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MIXED NORM INEQUALITIES FOR SOME DIRECTIONAL MAXIMAL OPERATORS

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Abstract

Mixed norm inequalities for directional operators are closely related to the boundedness problems of several important operators in harmonic analysis. In this paper we prove weighted inequalities for some one-dimensional one-sided maximal functions. Then by applying these results, we establish mixed norm inequalities for directional maximal operators which are defined from these one-dimensional maximal functions. We also estimate the constants in these inequalities.

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1. Introduction

Let Σ^{n-1} be the unit sphere in $\mathbb{R}^n$ and Ω be a given function over $\mathbb{R}^n \times \Sigma^{n-1}$. In [2] Calderón and Zygmund considered homogeneous singular integrals with variable kernel defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \Omega(x, y')|y|^{-n} f(x-y)\, dy,$$

where $y' = y/|y|$. If Ω is odd in its second variable, then T_Ω can be represented as

$$T_\Omega f(x) = \frac{1}{2} \int_{\Sigma^{n-1}} \Omega(x, \theta) H_\theta f(x)\, d\theta.$$

Here H_θ is the directional Hilbert transform defined by

$$H_\theta f(x) = \text{p.v.} \int_{\mathbb{R}} f(x - t\theta) t^{-1}\, dt, \qquad x \in \mathbb{R}^n.$$

The boundedness of T_Ω on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, can be obtained if $(\int_{\Sigma^{n-1}} \Omega(x, \theta)^{r^*}\, d\theta)^{1/r^*}$ is bounded as a function of x, where $1 \le r \le \infty$ and $1/r + 1/r^* = 1$, and if the mixed

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norm inequality

$$\left(\int_{\mathbb{R}^n}\left(\int_{\Sigma^{n-1}} |S_\theta f(x)|^r \, d\theta\right)^{q/r} dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p} \tag{1.1}$$

holds for $S_\theta = H_\theta$ and $q = p$; see [2, 3]. Here we let S_θ to be a directional operator defined from some one-dimensional operator. Consider the Riesz potentials

$$I_{\alpha,\Omega} f(x) = \int_{\mathbb{R}^n} \Omega(x, y')|y|^{\alpha-n} f(x-y)\, dy, \quad 0 < \alpha < n.$$

We can write

$$I_{\alpha,\Omega} f(x) = \frac{1}{2}\int_{\Sigma^{n-1}} \Omega(x,\theta) I_{\alpha,\theta} f(x)\, d\theta,$$

where $I_{\alpha,\theta} f(x) = \int_{\mathbb{R}} f(x-t\theta)|t|^{\alpha-1}\, dt$ is the directional Riesz potential. The boundedness of $I_{\alpha,\Omega}$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ is closely related to (1.1) for $S_\theta = I_{\alpha,\theta}$; see [10, 11]. The study of some types of maximal functions and maximal singular integrals rely on (1.1) for $S_\theta = M_\theta$, where

$$M_\theta f(x) = \sup_{h>0} \frac{1}{h}\int_0^h |f(x-t\theta)|\, dt;$$

see [6, 7, 10–12]. Several similar results can also be found in [1, 4, 5, 8, 9], and the references therein.

In this paper we extend M_θ to a more general form $\mathcal{M}^-_{\phi,\theta}$ and investigate the mixed norm inequality

$$\left(\int_{\mathbb{R}^n}\left(\int_{\Sigma^{n-1}} \Omega(x,\theta)\mathcal{M}^-_{\phi,\theta} f(x)^r \, d\theta\right)^{q/r} dx\right)^{1/q} \le C_\Omega\left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p} \tag{1.2}$$

for $1 \le r \le q \le p < \infty$, where Ω is a nonnegative function on $\mathbb{R}^n \times \Sigma^{n-1}$. The directional maximal operator $\mathcal{M}^-_{\phi,\theta}$ is defined as follows. Let ϕ be a nonnegative measurable function defined on $D = \{(z,t) \in \mathbb{R}^2 : t < z\}$. For any measurable function f on $\mathbb{R}$, we define the one-sided maximal function as

$$M^-_\phi f(z) := \sup_{s<z} \frac{1}{\int_s^z \phi(z,t)\, dt}\int_s^z \phi(z,t)|f(t)|\, dt, \quad z \in \mathbb{R}. \tag{1.3}$$

Let $\theta \in \Sigma^{n-1}$, $L_\theta = \{a\theta : a \in \mathbb{R}\}$, and let $L_\theta^\perp$ be the orthogonal complement of L_θ in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, there exists a unique $x_1 \in \mathbb{R}$ and $\bar{x} \in L_\theta^\perp$ such that $x = x_1\theta + \bar{x}$. For any measurable function f on $\mathbb{R}^n$, we define the directional maximal function $\mathcal{M}^-_{\phi,\theta} f$ from $M^-_\phi f$ by

$$\mathcal{M}^-_{\phi,\theta} f(x) := \sup_{h>0} \frac{1}{\int_0^h \phi(x_1, x_1-y)\, dy}\int_0^h \phi(x_1, x_1-y)|f(x-y\theta)|\, dy, \quad x \in \mathbb{R}^n.$$

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If $\phi \equiv 1$, then $\mathcal{M}^-_{\phi,\theta} = M_\theta$. Using Hölder's inequality, we see that the boundedness of operators of the form

$$\mathbb{T}_\Omega f(x) = \int_{\Sigma^{n-1}} \Omega(x, \theta)\mathcal{M}^-_{\phi,\theta} f(x)\, d\theta$$

from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ is closely related to (1.2). The purpose of this paper is to establish (1.2) for $1 \le r \le q \le p < \infty$. Based on the idea given in [9, Section 4.3], we show that (1.2) can be obtained by norm inequalities for M^-_ϕ. In general, we consider the weighted inequality

$$\left(\int_{\mathbb{R}} M^-_\phi f(z)^q u(z)\, dz\right)^{1/q} \le C_\phi \left(\int_{\mathbb{R}} |f(z)|^p v(z)\, dz\right)^{1/p}, \tag{1.4}$$

where $1 < p, q < \infty$, u and v are weights, and the constant C_ϕ is independent of f. Here a weight is a locally integrable function which is positive almost everywhere on $\mathbb{R}$. We prove (1.4) under some increasing conditions on ϕ and also give the estimates of C_ϕ. Then the particular case $p = q$ and $u = v \equiv 1$ of (1.4) is applied to obtain (1.2). We also establish the estimates of C_Ω.

Throughout this paper, we assume that all functions are measurable on their domains. For $0 < z < \infty$, we define z^* by $1/z + 1/z^* = 1$. We also take $0^0 = \infty^0 = 1$ and $\infty/\infty = 0/0 = 0 \cdot \infty = 0$.

2. Main results

We first show that (1.2) can be obtained by (1.4). The method of the proof is based on the idea given in [9, Section 4.3].

THEOREM 2.1. *Let $1 \le r \le q \le p < \infty$. Let Ω be a nonnegative function on $\mathbb{R}^n \times \Sigma^{n-1}$. Suppose that there exist $1 \le \beta \le q/r$, $0 \le m \le \beta^*$, and a positive function w on Σ^{n-1} such that*

$$U(\beta, m) := \left(\int_{\Sigma^{n-1}} \left(\int_{\mathbb{R}^n} A(x, \theta)^{p/(p-q)}\, dx\right)^{r\beta(p-q)/pq} w(\theta)^{1-\beta}\, d\theta\right)^{1/r\beta} < \infty, \tag{2.1}$$

where

$$A(x, \theta) = \Omega(x, \theta)^{(1-m/\beta^*)q/r} \left(\int_{\Sigma^{n-1}} \Omega(x, \tau)^m w(\tau)\, d\tau\right)^{q/r\beta^*}.$$

If (1.4) holds for $p = q$ and $u = v \equiv 1$ with constant C_ϕ, then we obtain (1.2) with

$$C_\Omega \le C_\phi U(\beta, m). \tag{2.2}$$

PROOF OF THEOREM 2.1. Since

$$\begin{aligned}
&\int_{\Sigma^{n-1}} \Omega(x, \theta)\mathcal{M}^-_{\phi,\theta} f(x)^r\, d\theta \\
&\quad = \int_{\Sigma^{n-1}} \Omega(x, \theta)^{1-m/\beta^* + m/\beta^*} \mathcal{M}_{\phi,\theta} f(x)^r w(\theta)^{1/\beta^* - 1/\beta^*}\, d\theta \\
&\quad \le \omega_m(x)^{1/\beta^*} \left(\int_{\Sigma^{n-1}} \Omega(x, \theta)^{(1-m/\beta^*)\beta} \mathcal{M}^-_{\phi,\theta} f(x)^{\beta r} w(\theta)^{1-\beta}\, d\theta\right)^{1/\beta},
\end{aligned}$$

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where $\omega_m(x) = \int_{\Sigma^{n-1}} \Omega(x,\tau)^m w(\tau)\,d\tau$, by Minkowski's integral inequality

$$\begin{aligned}
&\bigg(\int_{\mathbb{R}^n}\bigg(\int_{\Sigma^{n-1}} \Omega(x,\theta)\mathcal{M}^-_{\phi,\theta}f(x)^r\,d\theta\bigg)^{q/r}dx\bigg)^{1/q}\\
&\quad\le \bigg(\int_{\mathbb{R}^n}\bigg(\int_{\Sigma^{n-1}} \Omega(x,\theta)^{(1-m/\beta^*)\beta}\mathcal{M}^-_{\phi,\theta}f(x)^{\beta r}w(\theta)^{1-\beta}\,d\theta\bigg)^{q/r\beta}\omega_m(x)^{q/r\beta^*}\,dx\bigg)^{1/q} \qquad (2.3)\\
&\quad\le \bigg(\int_{\Sigma^{n-1}}\bigg(\int_{\mathbb{R}^n} \mathcal{M}^-_{\phi,\theta}f(x)^q A(x,\theta)\,dx\bigg)^{r\beta/q}w(\theta)^{1-\beta}\,d\theta\bigg)^{1/r\beta},
\end{aligned}$$

where $A(x,\theta) = \Omega(x,\theta)^{(1-m/\beta^*)q/r}\omega_m(x)^{q/r\beta^*}$. We have

$$\int_{\mathbb{R}^n} \mathcal{M}^-_{\phi,\theta}f(x)^q A(x,\theta)\,dx \le \bigg(\int_{\mathbb{R}^n} \mathcal{M}^-_{\phi,\theta}f(x)^p\,dx\bigg)^{q/p}\bigg(\int_{\mathbb{R}^n} A(x,\theta)^{p/(p-q)}\,dx\bigg)^{(p-q)/p}. \qquad (2.4)$$

If (1.4) holds for $p = q$ and $u = v \equiv 1$ with constant C_ϕ, then

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{M}^-_{\phi,\theta}f(x)^p\,dx &= \int_{L_\theta^\perp}\int_{\mathbb{R}} \mathcal{M}^-_{\phi,\theta}f(x_1\theta+\bar{x})^p\,dx_1\,d\bar{x}\\
&= \int_{L_\theta^\perp}\int_{\mathbb{R}} M^-_\phi(f(\cdot\theta+\bar{x}))(x_1)^p\,dx_1\,d\bar{x} \qquad (2.5)\\
&\le C_\phi^p\int_{L_\theta^\perp}\int_{\mathbb{R}} |f(x_1\theta+\bar{x})|^p\,dx_1\,d\bar{x} = C_\phi^p\int_{\mathbb{R}^n}|f(x)|^p\,dx.
\end{aligned}$$

Putting (2.3)–(2.5) together yields (1.2) and (2.2). □

In the case $p = q$, the item $(\int_{\mathbb{R}^n} A(x,\theta)^{p/(p-q)}\,dx)^{r\beta(p-q)/pq}$ in (2.1) is understood to be $(\sup_{x\in\mathbb{R}^n} A(x,\theta))^{r\beta/q}$. If $\beta = 1$, then (2.1) can be reduced to

$$U(1,m) = \bigg(\int_{\Sigma^{n-1}}\bigg(\int_{\mathbb{R}^n} \Omega(x,\theta)^{pq/(pr-qr)}\,dx\bigg)^{r(p-q)/pq}d\theta\bigg)^{1/r} < \infty.$$

If $\beta = q/r$, then $0 \le m \le (q/r)^*$ and (2.1) can be reduced to

$$U(q/r,m) = \bigg(\int_{\Sigma^{n-1}}\bigg(\int_{\mathbb{R}^n} A(x,\theta)^{p/(p-q)}\,dx\bigg)^{(p-q)/p}w(\theta)^{1-q/r}\,d\theta\bigg)^{1/q} < \infty,$$

where

$$A(x,\theta) = \Omega(x,\theta)^{m+(1-m)q/r}\bigg(\int_{\Sigma^{n-1}}\Omega(x,\tau)^m w(\tau)\,d\tau\bigg)^{q/r-1}.$$

On the other hand, if Ω is independent of x, then we simply write $\Omega(x,\theta) = \Omega(\theta)$. In the case $p = q$, $U(\beta,m)$ in (2.1) is reduced to

$$U(\beta,m) = \bigg(\int_{\Sigma^{n-1}}\Omega(\theta)^m w(\theta)\,d\theta\bigg)^{1/r\beta^*}\bigg(\int_{\Sigma^{n-1}}\Omega(\theta)^{(1-m/\beta^*)\beta}w(\theta)^{1-\beta}\,d\theta\bigg)^{1/r\beta}.$$

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If we choose $\beta = p/r$, then there are three particular cases:

$$\begin{cases} U(p/r, 0) = \Big(\int_{\Sigma^{n-1}} w(\theta)\, d\theta\Big)^{1/r-1/p} \Big(\int_{\Sigma^{n-1}} \Omega(\theta)^{p/r} w(\theta)^{1-p/r}\, d\theta\Big)^{1/p} \\ U(p/r, (p/r)^*) = \Big(\int_{\Sigma^{n-1}} \Omega(\theta)^{p/(p-r)} w(\theta)\, d\theta\Big)^{1/r-1/p} \Big(\int_{\Sigma^{n-1}} w(\theta)^{1-p/r}\, d\theta\Big)^{1/p} \\ U(p/r, 1) = \Big(\int_{\Sigma^{n-1}} \Omega(\theta) w(\theta)\, d\theta\Big)^{1/r-1/p} \Big(\int_{\Sigma^{n-1}} \Omega(\theta) w(\theta)^{1-p/r}\, d\theta\Big)^{1/p}. \end{cases}$$

The following theorem can be proved by a similar proof to that given in [13, Lemma 21.75 and Theorem 21.76].

THEOREM 2.2. *Let $1 < p < \infty$. Suppose that σ is a locally integrable function which is positive almost everywhere on $\mathbb{R}$. Let M_σ^- be defined as in (1.3) with $\phi(z, t)$ replaced by $\sigma(t)$. Then for any nonnegative f on $\mathbb{R}$,*

$$\Big(\int_{\mathbb{R}} M_\sigma^- f(z)^p \sigma(z)\, dz\Big)^{1/p} \le p^* \Big(\int_{\mathbb{R}} |f(z)|^p \sigma(z)\, dz\Big)^{1/p}.$$

In the following we establish (1.4) under some increasing conditions on ϕ. Suppose that $1 < p, q < \infty$, u and v are weights, and $\sigma = g^{p^*} v^{1-p^*}$. Then for $0 \le \epsilon \le \min\{1, p/q\}$ we define

$$U_\phi^\epsilon(z) = \sup_{s<z} \frac{1}{\int_s^z \phi(z, t)\, dt} \int_s^z \phi(z, t) \frac{\sigma}{g}(t) \Big(\int_t^z \sigma(y)\, dy\Big)^{(\epsilon-1)/p} dt, \tag{2.6}$$

$$U_\phi^\epsilon = \Big(\int_{\mathbb{R}} U_\phi^\epsilon(z)^{pq/(p-\epsilon q)} u(z)^{p/(p-\epsilon q)} \sigma(z)^{\epsilon q/(\epsilon q-p)}\, dz\Big)^{(p-\epsilon q)/pq}.$$

In the case $p \le q$ and $\epsilon = p/q$, it is understood that

$$U_\phi^\epsilon = \sup_{z\in\mathbb{R}} U_\phi^\epsilon(z) (u(z)/\sigma(z))^{1/q}.$$

THEOREM 2.3. *Let $1 < p, q < \infty$. Suppose that $\phi = g\psi$, where g is a function positive almost everywhere on $\mathbb{R}$, ψ is a nonnegative function defined on D, and $\psi(z, \cdot)$ is increasing and left continuous for each $z \in \mathbb{R}$. Suppose that u and v are weights such that $\sigma = g^{p^*} v^{1-p^*}$ is locally integrable. Then (1.4) holds with*

$$C_\phi \le \inf_{0\le\epsilon\le\min\{1,p/q\}} \Big(\frac{1}{p^*} + \frac{\epsilon}{p}\Big) (p^*)^\epsilon U_\phi^\epsilon. \tag{2.7}$$

PROOF. It suffices to prove (1.4) for nonnegative f. Let $a < z$. Let $\Lambda_{\psi(z,\cdot)}$ be the Lebesgue–Stieltjes measure on $(-\infty, z)$ generated by $\psi(z, \cdot)$ defined by $\Lambda_{\psi(z,\cdot)}([a, b)) = \psi(z, b) - \psi(z, a)$ for $[a, b) \subset (-\infty, z)$. Then $\psi(z, t) = \psi(z, a) + \int_{[a,t)} d\Lambda_{\psi(z,\cdot)}$ for all $a < t < z$. Let $h = (g/v)^{1-p^*} f$. By Fubini's theorem we see that, for any nonnegative f,

$$\int_a^z \phi(z, t) f(t)\, dt = \psi(z, a) \int_a^z \sigma(t) h(t)\, dt + \int_{[a,z)} \Big(\int_s^z \sigma(t) h(t)\, dt\Big)\, d\Lambda_{\psi(z,\cdot)}. \tag{2.8}$$

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Let M_σ^- be defined as in (1.3) with $\phi(z,t)$ replaced by $\sigma(t)$. Then

$$\int_s^z \sigma(t)h(t)\,dt \le \Big(\int_s^z \sigma(t)\,dt\Big) M_\sigma^- h(z)$$

for $a \le s < z$. On the other hand, by Hölder's inequality,

$$\int_s^z \sigma(t)h(t)\,dt \le \Big(\int_s^z \sigma(t)\,dt\Big)^{1/p^*}\Big(\int_{\mathbb{R}} h(t)^p\sigma(t)\,dt\Big)^{1/p}.$$

Therefore, for any $0 \le \epsilon \le \min\{1, p/q\}$,

$$\int_s^z \sigma(t)h(t)\,dt \le \Big(\int_s^z \sigma(t)\,dt\Big)^{1/p^*+\epsilon/p} M_\sigma^- h(z)^\epsilon \Big(\int_{\mathbb{R}} h(t)^p\sigma(t)\,dt\Big)^{(1-\epsilon)/p} \tag{2.9}$$

for $a \le s < z$. By (2.8)–(2.9),

$$\int_a^z \phi(z,t)f(t)\,dt \le \Psi(z) M_\sigma^- h(z)^\epsilon \Big(\int_{\mathbb{R}} h(t)^p\sigma(t)\,dt\Big)^{(1-\epsilon)/p},$$

where

$$\Psi(z) = \psi(z,a)\Big(\int_a^z \sigma(t)\,dt\Big)^{1/p^*+\epsilon/p} + \int_{[a,z)}\Big(\int_s^z \sigma(t)\,dt\Big)^{1/p^*+\epsilon/p} d\Lambda_{\psi(z,\cdot)}.$$

If $\psi(z,\cdot)$ is bounded on $(-\infty, z)$, then, using integration by parts,

$$\Psi(z) = \Big(\frac{1}{p^*} + \frac{\epsilon}{p}\Big)\int_a^z \psi(z,s)\sigma(s)\Big(\int_s^z \sigma(t)\,dt\Big)^{(\epsilon-1)/p} ds.$$

This implies that

$$\begin{aligned}\int_a^z \phi(z,t)f(t)\,dt \le \Big(\frac{1}{p^*} + \frac{\epsilon}{p}\Big) M_\sigma^- h(z)^\epsilon \Big(\int_{\mathbb{R}} h(t)^p\sigma(t)\,dt\Big)^{(1-\epsilon)/p} \\ \times \int_a^z \psi(z,s)\sigma(s)\Big(\int_s^z \sigma(t)\,dt\Big)^{(\epsilon-1)/p} ds.\end{aligned} \tag{2.10}$$

This inequality still holds when $\psi(z,\cdot)$ is not bounded on $(-\infty, z)$ since we can replace ψ by ψ_m in (2.10), where $\{\psi_m(z,\cdot)\}$ is an increasing sequence of bounded increasing and left continuous functions such that $\psi_m(z,\cdot) \to \psi(z,\cdot)$ as $m \to \infty$, and then by letting $m \to \infty$ and applying the monotone convergence theorem. Therefore

$$M_\phi^- f(z) \le \Big(\frac{1}{p^*} + \frac{\epsilon}{p}\Big) U_\phi^\epsilon(z) M_\sigma^- h(z)^\epsilon \Big(\int_{\mathbb{R}} h(t)^p\sigma(t)\,dt\Big)^{(1-\epsilon)/p}.$$

Now

$$\begin{aligned}&\Big(\int_{\mathbb{R}} M_\phi^- f(z)^q u(z)\,dz\Big)^{1/q} \\ &\quad\le \Big(\frac{1}{p^*} + \frac{\epsilon}{p}\Big)\Big(\int_{\mathbb{R}} M_\sigma^- h(z)^{q\epsilon} U_\phi^\epsilon(z)^q u(z)\,dz\Big)^{1/q}\Big(\int_{\mathbb{R}} h(z)^p\sigma(z)\,dz\Big)^{(1-\epsilon)/p}.\end{aligned}$$

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By Hölder's inequality,

$$\int_{\mathbb{R}} M_\sigma^- h(z)^{q\epsilon} U_\phi^\epsilon(z)^q u(z)\,dz \le (U_\phi^\epsilon)^q \Big(\int_{\mathbb{R}} M_\sigma^- h(z)^p \sigma(z)\,dz\Big)^{q\epsilon/p}$$
$$\le (p^*)^{q\epsilon}(U_\phi^\epsilon)^q \Big(\int_{\mathbb{R}} h(z)^p \sigma(z)\,dz\Big)^{q\epsilon/p}.$$

This implies (1.4) with $C_\phi \le (1/p^* + \epsilon/p)(p^*)^\epsilon U_\phi^\epsilon$. This estimate holds for arbitrary $0 \le \epsilon \le \min\{1, p/q\}$ and therefore (2.7) is obtained. □

If we choose $\epsilon = 0$, then (2.7) can be reduced to

$$C_\phi \le \frac{1}{p^*}\Big(\int_{\mathbb{R}} U_\phi^0(z)^q u(z)\,dz\Big)^{1/q},$$

where $U_\phi^0(z)$ is given in (2.6) with $\epsilon = 0$. In the case $q < p$, if we choose $\epsilon = 1$, then $U_\phi^1(z) = M_\phi^-(\sigma/g)(z)$ and (2.7) can be reduced to

$$C_\phi \le p^*\Big(\int_{\mathbb{R}} M_\phi^-(\sigma/g)(z)^{pq/(p-q)} u(z)^{p/(p-q)} \sigma(z)^{q/(q-p)}\,dz\Big)^{(p-q)/pq}.$$

In the case $p \le q$, if we choose $\epsilon = p/q$, then the estimate in (2.7) is reduced to

$$C_\phi \le \Big(\frac{1}{p^*} + \frac{1}{q}\Big)(p^*)^{p/q} \sup_{z\in\mathbb{R}} U_\phi^{p/q}(z)\Big(\frac{u(z)}{\sigma(z)}\Big)^{1/q}, \tag{2.11}$$

where $U_\phi^{p/q}(z)$ is given in (2.6) with $\epsilon = p/q$. In particular, if $p = q$ and $u = v$, then (2.11) is reduced to

$$C_\phi \le p^* \sup_{z\in\mathbb{R}} M_\phi^-(\sigma/g)(z)(g(z)/\sigma(z)).$$

If g/v is increasing, then $M_\phi^-(\sigma/g)(z) \le \sigma(z)/g(z)$ and so $C_\phi \le p^*$.

Corollary 2.4. *Let $1 < p < \infty$. Suppose that $\phi = g\psi$, where g is a locally integrable function which is positive almost everywhere on $\mathbb{R}$, ψ is a nonnegative function defined on D, and $\psi(z,\cdot)$ is increasing and left continuous for each $z \in \mathbb{R}$. Suppose that v is a weight such that g/v is increasing. Then*

$$\Big(\int_{\mathbb{R}} M_\phi^- f(z)^p v(z)\,dz\Big)^{1/p} \le p^*\Big(\int_{\mathbb{R}} |f(z)|^p v(z)\,dz\Big)^{1/p}.$$

The following corollary can be obtained by Theorem 2.1 and Corollary 2.4 with $v \equiv 1$.

Corollary 2.5. *Let $1 \le r \le q \le p < \infty$. Let Ω, β, m, and w be given as in Theorem 2.1. Suppose that $\phi = g\psi$, where g and ψ are as given in Corollary 2.4, and g is increasing. Then (1.2) holds with*

$$C_\Omega \le p^* U(\beta, m),$$

where $U(\beta, m)$ is defined in (2.1).

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