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THE MINIMAL S^3 WITH CONSTANT SECTIONAL CURVATURE IN CP^n

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Abstract

It is known that the minimal 3-spheres of CR type with constant sectional curvature have been classified explicitly, and also that the weakly Lagrangian case has been studied. In this paper, we provide some examples of minimal 3-spheres with constant curvature in the complex projective space, which are neither of CR type nor weakly Lagrangian, and give the adapted frame of a minimal 3-sphere of CR type with constant sectional curvature.

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1. Introduction

It is known that the minimal surfaces in the complex projective space CP^n have been studied by many geometricians in the past few decades, and various perfect results were obtained. Since the surface has a natural complex structure, most of the profound results more or less depend on the theory of a single complex variable. However, there is no complex structure on the general higher dimensional manifolds, which enhances the difficulty in studying the higher dimensional submanifolds in CP^n. It is known that complex analytic submanifolds and totally real submanifolds are two typical classes among all the submanifolds of a Kähler manifold. In this paper we wish to study a special family of higher dimensional submanifolds in CP^n, that is, S^3 with constant sectional curvature in CP^n.

Let $f : M \to CP^n$ be an immersion and let $F : TM \to TM$ be defined by $\langle F(X), Y\rangle = f^*\Omega(X, Y)$, where $f^*\Omega$ is the pullback of the Kähler form of CP^n; the submanifold M is a weakly Lagrangian (or totally real) submanifold if $F \equiv 0$. Also, M is a CR submanifold if there is a direct sum decomposition of an F-invariant subbundle $TM = V_1 \oplus V_2$ such that $F|_{V_1} = 0$ and $(F|_{V_2})^2 = -\mathrm{id}$. The immersion f is said to be weakly Lagrangian if M is a weakly Lagrangian submanifold. The immersion f is of CR type if M is a CR submanifold.

Bando and Ohnita [1] proved that minimal S^2 in CP^n with constant curvature must be homogeneous and Bolton *et al.* [2] proved some results about the curvature, Kähler angle pinching and rigidity of conformal minimal S^2 in CP^n. For the higher dimensional case, Li [6] studied weakly Lagrangian (totally real) and CR kinds of minimal S^3 with constant sectional curvature in CP^n. Li and Tao [8] studied the equivariant Lagrangian minimal immersion of S^3 into CP^3.

It is well known that the minimal S^2 immersed into CP^n with constant curvature has rigidity, which is one of the Veronese sequences. So, one may ask whether the minimal S^3 with constant sectional curvature isometrically immersed into CP^n also has rigidity. Li [6] and Li and Huang [7] proved that this is true for the minimal S^3 of CR type with constant sectional curvature in CP^n.

Fei *et al.* [5] characterized the minimal weakly Lagrangian (totally real) isometric immersions from S^3 to CP^n by some standard examples. In this paper, besides the CR-type examples in [6], we give some new examples by the unitary representation of SU(2), which are neither weakly Lagrangian nor of CR type. By using the method of moving frames in [4], we get the adapted frame of the minimal S^3 of CR type with constant sectional curvature in CP^n, and also give a new proof of the rigidity theorem in [7].

2. The unitary representation of SU(2)

In this section, we recall some results about the unitary representation of the special unitary group SU(2).

The special unitary group SU(2) is defined by

$$\mathrm{SU}(2) = \left\{ g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,\middle|\, |a|^2 + |b|^2 = 1, a, b \in \mathbb{C} \right\},$$

which is homeomorphic to $S^3 = \{(a, b) \in \mathbb{C}^2 \mid |a|^2 + |b|^2 = 1\}$ in the natural way. The Lie algebra of SU(2) is

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & y \\ -\bar{y} & -ix \end{pmatrix} \,\middle|\, x \in \mathbb{R}, y \in \mathbb{C} \right\},$$

where $i^2 = -1$. We define a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{su}(2)$ by

$$\varepsilon_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

which satisfy

$$[\varepsilon_1, \varepsilon_2] = 2\varepsilon_3, \quad [\varepsilon_3, \varepsilon_1] = 2\varepsilon_2, \quad [\varepsilon_2, \varepsilon_3] = 2\varepsilon_1.$$

The Maurer–Cartan form of SU(2) is given by

$$\Phi \doteq dgg^{-1} = \begin{pmatrix} i\omega_1 & \omega_2 + i\omega_3 \\ -\omega_2 + i\omega_3 & -i\omega_1 \end{pmatrix}.$$

The Maurer–Cartan equation $d\Phi = \Phi \wedge \Phi$ gives

$$d\omega_1 = 2\omega_2 \wedge \omega_3, \quad d\omega_2 = 2\omega_3 \wedge \omega_1, \quad d\omega_3 = 2\omega_1 \wedge \omega_2.$$

Let V_n be the $(n+1)$-dimensional complex vector space of all complex homogeneous polynomials of degree n with respect to the two complex variables z and w. The Hermitian inner product $\langle\, ,\rangle$ on V_n is defined by

$$\langle f, g\rangle \doteq \sum_{k=0}^{n} a_k \bar{b}_k k!(n-k)!$$

for $f = \sum_{k=0}^{n} a_k z^k w^{n-k}$, $g = \sum_{k=0}^{n} b_k z^k w^{n-k} \in V_n$. So, $\{u_{k,n} = z^k w^{n-k}/\sqrt{k!(n-k)!} \mid 0 \le k \le n\}$ is a unitary basis for V_n. The unitary representation ρ_n of SU(2) on V_n is defined by

$$\rho_n(g)f(z, w) \doteq f((z, w)g^{-1}) = f(\bar{a}z + \bar{b}w, -bz + aw)$$

for $g \in \mathrm{SU}(2)$ and $f \in V_n$. The action of $\mathfrak{su}(2)$ on V_n is described as follows:

$$u_{k,n} d\rho_n(\varepsilon) \doteq \frac{d}{dt}(\rho_n(\exp t\varepsilon)(u_{k,n}))|_{t=0}$$

for $0 \le k \le n$ and $\varepsilon \in \mathfrak{su}(2)$. In particular,

$$u_{k,n} d\rho_n(\varepsilon_1) = (n-2k) i u_{k,n}, \tag{2.1}$$

$$u_{k,n} d\rho_n(\varepsilon_2) = \sqrt{k(n-k+1)}\, u_{k-1,n} - \sqrt{(k+1)(n-k)}\, u_{k+1,n}, \tag{2.2}$$

$$u_{k,n} d\rho_n(\varepsilon_3) = \sqrt{k(n-k+1)}\, i u_{k-1,n} + \sqrt{(k+1)(n-k)}\, i u_{k+1,n} \tag{2.3}$$

for $0 \le k \le n$.

It is well known that every finite-dimensional complex representation of $\mathfrak{su}(2)$ can be extended uniquely to a complex representation of $\mathfrak{sl}(2, \mathbb{C})$. Thus, we obtain the representation of $\mathfrak{sl}(2, \mathbb{C})$, which will be denoted also by $d\rho_n$. Select the basis $\{\sigma_1, \sigma_2, \sigma_3\}$ of $\mathfrak{sl}(2, \mathbb{C})$, where

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and denote $d\rho_n(\sigma_1), d\rho_n(\sigma_2), -d\rho_n(\sigma_3)$ by H, A, B, respectively. By (2.1)–(2.3),

$$u_{k,n} H = (n-2k) u_{k,n}, \quad u_{k,n} A = \sqrt{(k+1)(n-k)}\, u_{k+1,n}, \quad u_{k,n} B = \sqrt{k(n-k+1)}\, u_{k-1,n} \tag{2.4}$$

for $0 \le k \le n$. From the Lie algebra homomorphism $d\rho_n$ between $\mathfrak{sl}(2, \mathbb{C})$ and $\mathrm{End}(V_n)$,

$$[H, A] = 2A, \quad [H, B] = -2B, \quad [A, B] = H.$$

The pullback of the Maurer–Cartan form of $\mathrm{Aut}(V_n)$ is

$$d\rho_n \rho_n^{-1} = iH\omega_1 + A\varphi - B\bar{\varphi},$$

where $\varphi = \omega_2 + i\omega_3$.

Set $\lambda_k \doteq n - 2k$, $v_{\lambda_k, n} = u_{k,n}$ for $0 \le k \le n$. An element in $\Delta_n = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is called a weight of the unitary representation ρ_n, and λ_0 is called the highest weight. The representation space V_n decomposes into $V_{\lambda_0,n} \oplus V_{\lambda_1,n} \oplus \cdots \oplus V_{\lambda_n,n}$, where

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$\dim_{\mathbb{C}} V_{\lambda_k,n} = 1$ and $V_{\lambda_k,n} = \mathrm{span}_{\mathbb{C}}\{v_{\lambda_k,n}\}$ is called the weight space with respect to the weight λ_k. By the terminology of weight, (2.4) can be rewritten as

$$v_{\lambda,n}H = \lambda v_{\lambda,n}, \quad v_{\lambda,n}A = a_{\lambda,n}v_{\lambda-2,n}, \quad v_{\lambda,n}B = b_{\lambda,n}v_{\lambda+2,n}, \quad \lambda \in \Delta_n,$$

where $a_{\lambda,n} = \sqrt{(n+1)^2 - (\lambda-1)}/2$ and $b_{\lambda,n} = \sqrt{(n+1)^2 - (\lambda+1)}/2$. Clearly, we have $a_{\lambda+2,n} = b_{\lambda,n}, a_{-n,n} = b_{n,n} = 0$.

It is well known that $\{(V_n, \rho_n) \mid n = 0, 1, 2, \ldots\}$ are all inequivalent irreducible unitary representations of SU(2), and any unitary representation of SU(2) is completely reducible. Up to an isomorphism, a unitary representation ρ of SU(2) can be written as $\rho = \rho_{n_1} \oplus \rho_{n_2} \oplus \cdots \oplus \rho_{n_s}$ for some nonnegative integers $n_1 \geq n_2 \geq \cdots \geq n_s$. The corresponding representation space is $V = V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_s}$. Similarly, we have the weight space decomposition $V = \bigoplus_{\alpha=1}^{s}(\bigoplus_{\lambda} V_{\lambda,n_\alpha}) = \bigoplus_{\lambda} V_\lambda$, where $V_\lambda = \bigoplus_{\alpha=1}^{s} V_{\lambda,n_\alpha}$ is the weight space with respect to the weight λ. There are also similar operators $H, A, B \in \mathrm{End}(V)$ associated to the representation ρ.

3. The minimal 3-folds in CP^n

In this section, we follow the Einstein convention and the ranges of indices:

$$0 \leq A, B, C, \ldots \leq n, \quad 1 \leq \alpha, \beta, \gamma, \ldots \leq n, \quad 1 \leq i, j, k, \ldots \leq 3.$$

Let $f : M \to CP^n$ be an isometric immersion, where $(M, d\tilde{s}^2)$ is a three-dimensional manifold and $d\tilde{s}^2 = \sum_{i=1}^{3} \tilde{\omega}_i^2$. Choose a unitary frame $\{Z_0, Z_1, \ldots, Z_n\}$ of $\mathbb{C}^{n+1}$, $\langle Z_A, Z_B\rangle = \delta_{AB}$, such that $f = [Z_0]$. Suppose that $dZ_A = \theta_{AB}Z_B$; then

$$d\theta_{AB} = \theta_{AC} \wedge \theta_{CB}, \quad \theta_{AB} + \bar{\theta}_{BA} = 0.$$

The Fubini–Study metric on CP^n is $ds^2_{\mathrm{FS}} = \theta_{0\alpha}\bar{\theta}_{0\alpha}$ and its Kähler form is $\Omega = \sqrt{-1}/2\theta_{0\alpha} \wedge \bar{\theta}_{0\alpha}$. Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame of TM and $\{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3\}$ its dual frame. Suppose that $f^*\Omega = J_{ij}\tilde{\omega}_i \wedge \tilde{\omega}_j$, where $J_{ij} = f^*\Omega(e_i, e_j)$. Clearly,

$$0 \leq |f^*\Omega|^2 \doteq \sum_{ij} J_{ij}^2 \leq 2.$$

We know that $|f^*\Omega|^2 = 0$ if and only if f is weakly Lagrangian (or totally real) and $|f^*\Omega|^2 = 2$ if and only if f is of CR type.

In the following, we will analyze the minimal condition of f. By the isometric condition,

$$d\tilde{s}^2 = f^*ds^2_{\mathrm{FS}} = \sum_{i=1}^{3} \tilde{\omega}_i^2.$$

Then the first structure equation of ds^2 is

$$d\tilde{\omega}_i = -\tilde{\omega}_{ij} \wedge \tilde{\omega}_j. \tag{3.1}$$

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Without fear of confusion, set

$$\theta_{0\alpha} = f^*\theta_{0\alpha} = f_{\alpha i}\tilde{\omega}_i, \tag{3.2}$$

where the $f_{\alpha i}$ are some complex functions. Taking the exterior differentiation of (3.2) and, by (3.1),

$$Df_{\alpha i} \wedge \tilde{\omega}_i \doteq f_{\alpha ij}\tilde{\omega}_j \wedge \tilde{\omega}_i = 0,$$

where, by definition,

$$f_{\alpha ij}\tilde{\omega}_j = df_{\alpha i} - f_{\alpha j}\tilde{\omega}_{ji} + f_{\beta i}\theta_{\beta\alpha} - f_{\alpha i}\theta_{00},$$
$$f_{\alpha ij} = f_{\alpha ji}.$$

We know that f is minimal if and only if $\sum_{i=1}^{3} f_{\alpha ii} = 0, \alpha = 1, \ldots, n$; see [3].

In the rest of this section, we assume that $M = S^3$, which, through the homeomorphism to SU(2), is endowed with the bi-invariant metric with constant sectional curvature c. So,

$$\tilde{\omega}_i = \frac{1}{\sqrt{c}}\omega_i, \quad \tilde{\omega}_{12} = \sqrt{c}\omega_3, \quad \tilde{\omega}_{31} = \sqrt{c}\omega_2, \quad \tilde{\omega}_{23} = \sqrt{c}\omega_1. \tag{3.3}$$

Define $\tilde{\varphi} = \tilde{\omega}_2 + i\tilde{\omega}_3$ and let

$$dZ_0 \equiv X_1\tilde{\omega}_1 + X_2\tilde{\omega}_2 + X_3\tilde{\omega}_3 \mod Z_0,$$
$$X_i = f_{\alpha i}Z_\alpha;$$

then

$$\begin{aligned} DX_i &\doteq dX_i - X_j\tilde{\omega}_{ji} - \theta_{00}X_i \\ &= df_{\alpha i}Z_\alpha + f_{\beta i}\theta_{\beta\alpha}Z_\alpha + f_{\beta i}\theta_{\beta 0}Z_0 - f_{\alpha j}\tilde{\omega}_{ji}Z_\alpha - f_{\alpha i}\theta_{00}Z_\alpha \\ &= Df_{\alpha i}Z_\alpha + f_{\beta i}\theta_{\beta 0}Z_0 \\ &= f_{\alpha ij}\tilde{\omega}_j Z_\alpha + f_{\beta i}\theta_{\beta 0}Z_0 \\ &\equiv f_{\alpha ij}\tilde{\omega}_j Z_\alpha \mod Z_0 \end{aligned}$$

and set

$$X = X_1, Y = (X_2 - iX_3)/2, W = (X_2 + iX_3)/2.$$

By the relation (3.3),

$$\begin{aligned} dZ_0 &\equiv X_1\tilde{\omega}_1 + X_2\tilde{\omega}_2 + X_3\tilde{\omega}_3 \mod Z_0 \\ &\equiv X\tilde{\omega}_1 + Y\tilde{\varphi} + W\bar{\tilde{\varphi}} \mod Z_0, \\ DX &= DX_1 = dX_1 - X_2\tilde{\omega}_{21} - X_3\tilde{\omega}_{31} - \theta_{00}X_1 \\ &= dX + \sqrt{c}X_2\tilde{\omega}_3 - \sqrt{c}X_3\tilde{\omega}_2 - \theta_{00}X \\ &= dX - i\sqrt{c}Y\tilde{\varphi} + i\sqrt{c}W\bar{\tilde{\varphi}} - \theta_{00}X, \\ DX_2 &= dX_2 - X_1\tilde{\omega}_{12} - X_3\tilde{\omega}_{32} - \theta_{00}X_2 \\ &= dX_2 - \sqrt{c}X_1\tilde{\omega}_3 + \sqrt{c}X_3\tilde{\omega}_1 - \theta_{00}X_2, \end{aligned}$$

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$$\begin{aligned}
DX_3 &= dX_3 - X_1\tilde\omega_{13} - X_3\tilde\omega_{23} - \theta_{00}X_3\\
&= dX_3 + \sqrt{c}X_1\tilde\omega_2 - \sqrt{c}X_2\tilde\omega_1 - \theta_{00}X_3,\\
DY &= (DX_2 - iDX_3)/2\\
&= dY - \frac{i}{2}\sqrt{c}X\bar{\tilde\varphi} + i\sqrt{c}Y\tilde\omega_1 - Y\theta_{00},\\
DW &= (DX_2 + iDX_3)/2\\
&= dW + \frac{i}{2}\sqrt{c}X\tilde\varphi - i\sqrt{c}W\tilde\omega_1 - W\theta_{00}.
\end{aligned}$$

Set

$$\begin{aligned}
DX &\equiv p_1\tilde\omega_1 + q_1\tilde\varphi + r_1\bar{\tilde\varphi} \mod Z_0,\\
DY &\equiv p_2\tilde\omega_1 + q_2\tilde\varphi + r_2\bar{\tilde\varphi} \mod Z_0,\\
DW &\equiv p_3\tilde\omega_1 + q_3\tilde\varphi + r_3\bar{\tilde\varphi} \mod Z_0;
\end{aligned}$$

then

$$\begin{gathered}
p_1 = f_{11}, q_1 = (f_{12} - if_{13})/2, r_1 = (f_{12} + if_{13})/2,\\
p_2 = (f_{21} - if_{31})/2, q_2 = (f_{22} - f_{33} - 2if_{23})/4, r_2 = (f_{22} + f_{33})/4,\\
p_3 = (f_{21} + if_{31})/2, q_3 = (f_{22} + f_{33})/4, r_2 = (f_{22} - f_{33} + 2if_{23})/4,
\end{gathered}$$

where $f_{ij} = f_{\alpha ij}Z_\alpha$. Consequently, we get the following theorem.

THEOREM 3.1. *Let $f : S^3 \to CP^n$ be an isometric immersion; then f is minimal if and only if one of the following conditions holds:* (1) $p_1 + 4r_2 = 0$; (2) $p_1 + 4q_3 = 0$.

PROOF. It is enough to just note that $p_1 + 4r_2 = p_1 + 4q_3 = \sum_{i=1}^3 f_{ii} = \sum_{i=1}^3 f_{\alpha ii}Z_\alpha$. □

4. The minimal S^3 of CR type in CP^n

Two maps $f, g : S^3 \to CP^n$ are said to be equivalent if there is a holomorphic isometry $A : CP^n \to CP^n$ such that $f = A \circ g$, and f is said to be equivariant if there is a homomorphism $E : S^3 \to U(n+1)$ of a Lie group such that f is equivalent to $\pi \circ E$, where $\pi : U(n+1) \to CP^n = U(n+1)/U(1) \times U(n)$.

In [6], Li provided the following example, which is an equivariant minimal immersion of CR type with constant sectional curvature $c = 1/(m^2 - 1)$.

EXAMPLE 4.1. For a given integer $m \geq 2$, put

$$k = (m-2)(m+1), \quad l = (m+2)(m-1),$$
$$\cos^2 t = \frac{m-1}{2m}, \quad \sin^2 t = \frac{m+1}{2m},$$

where $t \in (0, \pi/2)$. Let

$$f_1 = \sum_{j=0}^{k} \sqrt{\binom{k}{j}} z^j w^{k-j}\varepsilon_j, \quad f_2 = \sum_{j=0}^{l} \sqrt{\binom{l}{j}} z^j w^{l-j}\varepsilon'_j,$$

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where $(z, w) \in S^3 = \{(z, w) \in \mathbb{C}^2 : z\bar{z} + w\bar{w} = 1\}$ and $\{\varepsilon_0, \ldots, \varepsilon_k, \varepsilon'_0, \ldots, \varepsilon'_l\}$ is the natural basis of $\mathbb{C}^{k+l+2} = \mathbb{C}^{k+1} \oplus \mathbb{C}^{l+1}$. Define $f = [e_0] : S^3 \to CP^{k+l+1}$, where $e_0 = (\cos t f_1, \sin t f_2)$.

In [6], Li proved the following result.

THEOREM 4.2. *Let $f : S^3 \to CP^n$ be an equivariant minimal immersion of CR type with constant sectional curvature c. If f is linearly full, then $c = 2/(n+1)$, where $n = 2m^2 - 3$ for some integer $m \geq 2$. Moreover, up to an isometry of S^3, f is equivalent to the immersion defined in Example 4.1.*

Suppose that $f : S^3 \to CP^n$ is a minimal immersion of CR type which is linearly full, and that the induced metric by f is

$$ds^2 = \sum_{j=1}^{3} \tilde{\omega}_j \tilde{\omega}_j = \frac{1}{c} \sum_{j=1}^{3} \omega_j \omega_j.$$

Choose a unitary frame $\{e_0, e_1, \ldots, e_n\}$ on S^3 such that $f = [e_0]$. Since f is of CR type, we assume that

$$\theta_{00} = \frac{i}{c}\omega_1 = \frac{i}{\sqrt{c}}\tilde{\omega}_1.$$

Set

$$\theta_{0\alpha} = a_{\alpha j}\tilde{\omega}_j = \frac{1}{\sqrt{c}} a_{\alpha j}\omega_j, \quad 1 \leq \alpha \leq n,$$
$$e'_j = a_{\alpha j} e_\alpha, \quad 1 \leq j \leq 3;$$

then, by the CR-type condition,

$$de_0 = \theta_{00} e_0 + \theta_{0\alpha} e_\alpha = \frac{i}{c}\omega_1 e_0 + \frac{1}{\sqrt{c}}\omega_1 e'_1 + \frac{1}{\sqrt{c}}\varphi e'_2.$$

So, in the following, we assume that the unitary frame $\{e_0, e_1, \ldots, e_n\}$ on S^3 satisfies

$$de_0 = \frac{i}{c}\omega_1 e_0 + p_1\omega_1 e_1 + r_2\varphi e_2,$$

where $p_1 = r_2 = 1/\sqrt{c}$. Taking the exterior differentiation of θ_{01}, θ_{02},

$$ip_1\varphi \wedge \bar{\varphi} = p_1(\theta_{00} - \theta_{11}) \wedge \omega_1 + r_2\varphi \wedge \theta_{21},$$
$$2ir_2\omega_1 \wedge \varphi = r_2(\theta_{00} - \theta_{22}) \wedge \varphi + p_1\omega_1 \wedge \theta_{12},$$

that is,

$$\theta_{12} = p_2\varphi, \quad p_2 = i\frac{p_1}{r_2},$$
$$(\theta_{00} - \theta_{11}) \wedge \omega_1 = 0,$$

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$$\theta_{00} - \theta_{22} = i\Big(2 - \frac{p_1^2}{r_2^2}\Big)\omega_1.$$

Taking the exterior differentiation of $\theta_{0\alpha} = 0, 3 \le \alpha \le n$,

$$p_1\omega_1 \wedge \theta_{1\alpha} + r_2\varphi \wedge \theta_{2\alpha} = 0. \tag{4.1}$$

Also, we note that Theorem 3.1 leads to

$$\begin{aligned} \theta_{00} - \theta_{11} &= 2i\omega_1, \\ \theta_{1\alpha} \wedge \varphi &= 0. \end{aligned} \tag{4.2}$$

Choose e_3 such that

$$\theta_{13} = r_2p_3\varphi, \quad \theta_{1\alpha} = 0, \quad 4 \le \alpha \le n;$$

by (4.1) and (4.2), we can choose e_4 such that

$$\theta_{23} = p_1p_3\omega_1 + p_1r_3\varphi,$$
$$\theta_{24} = p_1r_4\varphi, \quad \theta_{2\alpha} = 0, \quad 5 \le \alpha \le n.$$

Taking the exterior differentiation of $\theta_{11}, \theta_{12}, \theta_{13}$,

$$|r_2p_3|^2 = \frac{1}{c} - 2 - |p_2|^2, \tag{4.3}$$
$$r_3 = 0, \quad \theta_{23} = p_1p_3\omega_1, \tag{4.4}$$
$$\theta_{11} - \theta_{33} = i\Big(2 + \frac{p_1^2}{r_2^2}\Big)\omega_1. \tag{4.5}$$

Taking the exterior differentiation of $\theta_{1\alpha} = 0, 4 \le \alpha \le n$,

$$\theta_{3\alpha} \wedge \varphi = 0, \tag{4.6}$$

so we can set $\theta_{34} = p_4\varphi$. Taking the exterior differentiation of $\theta_{22}, \theta_{23}, \theta_{24}$,

$$|p_1r_4|^2 = \frac{1}{c} + |r_2|^2 + |p_2|^2 - \Big(2 - \frac{p_1^2}{r_2^2}\Big),$$
$$p_4 = 2i\frac{p_3}{r_4},$$
$$\theta_{22} - \theta_{44} = i\Big(2 - 2\frac{p_3^2}{r_4^2}\Big)\omega_1.$$

Taking the exterior differentiation of $\theta_{2\alpha} = 0, 5 \le \alpha \le n$,

$$p_3\omega_1 \wedge \theta_{3\alpha} + r_4\varphi \wedge \theta_{4\alpha} = 0.$$

By (4.5), we can choose e_5 such that

$$\theta_{35} = r_4p_5\varphi, \quad \theta_{3\alpha} = 0, \quad 6 \le \alpha \le n$$

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and, by (4.6), we can choose e_6 such that

$$\theta_{45} = p_3 p_5 \omega_1 + p_3 r_5 \varphi,$$
$$\theta_{46} = p_3 r_6 \varphi, \quad \theta_{4\alpha} = 0, \quad 7 \le \alpha \le n.$$

Taking the exterior differentiation of $\theta_{33}, \theta_{34}, \theta_{35}$,

$$|r_4 p_5|^2 = \frac{1}{c} + |r_2 p_3|^2 + |p_4|^2 - \Big(4 + \frac{p_1^2}{r_2^2}\Big),$$
$$r_5 = 0, \quad \theta_{45} = p_3 p_5 \omega_1,$$
$$\theta_{33} - \theta_{55} = i\Big(2 + 2\frac{p_3^2}{r_4^2}\Big)\omega_1.$$

Taking the exterior differentiation of $\theta_{3\alpha} = 0, 6 \le \alpha \le n$,

$$\theta_{5\alpha} \wedge \varphi = 0, \tag{4.7}$$

so we can set $\theta_{56} = p_6\varphi$. Taking the exterior differentiation of $\theta_{44}, \theta_{45}, \theta_{46}$,

$$|p_3 r_6|^2 = \frac{1}{c} + |p_1 r_4|^2 + |p_4|^2 - \Big(4 - \frac{p_1^2}{r_2^2} - 2\frac{p_3^2}{r_4^2}\Big),$$
$$p_6 = 3i\frac{p_5}{r_6},$$
$$\theta_{44} - \theta_{66} = i\Big(2 - 3\frac{p_5^2}{r_6^2}\Big)\omega_1.$$

Taking the exterior differentiation of $\theta_{4\alpha} = 0, 7 \le \alpha \le n$,

$$p_5 \omega_1 \wedge \theta_{5\alpha} + r_6 \varphi \wedge \theta_{6\alpha} = 0.$$

By (4.7), we can choose e_7 such that

$$\theta_{57} = r_6 p_7 \varphi, \quad \theta_{5\alpha} = 0, \quad 8 \le \alpha \le n \tag{4.8}$$

and, by (4.8), we can choose e_8 such that

$$\theta_{67} = p_5 p_7 \omega_1 + p_5 r_7 \varphi,$$
$$\theta_{68} = p_5 r_8 \varphi, \quad \theta_{6\alpha} = 0, \quad 9 \le \alpha \le n.$$

Iterating the above process, we reach the following result.

THEOREM 4.3. *Let $f : S^3 \to CP^n$ be a minimal immersion of CR type with constant sectional curvature c which is linearly full; then there is a unitary frame $\{e_0, e_1, \dots, e_n\}$ of $\mathbb{C}^{n+1}$ such that the pullbacks of the Maurer–Cartan forms of $U(n+1)$ are*

$$\theta_{00} = \frac{i}{c}\omega_1, \quad \theta_{01} = p_1\omega_1 = \frac{1}{\sqrt{c}}\omega_1, \quad \theta_{02} = r_2\varphi = \frac{1}{\sqrt{c}}\varphi,$$
$$\theta_{2k-1,2k} = p_{2k}\varphi, \quad \theta_{2k-1,2k+1} = r_{2k} p_{2k+1}\varphi,$$

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$$\theta_{2k,2k+1} = p_{2k-1}p_{2k+1}\omega_1, \quad \theta_{2k,2k+2} = p_{2k-1}r_{2k+2}\varphi,$$
$$\theta_{00} - \theta_{2k-1,2k-1} = i\Big(\sum_{j=1}^{k-1} j\frac{p_{2j-1}^2}{r_{2j}^2} + 2k\Big)\omega_1,$$
$$\theta_{00} - \theta_{2k,2k} = i\Big(-\sum_{j=1}^{k} j\frac{p_{2j-1}^2}{r_{2j}^2} + 2k\Big)\omega_1,$$

where

$$\begin{aligned}
|r_{2k}p_{2k+1}|^2 &= \frac{1}{c} + |r_{2k-2}p_{2k-1}|^2 - |p_{2k}|^2 - \sum_{j=1}^{k-1} j\frac{p_{2j-1}^2}{r_{2j}^2} - 2k,\\
|p_{2k-1}r_{2k+2}|^2 &= \frac{1}{c} + |p_{2k-3}r_{2k}|^2 + |p_{2k}|^2 + \sum_{j=1}^{k} j\frac{p_{2j-1}^2}{r_{2j}^2} - 2k, \qquad (4.9)\\
p_{2k} &= k\frac{p_{2k-1}}{r_{2k}}i, \quad r_0 = 0, \quad p_{-1} = 1, \quad k \geq 1.
\end{aligned}$$

From Theorem 4.3, we conclude that a minimal immersion of CR type is equivariant and, when $k = 1$, from (4.9) we know that $0 \leq c \leq 1/3$. Together with Theorem 4.2, we also get the following rigidity result [7].

THEOREM 4.4. *Let $f : S^3 \to CP^n$ be a minimal immersion of CR type with constant curvature c. If f is linearly full, then $c = 2/(n+1)$, where $n = 2m^2 - 3$ for some integer $m \geq 2$. Moreover, up to an isometry of S^3, f is equivalent to the immersion defined in Example 4.1.*

5. The construction of examples

In this section, we will construct some examples of minimal S^3 in CP^n based on the unitary representation of SU(2). Let

$$Z_0 = av_{t,l}\rho_l + bv_{s,k}\rho_k,$$

where $|a|^2 + |b|^2 = 1$, $l + k - 1 = n$ and $v_{t,l}, v_{s,k}$ are as in the Section 2. Then

$$\begin{aligned}
dZ_0 &= av_{t,l}d\rho_l + bv_{s,k}d\rho_k\\
&= av_{t,l}(iH\omega_1 + A\varphi - B\bar{\varphi})\rho_l + bv_{s,k}(iH\omega_1 + A\varphi - B\bar{\varphi})\rho_k\\
&= i(tav_{t,l} + sbv_{s,k})\omega_1 + (a_{t,l}av_{t-2,l} + a_{s,k}bv_{s-2,k})\varphi\\
&\quad - (b_{t,l}av_{t+2,l} + b_{s,k}bv_{s+2,k})\bar{\varphi},\\
\theta_{00} &= \langle dZ_0, Z_0\rangle = i\,(t|a|^2 + s|b|^2)\omega_1 \doteq iN\omega_1,\\
\langle dZ_0, dZ_0\rangle &= (t^2|a|^2 + s^2|b|^2)\omega_1^2 + [(a_{t,l}^2 + b_{t,l}^2)|a|^2 + (a_{s,k}^2 + b_{s,k}^2)|b|^2]\varphi\bar{\varphi}\\
&= (t^2|a|^2 + s^2|b|^2)\omega_1^2 + \Big(\frac{l^2 + 2l - t^2}{2}|a|^2 + \frac{k^2 + 2k - s^2}{2}|b|^2\Big)\varphi\bar{\varphi},
\end{aligned}$$

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$$
\begin{aligned}
ds^2 &= [t^2|a|^2 + s^2|b|^2 - (t|a|^2 + s|b|^2)^2]\omega_1^2 \\
&\quad + \Big(\frac{l^2+2l-t^2}{2}|a|^2 + \frac{k^2+2k-s^2}{2}|b|^2\Big)\varphi\bar{\varphi}, \\
dZ_0 &\equiv X\tilde{\omega}_1 + Y\tilde{\varphi} + W\bar{\tilde{\varphi}} \quad \bmod Z_0,
\end{aligned}
$$

where

$$
\begin{aligned}
X &= \sqrt{c}[i(t-N)a\nu_{t,l}\rho_l + i(s-N)b\nu_{s,k}\rho_k], \\
Y &= \sqrt{c}[a_{t,l}a\nu_{t-2,l}\rho_l + a_{s,k}b\nu_{s-2,k}\rho_k], \\
W &= \sqrt{c}[-b_{t,l}a\nu_{t+2,l}\rho_l - b_{s,k}b\nu_{s+2,k}\rho_k],
\end{aligned}
$$

$$
\begin{aligned}
DX &= dX - i\sqrt{c}Y\tilde{\varphi} + i\sqrt{c}W\bar{\tilde{\varphi}} - \theta_{00}X \\
&= -c[(t-N)^2a\nu_{t,l}\rho_l + (s-N)^2b\nu_{s,k}\rho_k]\tilde{\omega}_1 \\
&\quad + ic[(t-N-1)a_{t,l}a\nu_{t-2,l}\rho_l + (s-N-1)a_{s,k}b\nu_{s-2,k}\rho_k]\tilde{\varphi} \\
&\quad - ic[(t-N+1)b_{t,l}a\nu_{t+2,l}\rho_l + (s-N+1)b_{s,k}b\nu_{s+2,k}\rho_k]\bar{\tilde{\varphi}}, \\
DY &= dY - \frac{i}{2}\sqrt{c}X\bar{\tilde{\varphi}} + i\sqrt{c}Y\tilde{\omega}_1 - \theta_{00}Y \\
&= ic[(t-N-1)a_{t,l}a\nu_{t-2,l}\rho_l + (s-N-1)a_{s,k}b\nu_{s-2,k}\rho_k]\tilde{\omega}_1 \\
&\quad + c[a_{t-2,l}a_{t,l}a\nu_{t-4,l}\rho_l + a_{s-2,k}a_{s,k}b\nu_{s-4,k}\rho_k]\tilde{\varphi} \\
&\quad + c[(\tfrac{1}{2}t - \tfrac{1}{2}N - b_{t-2,l}a_{t,l})a\nu_{t,l}\rho_l + (\tfrac{1}{2}s - \tfrac{1}{2}N - b_{s-2,k}a_{s,k})b\nu_{s,k}\rho_k]\bar{\tilde{\varphi}}, \\
DW &= dW + \frac{i}{2}\sqrt{c}X\tilde{\varphi} - i\sqrt{c}W\tilde{\omega}_1 - \theta_{00}W \\
&= -ic[(t-N+1)b_{t,l}a\nu_{t+2,l}\rho_l + (s-N+1)b_{s,k}b\nu_{s+2,k}\rho_k]\tilde{\omega}_1 \\
&\quad - c[(\tfrac{1}{2}t - \tfrac{1}{2}N + a_{t+2,l}b_{t,l})a\nu_{t,l}\rho_l + (\tfrac{1}{2}s - \tfrac{1}{2}N + a_{s+2,k}b_{s,k})b\nu_{s,k}\rho_k]\tilde{\varphi} \\
&\quad + c[b_{t+2,l}b_{t,l}a\nu_{t+4,l}\rho_l + b_{s+2,k}b_{s,k}b\nu_{s+4,k}\rho_k]\bar{\tilde{\varphi}}.
\end{aligned}
$$

So,

$$
\begin{aligned}
p_1 &= -c[(t-N)^2a\nu_{t,l}\rho_l + (s-N)^2b\nu_{s,k}\rho_k] + c(t-s)^2|a|^2|b|^2(a\nu_{t,l}\rho_l + b\nu_{s,k}\rho_k), \\
q_1 &= ic[(t-N-1)a_{t,l}a\nu_{t-2,l}\rho_l + (s-N-1)a_{s,k}b\nu_{s-2,k}\rho_k], \\
r_1 &= -ic[(t-N+1)b_{t,l}a\nu_{t+2,l}\rho_l + (s-N+1)b_{s,k}b\nu_{s+2,k}\rho_k], \\
p_2 &= ic[(t-N-1)a_{t,l}a\nu_{t-2,l}\rho_l + (s-N-1)a_{s,k}b\nu_{s-2,k}\rho_k], \\
q_2 &= c(a_{t-2,l}a_{t,l}a\nu_{t-4,l}\rho_l + a_{s-2,k}a_{s,k}b\nu_{s-4,k}\rho_k), \\
r_2 &= c(\tfrac{1}{2}t - \tfrac{1}{2}N - b_{t-2,l}a_{t,l})a\nu_{t,l}\rho_l + (\tfrac{1}{2}s - \tfrac{1}{2}N - b_{s-2,k}a_{s,k})b\nu_{s,k}\rho_k \\
&\qquad + c(b_{t-2,l}a_{t,l}|a|^2 + b_{s-2,k}a_{s,k}|b|^2)(a\nu_{t,l}\rho_l + b\nu_{s,k}\rho_k), \\
p_3 &= -ic[(t-N+1)b_{t,l}a\nu_{t+2,l}\rho_l + (s-N+1)b_{s,k}b\nu_{s+2,k}\rho_k], \\
q_3 &= -c[(\tfrac{1}{2}t - \tfrac{1}{2}N + a_{t+2,l}b_{t,l})a\nu_{t,l}\rho_l + (\tfrac{1}{2}s - \tfrac{1}{2}N + a_{s+2,k}b_{s,k})b\nu_{s,k}\rho_k] \\
&\qquad + c(a_{t+2,l}b_{t,l}|a|^2 + a_{s+2,k}b_{s,k}|b|^2)(a\nu_{t,l}\rho_l + b\nu_{s,k}\rho_k), \\
r_3 &= c(b_{t+2,l}b_{t,l}a\nu_{t+4,l}\rho_l + b_{s+2,k}b_{s,k}b\nu_{s+4,k}\rho_k).
\end{aligned}
$$

From Theorem 3.1, we have the following result.

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THEOREM 5.1. *Let $f = [Z_0] : S^3 \to CP^n$ be an immersion; then f is isometrically minimal if and only if the following conditions hold:*

(1) $t^2|a|^2 + s^2|b|^2 - (t|a|^2 + s|b|^2)^2 = \dfrac{l^2 + 2l - t^2}{2}|a|^2 + \dfrac{k^2 + 2k - s^2}{2}|b|^2;$

(2) $(t - s)^2(|a|^2 - |b|^2) - l^2 - 2l + t^2 + k^2 + 2k - s^2 = 0,$

where $|a|^2 + |b|^2 = 1$.

The above theorem gives

$$|a|^2 = \frac{(t - s)^2 + l^2 + 2l - t^2 - k^2 - 2k + s^2}{2(t - s)^2},$$
$$|b|^2 = \frac{(t - s)^2 - l^2 - 2l + t^2 + k^2 + 2k - s^2}{2(t - s)^2}.$$

In the following examples, we denote by $\{\phi_{i,n}, 0 \le i \le n\}$ the Veronese surfaces determined by the holomorphic map $[\phi_{0,n}] : S^2 \to CP^n$.

EXAMPLE 5.2. For a given integer $m \ge 2$, we put

$$t = l = (m + 2)(m - 1), \quad s = k = (m - 2)(m + 1);$$

then

$$|a|^2 = \frac{m + 1}{2m}, \quad |b|^2 = \frac{m - 1}{2m},$$
$$Z_0 = a\nu_{l,l}\rho_l + b\nu_{k,k}\rho_k = a\phi_{0,l} + b\phi_{0,k}.$$

The curvature is

$$c = \frac{1}{t^2|a|^2 + s^2|b|^2 - (t|a|^2 + s|b|^2)^2} = \frac{1}{m^2 - 1}$$

and the pullback of the Kähler form is

$$\Omega = f^*\Omega = -\frac{i}{2}d\theta_{00} = \frac{1}{2}Nd\omega_1$$
$$= N\omega_2 \wedge \omega_3 = Nc\tilde{\omega}_2 \wedge \tilde{\omega}_3 = \tilde{\omega}_2 \wedge \tilde{\omega}_3.$$

So, we know that $|\Omega|^2 = 2$, that is, $f = [Z_0]$, is of CR type. Note that this example is the one in Li [6].

EXAMPLE 5.3. For an integer m such that $l, k \ge 0$, we put

$$t = l - 2, \quad l = 3m^2 + 13m + 8 \quad (\text{or} = 3m^2 - 7m - 2),$$
$$s = k - 2, \quad k = 3m^2 + 7m - 2 \quad (\text{or} = 3m^2 - 13m + 8);$$

then

$$|a|^2 = \frac{m + 1}{2m}, \quad |b|^2 = \frac{m - 1}{2m},$$
$$Z_0 = a\nu_{l-2,l}\rho_l + b\nu_{k-2,k}\rho_k = a\phi_{1,l} + b\phi_{1,k}.$$

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The curvature is

$$c = \frac{1}{t^2|a|^2 + s^2|b|^2 - (t|a|^2 + s|b|^2)^2} = \frac{1}{(3m+2)(3m+8)} \quad \Big(\text{or } = \frac{1}{(3m-2)(3m-8)}\Big)$$

and the square of the pullback of the Kähler form is

$$|\Omega|^2 = 2|Nc|^2 = 2\left|\frac{3m^2+10m+4}{(3m+2)(3m+8)}\right|^2 \quad \Big(\text{or } = 2\left|\frac{3m^2-10m+4}{(3m-2)(3m-8)}\right|^2\Big).$$

It is easy to verify that $0 < |\Omega|^2 < 2$, so the immersion $f = [Z_0]$ is neither weakly Lagrangian nor of CR type.

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