. A theorem on fibre bundles. In the following we shall call an n-frame of a differentiable manifold M of dimension n a frame of M; the set of all such frames forms a principal bundle P over M. It is known that the fibre bundle E is associated with P, the fibre being the Grassmann manifold $G_{m,n-m}$ of oriented m-planes in the n-dimensional number space R^n, and the action of L_n being induced by its usual action on R^n. We shall need a theorem concerning E which is true for general bundles associated with P. Thus, for the moment, let E denote a general bundle associated with P, with fibre $F = L_n/H$ of dimension N and, as before, let π denote the projection $E \to M$. We use p, f to denote general elements of P, F respectively and f_0 to denote the particular element H. Now we recall that, by definition **(10**, p. 54**)**, E is the set of pairs (p, f) modulo the equivalence relation $(p, f) \sim (pl, l^{-1}f)$ where $l \in L_n$. We shall write $\{p, f\}$ for the equivalence class of (p, f). It is known **(10**, Prop. 5.4**)** that the elements of P can be identified with a set of diffeomorphisms of F onto the fibres of E, the diffeomorphism corresponding to p being given by $f \to pf = \{p, f\}$. It is obvious that

$$(pl)f = p(lf). \tag{2.1}$$

We are now concerned with two principal bundles over E. The first is the induced bundle $\pi^{-1}(P)$, which is, by definition, the set of pairs (p, u) where u is an element of E and p is a frame at πu; we shall call such a pair an induced frame at u. Secondly we consider the bundle Q of N-frames of E whose vectors are tangent to the fibres of E. It is sometimes convenient to use an identification of the m-frames of a differentiable manifold M with the vector space isomorphisms of R^m into the tangent spaces to M, a frame e being identified with the isomorphism which takes the canonical basis of R^m into e. With these preliminaries we can now state

THEOREM 2.1. (i) *$\pi^{-1}(P)$ has a reduced structure P' with group H.*

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(ii) *Q has a reduced structure Q' with group the linear isotropy group H' of F at H.*

(iii) *There is a bundle homomorphism (ϕ', ϕ'') of P' onto Q' with ϕ'' the canonical homomorphism of H onto H'.*

Proof. To prove (i) consider, for a given point u of E, the induced frames (q, u) such that $qf_0 = u$, and define P' to be the union of these frames for all $u \in E$. It is clear from (2.1) that H acts freely and transitively on the elements of P' at u; we omit the remainder of the detail involved in proving that P' is a reduced structure of $\pi^{-1}(P)$.

To prove (ii) it is convenient to identify R^N with the tangent space to F at f_0. Then consider the N-frames at u defined by q_* (restricted to R^N) where q is an element of P such that $qf_0 = u$, and define Q' to be the union of these frames for all $u \in E$. Clearly $Q' \subset Q$ and further, if two elements of Q' at u are defined by q_*, q'_* respectively, we must have $q = q'h$ for some $h \in H$ and hence $q_* = q'_* h_*$. Since H' consists of the isomorphisms h_* restricted to R^N, this shows that H' acts freely and transitively on the elements of Q' at u; again we omit the rest of the detail involved in proving that Q' is a reduced structure of Q.

Finally we define ϕ': $P' \to Q'$ by $\phi'(q, u) = q_*$. Then

$$\phi'(qh, u) = q_* h_* = \phi'(q)\phi''(h)$$

so that the pair (ϕ', ϕ'') gives a bundle homomorphism with ϕ'' the canonical projection of H onto H'.

We now obtain H and H' in case E is the bundle of oriented tangent m-planes of M. Then F is the Grassmann manifold $G_{m,n-m}$ of oriented m-planes through the origin in R^n, and we take f_0 to be the m-plane defined by the first m vectors of the canonical basis of R^n. Thus H is the subgroup of L_n defined by matrices of the form

$$\begin{bmatrix} r & t \\ 0 & s \end{bmatrix}$$

where r is of order $m \times m$ and has positive determinant. The reduced structure P' is formed by the set of induced frames at the elements of E such that the first m vectors of the frame lie in the plane u.

In order to obtain H' we have to identify R^N $(N = m(n - m))$ with the tangent space to F at f_0. In fact we shall use the tensor product $R^m \otimes R^{n-m}$ instead of R^N. Let $e_1, \ldots, e_n$ be the canonical basis for R^n and consider the m-planes defined by the m-frames

$$\{e_\alpha + u_\alpha{}^a e_a\} \qquad (a = m + 1, \ldots, n)$$

where the $u_\alpha{}^a$ are arbitrary real numbers. There is a unique m-frame of the above form corresponding to each such m-plane and the functions thus defined give a coordinate system on a neighbourhood of f_0. We identify the tangent space to F at f_0 with $R^m \otimes R^{n-m}$ by identifying the basis $\partial/\partial u_\alpha{}^a$ with the basis $e_\alpha \otimes e_a$.

A calculation shows that the homomorphism ϕ'' is

$$\begin{bmatrix} r & t \\ 0 & s \end{bmatrix} \to r^* \otimes s$$

where $r^* = (r^{-1})^T$.

3. Transversality, angular metric, and a scalar product in T′. The purpose of this section is to define an intrinsic scalar product in T'. In order to do this we use some ideas from the Calculus of Variations. The proofs of our statements are omitted but they are easy consequences of the definition of an area measure; we refer to **(2, 6)** for more detail.

We first introduce some local coordinate functions on E'. Let U denote a coordinate neighbourhood on M with coordinate functions x^i $(i = 1, \ldots, n)$, and let $e = \{e_\alpha\}$ be an m-frame $\in (\pi')^{-1}(U)$. Coordinate functions x^i, $p_\alpha{}^i$ on $\pi'^{-1}(U)$ are then defined by the equations

$$x^i(e) = x^i(\pi' e), \qquad e_\alpha = p_\alpha{}^i(e)\partial/\partial x^i.$$

The symbol x^i now has two meanings but, whenever it is used, the particular meaning will be clear from the context.

The local differential form on E', $\omega^\alpha = (\partial L/\partial p_\alpha{}^i)dx^i$, is independent of these local coordinates and so is defined globally on E'. Further, the forms $\omega^1, \ldots, \omega^m$ are linearly independent, and we consider the non-zero form of degree m, $(1/L^{m-1})\omega^1 \wedge \ldots \wedge \omega^m$. Let ρ denote the projection of E' onto E; then this form is the image, under ρ^*, of a form Ω on E. For a given point u of E, $\Omega(u)$ is the image, under π^*, of a form Ω_u at πu. We define the concepts of transversality and angular metric in terms of Ω_u.

Transversality. The (non-oriented) plane transversal to an m-plane u is the plane spanned by the vectors λ satisfying $i(\lambda)\Omega_u = 0$ (i denotes the interior product). This plane is of dimension $n - m$ and intersects u in the zero vector only.

Angular metric. For a given point u in E we define a function F_u on the fibre of E through u by

$$F_u(v) = \Omega_u(e_1, \ldots, e_m)/\Omega_v(e_1, \ldots, e_m)$$

where $\{e_\alpha\}$ is any frame in the m-plane v. F_u has a critical point at $v = u$, and the Hessian at u **(12**, p. 74**)** is a quadratic differential form on the fibre at u. This form is differentiable in u and is called the angular metric or the Legendre form.

An area measure is said to be regular if the angular metric is positive definite. In this case the angular metric is a Riemannian metric on the fibres of E.

In order to define an intrinsic positive definite scalar product in the vector bundle T' it is sufficient (and necessary) to obtain an intrinsic reduced structure of the principal bundle $\pi^{-1}(P)$ with an orthogonal group. We have already obtained a reduced structure P' by restricting the discussion to the set of induced frames at the elements u of E such that the first m vectors of the frame lie in the plane u. We obtain a further reduced structure P'' by restricting the discussion to those induced frames $\{e_1, \ldots, e_n\}$ such that

(i) $L(\{e_1, \ldots, e_m\}) = 1$,

(ii) $e_{m+1}, \ldots, e_n$ lie in the plane transversal to u. It is clear that the group of P'' is the subgroup of H formed by the matrices

$$\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}, \qquad \det r = 1.$$

We call this group G. The bundle homomorphism (ϕ', ϕ'') introduced in Theorem 2.1 gives a reduced structure $Q'' = \phi'(P'')$ of Q' with group $\phi''(G)$.

An element q_* of Q' is an isomorphism of $R^m \otimes R^{n-m}$ into the tangent space to the fibre of E through some point u. In the case of a regular area measure the angular metric defines a positive definite scalar product on the tangent space to the fibre at u. Using the isomorphism q_*^{-1} we obtain, for each element of Q', a positive definite scalar product in $R^m \otimes R^{n-m}$ and hence a symmetric positive definite $N \times N$ matrix

$$X_{\alpha a, \beta b} = \langle e_\alpha \otimes e_a, e_\beta \otimes e_b \rangle.$$

Let $\mathfrak{P}$ denote the space of symmetric positive definite matrices of order $N \times N$; the above procedure defines a differentiable mapping ρ' of Q' into $\mathfrak{P}$, and the mapping is equivariant if a right action of H' on $\mathfrak{P}$ is defined by

$$X \to Y^T X Y$$

for $Y \in H'$. Thus, defining a right action of $g \in G$ on Q'' and $\mathfrak{P}$ by the action of $\phi''(g)$, we have differentiable equivariant mappings

$$P'' \xrightarrow{\phi'} Q'' \xrightarrow{\rho'} \mathfrak{P}.$$

We now use the following theorem, which we shall prove in §5.

THEOREM 3.1. *There exists a differentiable map τ of $\mathfrak{P}$ into the space of right cosets $K \backslash G$, where K is the orthogonal subgroup of G, such that*

(i) *τ is equivariant,*

(ii) *$\tau(I_N) = K$, where I_N is the identity matrix of order $N \times N$.*

THEOREM 3.2. *There exists a positive definite scalar product in T' intrinsically associated with a regular area measure.*

Proof. The mapping $\sigma = \tau\rho'\phi'$ is a differentiable equivariant mapping of P'' onto K/G; it is easy to show that the subset $\sigma^{-1}(K)$ of P'' is a reduced structure with the orthogonal group K. As we have mentioned, this defines a positive definite scalar product in T' and, since all our constructions are intrinsic, the scalar product is intrinsic.

Remarks. In the geometries of Finsler and E. Cartan $(m = 1, n - 1)$ $\mathfrak{P}$ consists of a single orbit under the action of G and there is clearly a unique differentiable map τ satisfying the conditions of Theorem 3.1. It can be shown that the scalar products given by Theorem 3.2 coincide with those given in **(3, 4)**.

For a particular area measure, τ has only to be defined on the image of ρ'. Thus, if the image of ρ' consists of the single orbit containing I_N, τ is uniquely defined on this orbit. However, except for the cases $m = 1, n - 1$, the theorem of K. Tandai **(14)** then implies that the area measure is obtained from a Riemannian metric on M. For a direct proof of this result see **(1**, Theorem 6.2**)**.

4. An intrinsic Riemannian metric and some applications. We have introduced an intrinsic positive definite scalar product into the vector bundle $T' = \pi^{-1}(T(M))$ where $T(M)$ is the tangent bundle to M.

T' is the set of pairs (v, λ) where $v \in E$ and λ is a tangent vector at πv. Thus, given a tangent vector λ to M of origin x_0 we can define vectors $\lambda_v = (v, \lambda)$ at all points v in $\pi^{-1}(x_0) = E(x_0)$. Now orientate the fibres $E(x)$ continuously in a neighbourhood of x_0 and choose the volume element dV of the angular metric to agree with this orientation. Then we define a positive definite Riemannian metric in $T(M)$ by

$$\langle\lambda, \mu\rangle = \int_{E(x)} \langle\lambda_V, \mu_V\rangle dV / \int_{E(x)} dV.$$

This metric is differentiable and, since it is independent of the chosen local orientation, is defined globally on M. It is also intrinsically associated with the area measure and we call it the associated Riemannian metric.

THEOREM 4.1. *The group of automorphisms of a regular area measure is a Lie transformation group with respect to the compact-open topology.*

Proof. It is well known that the group of isometries of a Riemannian manifold is a Lie transformation group with respect to the compact-open topology. The group of automorphisms of a regular area measure is clearly a closed subgroup of the group of isometries of the associated Riemannian metric and is thus a Lie transformation group.

Remark. In E. Cartan's treatment of the cases $m = 1$ and $m = n - 1$ the area measure was not required to be defined for all m-frames but only on some open set of E' covering M. An example due to E. Cartan **(4**, p. 3**)** shows that Theorem 4.1 is not in general true in this situation. However it is still valid in the case $m = 1$; this follows from S. Chern's solution of the local equivalence problem **(5)** and a theorem of S. Kobayashi **(9)**.

In §3 we expressed the function L in terms of local coordinates x^i, $p_\alpha{}^i$ and now we use this expression to define the concept of local triviality. An area measure is said to be locally trivial if each point of M lies in the domain of a local coordinate system x^i such that L is a function of the coordinates $p_\alpha{}^i$ only.

THEOREM 4.2. *A manifold which admits a regular, locally trivial, area measure also admits a locally Euclidean metric.*

Proof. By following through the computations one sees that, if the function L depends only on the coordinates $p_\alpha{}^i$, then the associated Riemannian scalar product is independent of the coordinates x^i. Thus the associated Riemannian metric is locally Euclidean. Since a regular, locally trivial, area measure of dimension one is the same as a locally Minkowskian space, this theorem generalizes a remark of Lichnerowicz (**11**, p. 297).

THEOREM 4.3. *Let A be the group of automorphisms of a regular area measure on a connected differentiable manifold M of dimension n, and suppose that either $n \neq 4$ and* $\dim A > \frac{1}{2}n(n-1)+1$, *or $n = 4$ and* $\dim A > 8$. *Then the area measure is that defined by a Riemannian metric of constant curvature.*

Proof. A is a subgroup of the group of isometries of the associated Riemannian metric and thus the isotropy group A_0 at a point O of M is isomorphic with the linear isotropy group A_0'. Choose an n-frame P_0 at O, which is orthonormal with respect to the associated Riemannian metric. Using this frame as basis, A_0' is represented by a group B of orthogonal matrices. For $n = 4$ we have

$$\dim B = \dim A_0 > \tfrac{1}{2}(n-1)(n-2)$$

and, for $n = 4$, $\dim B = \dim A_0 > 4$. It is well known that these inequalities imply that B is either the orthogonal group O_{n-1} or the rotation group SO_{n-1}, and thus A_0' is transitive on the directions at O. The argument given on pp. 262–263 of (**17**) shows that A is transitive on M and has dimension $\frac{1}{2}n(n+1)$.

We now take all images of the frame P_0 under the action of A and thus obtain a reduced structure of the bundle P with group B. Since A_0' is transitive on oriented m-planes, the area measure must take the form

$$L(e) = w(x) \text{ (sum of the squares of all } m \times m \text{ minors of the matrix } P_\alpha{}^i)^{\frac{1}{2}}$$

where e is any m-frame at x whose vectors e_α have components $P_\alpha{}^i$ with respect to a frame of the reduced structure. L is invariant under the action of A and this implies that $w(x)$ is constant on M. Thus the area measure coincides with that derived from a constant multiple of the associated Riemannian metric and, as this metric admits a group of isometries of dimension $\frac{1}{2}n(n+1)$, it must have constant curvature. We remark that this theorem is well known for Finsler geometry and is due to H. C. Wang (**15**). A similar theorem for Cartan geometry is stated by Y. Tashiro (**15**, Theorem 8.1) but is proved under different hypotheses.

5. Proof of Theorem 3.1. It is convenient to identify G with $SL_m \times L_{n-m}$ so that K is identified with $SO_m \times O_{n-m}$. We recall that $\mathfrak{P}$ is the space of positive definite matrices of order $N \times N$ and that G acts on $\mathfrak{P}$ according to the rule $X \to Y^T X Y$ where $Y = r^* \otimes s = \phi''(r \times s)$ and $r \times s \in G$. We write $G'' = \phi''(G)$.

Our aim is to construct an equivariant map of $\mathfrak{P}$ into the space of right cosets $K\backslash G$. We shall use some general results, due to R. S. Palais (**13**), concerning a differentiable manifold on which there is a differentiable action by a Lie group G. We remark that Palais' results are formulated in terms of a left action but are also true for a right action. In order to be able to apply these results we must prove

LEMMA 5.1. *The space $\mathfrak{P}$ is proper under the action of G.*

Proof. For the definition of proper see (**13**, Definition 1.2.2). Since the space $\mathfrak{P}$ is locally compact, it is sufficient to prove (**13**, Theorem 1.2.9) that, if $\mathfrak{W}$ is any compact subset of $\mathfrak{P}$, then the set of elements g of G such that $\mathfrak{W}g$ intersects $\mathfrak{W}$ is compact. Let $\mathfrak{R}$ denote the set of matrices Y such that $Y^T W Y \in \mathfrak{W}$ for some $W \in \mathfrak{W}$; we have to show that $(\phi'')^{-1}(\mathfrak{R} \cap G'')$ is compact and, since ϕ'' is at most a double covering, it is sufficient to show that $\mathfrak{R} \cap G''$ is a compact subset of G''. The correspondence $X \to X^{\frac{1}{2}}$ is a homeomorphism of $\mathfrak{P}$ onto itself and we denote the image of $\mathfrak{W}$ by $\mathfrak{W}'$. Then $\mathfrak{R}$ is the set of matrices Y such that $Y^T P^2 Y = Q^2$ for some P, Q in the compact set $\mathfrak{W}'$. This means that $\mathfrak{R}$ is the set $(\mathfrak{W}')^{-1} O_N \mathfrak{W}'$ and is thus a compact subset of L_N. Since G'' is a closed subset of L_N, $\mathfrak{R} \cap G''$ is compact in G''.

LEMMA 5.2. *Let G_1, G_2 denote two Lie groups; then every compact subgroup C of the direct product $G_1 \times G_2$ is contained in a direct product $C_1 \times C_2$ where C_1 and C_2 are compact subgroups of G_1 and G_2 respectively.*

Proof. Define C_1 to be the subgroup of G_1 formed by those elements g_1 such that $g_1 \times g_2$ is contained in C for some g_2. Define C_2 as a subgroup of G_2 in a similar way. Now consider a sequence g_{1r} in C_1; the sequence $g_{1r} \times g_{2r}$ lies in C and, by compactness, has a convergent subsequence $g'_{1r} \times g'_{2r}$ in C. Thus the sequence g_{1r} has a convergent subsequence g'_{1r} in C_1 so that C_1 is compact. Similarly C_2 is compact and C is contained in $C_1 \times C_2$.

LEMMA 5.3. *Each orbit of $\mathfrak{P}$ under G contains an element whose isotropy group is a subgroup of K.*

Proof. By Lemma 5.1 the isotropy group G_X of an element X of $\mathfrak{P}$ is compact and, by Lemma 5.2, is contained in a direct product $C_1 \times C_2$ where C_1, C_2 are compact subgroups of SL_m, L_{n-m} respectively. It is well known that C_1 and C_2 are conjugate to subgroups of SO_m and O_{n-m} so that G_X is conjugate to a subgroup of K. This completes the proof.

To continue with the proof of our theorem, we consider an orbit of $\mathfrak{P}$ under G and choose a point X in this orbit so that the isotropy group G_X of X is a subgroup of K. Since G_X is compact, we can choose a Riemannian metric on $\mathfrak{P}$ invariant under G_X. Let $S(X, a)$ be the union of all geodesic segments of length $<a$ at X orthogonal to the orbit of X. Because of our Lemma 5.1, Section 2.2 and Proposition 2.1.7 of (**13**) show that, for a sufficiently small, $S(X, a)G$ is open in $\mathfrak{P}$ and the correspondence $xg \to G_X\, g$ is a well-defined equivariant map of $S(X, a)G$ onto $G_X\backslash G$. Combining this with the canonical mapping of $G_X\backslash G$ onto $K\backslash G$ we have an equivariant differentiable map of $S(X, a)G$ onto $K\backslash G$.

We wish to show that we can glue local equivariant maps together to give a global one. Our first step is to show that we can choose a countable, locally finite, covering of $\mathfrak{P}$ from those sets $S(X, a)G$ which admit equivariant maps onto $K\backslash G$. Since these sets are open, their projections $U(X, a)$ onto the orbit space $\mathfrak{P}/G$ are open. $\mathfrak{P}$ is locally compact and proper by Lemma 5.1, so that Theorem 1.2.9 of **(13)** applies to show that $\mathfrak{P}/G$ is Hausdorff. Then, since the projection ψ of $\mathfrak{P}$ onto $\mathfrak{P}/G$ is open, it follows that $\mathfrak{P}/G$ is locally compact. Further, because $\mathfrak{P}$ is the union of a countable number of compact subsets, the same is true for $\mathfrak{P}/G$ and, as shown in **(7**, Theorem 2-65**)**,

$$\mathfrak{P}/G = \bigcup_{i=1}^{\infty} W_i$$

where the W_i are open sets with compact closures $\bar{W}_i$ such that $W_i \subset \bar{W}_i \subset W_{i+1}$ for all i. For every point w of the compact set $\bar{W}_i - W_{i-1}$ we can choose sets $U(X, a)$, $U(X, a')$ with $a' < a$ and $\psi X = w$ such that both are contained in the open set $W_{i+1} - \bar{W}_{i-2}$. Since $\bar{W}_i - W_{i-1}$ is compact, we may cover it by a finite number of sets $U(X, a')$, and doing this for each i gives two countable, locally finite, coverings of $\mathfrak{P}/G$. The corresponding sets in $\mathfrak{P}$ will be denoted by $U'_i = S(X_i, a'_i)G$, $U_i = S(X_i, a_i)G$ and both these coverings of $\mathfrak{P}$ are locally finite.

We shall glue the local equivariant mappings together by non-negative differentiable functions ϕ_i on $\mathfrak{P}$ which are invariant under G and have the additional properties:

(i) $\phi_i > 0$ on U'_i,

(ii) the support of ϕ_i is contained in U_i.

The following is one method of constructing the functions ϕ_i. With the invariant Riemannian metric used to construct $S(X_i, a_i)$ define ϕ_i on $S(X_i, a_i)$, as a function of the geodesic distance from X_i, so that it is positive on $S(X_i, a'_i)$ and has support in $S(X_i, a_i)$. Then extend it to U_i by $\phi_i(xg) = \phi_i(x)$ for $x \in S(X_i, a_i)$ and to the whole of $\mathfrak{P}$ by defining it to be zero on $\mathfrak{P} - U_i$.

So far the argument is rather general but the gluing together depends on the special nature of $K\backslash G$. We can identify $K\backslash G$ (as a differentiable manifold with a right action by G) with the space $P \times Q$ where P is the space of positive definite $m \times m$ matrices with unit determinant and Q is the space of positive definite $(n - m) \times (n - m)$ matrices. The right action of G on $P \times Q$ is defined by

$$p \times q \to r^T p r \times s^T q s$$

for $r \times s$ in G, and the identification is

$$\text{coset } (r \times s) \to r^T r \times s^T s.$$

Let f_i be the local equivariant map $U_i \to P \times Q$ and write $f_i(X) = h_i(X) \times k_i(X)$. Then we define an equivariant map $\mathfrak{P} \to P \times Q$ by

$$f(X) = \{(h(X)/(\det h(X))^{1/m}) \times k(X)\},$$

where

$$h(X) = \sum_{i=1}^{\infty} \phi_i(X) h_i(X) \quad \text{and} \quad k(X) = \sum_{i=1}^{\infty} \phi_i(X) k_i(X).$$

In order to complete the proof of Theorem 3.1 we shall show that any equivariant map $f: \mathfrak{P} \to K\backslash G$ can be modified to satisfy condition (ii).

LEMMA 5.5. $f(I_N) = \text{coset } (I_m \times \mu I_{n-m})$ *where* μ *is a positive scalar.*

Proof. Suppose that $f(I_N) = \text{coset } A \times B$ with $A \in SL_m$, $B \in L_{n-m}$. I_N is invariant under K so that A belongs to the normalizer of SO_m in SL_m and B to the normalizer of O_{n-m} in L_{n-m}. Since these normalizers are, respectively SO_m and μO_{n-m} ($\mu > 0$), the result follows.

We now write $\alpha = I_m \times \mu I_{n-m}$ and define $\tau(X) = f(X)\alpha^{-1}$. Obviously $\tau(I_N) = K$ so that the proof of Theorem 3.1 is completed by showing that τ is equivariant. This follows because, for $g \in G$,

$$\tau(Xg) = f(Xg)\alpha^{-1} = f(X)g\alpha^{-1} = f(X)\alpha^{-1}g = \tau(X)g$$

since α commutes with all elements of G.

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