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Infinitely Many Rotationally Symmetric Solutions to a Class of Semilinear Laplace–Beltrami Equations on Spheres

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Abstract. We show that a class of semilinear Laplace–Beltrami equations on the unit sphere in $\mathbb{R}^n$ has infinitely many rotationally symmetric solutions. The solutions to these equations are the solutions to a two point boundary value problem for a singular ordinary differential equation. We prove the existence of such solutions using energy and phase plane analysis. We derive a Pohozaev-type identity in order to prove that the energy to an associated initial value problem tends to infinity as the energy at the singularity tends to infinity. The nonlinearity is allowed to grow as fast as $|s|^{p-1}s$ for $|s|$ large with $1 < p < (n+5)/(n-3)$.

1 Introduction

The Laplace–Beltrami operator is a generalization to Riemannian manifolds of the Laplacian. For a differentiable function f defined on a Riemannian manifold M, the Laplace–Beltrami operator acting on f is defined as the Laplacian of the extension of f that is constant on normal directions to M (see [6]). If u is a differentiable function defined on the unit sphere in $\mathbb{R}^n$, S^{n-1}, that is rotationally symmetric with respect to the z-axis, an elementary calculation shows that the Laplace–Beltrami operator is given by

$$\Delta_s u(x_1, x_2, \ldots, x_{n-1}, z) = (1-z^2)u'' + (1-n)zu', \tag{1.1}$$

where u' and u'' denote the first and second derivative of u with respect to z. The goal of this paper is to give sufficient conditions for the semilinear Laplace–Beltrami equation

$$\Delta_s u + (1-|z|)f(u) = 0 \tag{1.2}$$

to have infinitely many rotationally symmetric solutions. Throughout this paper we assume that f is super linear, *i.e.*,

$$\lim_{|u|\to+\infty} \frac{f(u)}{u} = +\infty. \tag{1.3}$$

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Also, for the sake of simplicity in the calculations, we assume that f is nondecreasing and that $f(0) = 0$.

Our main result is the following theorem.

Theorem 1.1 *Let $n \geq 3$ and $F(u) = \int_0^u f(s)ds$. If there exists $\theta \in (2, \frac{2n+2}{n-3})$ and $k \in (0,1)$ such that $\theta F(d) - df(d)$ is bounded below and*

$$\lim_{d\to+\infty}\Big(\frac{d}{f(d)}\Big)^{(n+1)/2}\big(\theta F(kd) - df(d)\big) = +\infty, \tag{1.4}$$

then the boundary value problem (1.2) has infinitely many rotationally symmetric solutions on the unit sphere.

From (1.1) we see that finding classical solutions to (1.2) is equivalent to finding solutions to

$$\begin{cases}(1-z^2)u'' + (1-n)zu' + (1-|z|)f(u) = 0,\\ u'(1) = u'(-1) = 0.\end{cases} \tag{1.5}$$

Since every solution to

$$\begin{cases}(1-z^2)u'' + (1-n)zu' + (1-|z|)f(u) = 0,\\ u'(-1) = u'(0) = 0\end{cases} \tag{1.6}$$

yields an even solution to (1.5), we prove Theorem 1.1 by showing that (1.6) has infinitely many solutions. It is easily verified that if $f(u) = |u|^{p-1}u$ for $u \geq 0$, and $f(u) = |u|^{q-1}u$ for $u \leq 0$, with $1 < p, q < \frac{n+5}{n-3}$, then f satisfies the hypotheses of Theorem 1.1. In this case we say that f has *subcritical* growth. If $p > \frac{n+5}{n-3}$ or $q > \frac{n+5}{n-3}$, we say that f has *supercritical growth*. Suggested by the result in [2], we believe that Theorem 1.1 also holds requiring subcritical growth for $u > 0$ while allowing supercritical growth for $u < 0$. Our results extend to the case where the right-hand side in (1.2) is replaced by a rotationally symmetric function $q \in L_\infty$. Again, for the sake of clarity we leave the corresponding calculations for the reader.

Standard contraction mapping principle arguments show that, for each $d \in \mathbb{R}$, the initial value problem

$$\begin{cases}(1-z^2)u'' + (1-n)zu' + (1+z)f(u) = 0, \quad z \in [-1,0],\\ u(-1) = d, u'(-1) = 0\end{cases} \tag{1.7}$$

has a unique solution, and that such a solution depends continuously on d in the $C^1([-1,0])$ topology.

For $u(z,d)$ solution to (1.7) we define the *energy function* by

$$E(z,d) = \frac{(u'(z,d))^2}{2} + \frac{1}{1-z}F(u(z,d)). \tag{1.8}$$

The first step in proving the existence of infinitely many soltions to (1.6) is to estimate, in terms of d, the value t_0 for which $u(t_0,d) = kd$, and $d \geq u(t,d) \geq kd$ for $t \in [-1,t_0]$ (see Lemma 2.1). Next, in Lemma 2.2, we establish a version of the Pohozaev identity for (1.7). Using the estimate for t_0 and our version of the Pohozaev identity, we prove that

$$\lim_{d\to\infty} E(z,d) = \infty. \tag{1.9}$$

To finish the proof, we consider the (u,u') phase plane. From (1.9) we have that for sufficiently large d, we can define a continuous argument function $\eta(z,d)$ such that $\eta(-1,d) = 0$. We prove that

$$\lim_{d\to\infty} \eta(0,d) = \infty.$$

Then, by the intermediate value theorem, we can say that there are infinitely many solutions where $u'(0,d) = 0$, and hence there are infinitely many rotatationally symmetric solutions to (1.2). Our argument resembles those in [3], where radial solutions to similar semilinear elliptic equations in balls were considered. Despite the similarity in the equations, the critical exponent arising here is surprisingly bigger than the one in [3]. The reader is referred to [7] for recent results on the existence of multiple solutions to semilinear Laplace–Beltrami equations on general compact Riemannian manifolds using variational methods. Also, the reader is referred to [4] for the role of hypotheses such as (1.4) in the existence of *dead core* and *bursts* solutions to quasilinear equations.

2 Energy Analysis

First we estimate the quantity t_0 in terms of d.

Lemma 2.1 *Let $k \in (0,1)$ be as in Theorem 1.1 and let $u(z,d) := u(z)$ be the solution to (1.7). There exists $D_1 > 0$, $k_2 > k_1 > 0$ such that if $d > D_1$, $u(t_0) = kd$, and $u(z) \ge kd$ for all $z \in [-1, t_0]$, then*

$$k_1\big(d/f(d)\big)^{1/3} \le t_0 + 1 \le k_2\big(d/f(kd)\big)^{1/3}. \tag{2.1}$$

Proof Let $D_1 > 0$ be such that $f(kd) > 0$ for $d > D_1$. Multiplying the second order differential equation in (1.7) by $(1-z^2)^{(n-1)/2}$, and integrating on $[-1,t)$ we have

$$\begin{aligned}(1-t^2)^{(n-1)/2}u'(t) &= -\int_{-1}^{t}(1-z^2)^{(n-1)/2}(1+z)f(u(z))dz\\ &\ge -f(d)2^{(n-1)/2}\int_{-1}^{t}(1+z)^{(n+1)/2}dz\\ &= -\frac{2^{(n+1)/2}}{n+3}f(d)(1+t)^{(n+3)/2}.\end{aligned}$$

Hence there exists a constant $c_1 > 0$, independent of d, such that

$$u'(t) \ge -c_1 f(d)(1+t)^2.$$

Integration on $[-1,t_0]$ yields

$$t_0 + 1 \ge 3c_1(d/f(d))^{1/3} := k_1(d/f(d))^{1/3}.$$

The existence of k_2 follows similarly. ∎

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In order to prove (1.9), we develop a form of the Pohozaev identity for our problem in the next lemma.

Lemma 2.2 *If $u(z) := u(z,d)$ is the solution to (1.7), then*

$$
\begin{aligned}
P(z) &:= (1-z^2)^{\frac{n+1}{2}} h(z)(u')^2 + (1-z^2)^{\frac{n-1}{2}} uu' + 2(1-z^2)^{\frac{n-1}{2}}(1+z)h(z)F(u) \\
&= \int_{-1}^{z} \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y}\Big[F(u)\big(h(y)(6y-4ny+2)-2\big) - uf(u)\Big]dy, \qquad (2.2)
\end{aligned}
$$

where $h(z) = (1-z^2)^{\frac{n-3}{2}} \int_z^0 (1-y^2)^{\frac{1-n}{2}} dy$.

Proof First we observe that

$$
\lim_{z\to -1^+} h(z) = \frac{1}{n-3} \text{ for } n>3 \quad \text{and} \quad \lim_{z\to -1^+} \frac{h(z)}{\ln\big(\frac{1}{2(1+z)}\big)} = \frac{1}{2} \text{ for } n=3. \qquad (2.3)
$$

Let $p(z) = (1-z^2)^{\frac{n-3}{2}}$ and $q(z) = 2(1-z^2)^{\frac{n-1}{2}} h(z)$. Multiplying (1.7) by $p(z)u$ gives us

$$
(1-z^2)^{\frac{n-1}{2}} uu'' + (1-n)(1-z^2)^{\frac{n-3}{2}} zuu' + (1-z^2)^{\frac{n-3}{2}}(1+z)uf(u) = 0.
$$

Then integrating, we have

$$
(1-z^2)^{\frac{n-1}{2}} uu' - \int_{-1}^{z} (1-y^2)^{\frac{n-1}{2}} (u')^2 dy = -\int_{-1}^{z} \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y} uf(u)dy. \qquad (2.4)
$$

On the other hand, multiplying (1.7) by $q(z)u'$ yields

$$
\begin{aligned}
&(1-z^2)^{\frac{n+1}{2}} h(z)u'u'' + (1-n)(1-z^2)^{\frac{n-1}{2}} h(z)z(u')^2 \\
&\qquad + (1-z^2)^{\frac{n-1}{2}}(1+z)h(z)u'f(u) = 0. \qquad (2.5)
\end{aligned}
$$

A simple calculation shows that

$$
h'(z) = -(1-z^2)^{-1} - (n-3)(1-z^2)^{\frac{n-5}{2}} z \int_{-1}^{z} (1-y^2)^{\frac{1-n}{2}} dy,
$$

so

$$
(1-z^2)h'(z) = -1-(n-3)zh(z). \qquad (2.6)
$$

Integrating (2.5) and using (2.6) gives us

$$
\begin{aligned}
&\frac{1}{2}(1-z^2)^{\frac{n+1}{2}} h(z)(u')^2 \\
&\quad + \frac{1}{2}\int_{-1}^{z} (1-y^2)^{\frac{n-1}{2}} (u')^2 dy + (1-z^2)^{\frac{n-1}{2}}(1+z)h(z)F(u) \\
&\qquad = \int_{-1}^{z} \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y} F(u)\big[h(y)(3y-2ny+1)-1\big]dy. \qquad (2.7)
\end{aligned}
$$

Multiplying (2.7) by 2, adding to (2.4), and simplifying, (2.2) follows. ∎

Now, from (2.2), we estimate the energy defined in (1.8) as d tends to $+\infty$.

Lemma 2.3 *If n, f are as in Theorem 1.1, then $\lim_{d\to\infty} E(z,d) = \infty$ uniformly for $z \in [-1,0]$.*

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Proof First we consider the case where $n > 3$. From (2.3) we see that there exists $T \in (-1, -1/2)$ such that if $y \in [-1, T]$; then $h(y)(6y - 4ny + 2) - 2 \geq \theta$ (see (1.4)). Let $D_2 > D_1$ be such that for $d > D_2$, $t_0 < T < -1/2$ (see (2.1)). Replacing these in (2.2) we have

$$\begin{aligned} P(t_0) &\geq \big(\theta F(kd) - df(d)\big) \int_{-1}^{t_0} (1+y)^{\frac{n-1}{2}} dy \\ &\geq \big(\theta F(kd) - df(d)\big) \frac{2}{n+1} (1+t_0)^{\frac{n+1}{2}}. \end{aligned} \tag{2.8}$$

Let $M < 0$ be such that $\theta F(s) - sf(s) \geq M$ for all $s \in \mathbb{R}$. Hence, for $t \in [t_0, T]$,

$$\begin{aligned} P(t) &\geq P(t_0) + M \int_{t_0}^{t} (1-y)^{(n-3)/2} (1+y)^{(n-1)/2} dy \\ &\geq P(t_0) + M 2^{(n-3)/2}. \end{aligned} \tag{2.9}$$

From equations (1.4), (2.1), and (2.8), $\lim_{d \to +\infty} P(t_0) = +\infty$. This and (2.9) give $\lim_{d \to \infty} P(z, d) = +\infty$ uniformly for $z \in [t_0, T]$. Thus, from the definition of P we have $\lim_{d \to \infty} E(z, d) = +\infty$ uniformly for $z \in [t_0, T]$.

From (1.8), for $t \in [-1, t_0]$, $E(t, d) \geq F(kd)$. Hence,

$$\lim_{d \to \infty} E(z, d) = +\infty \quad \text{uniformly for } z \in [-1, T]. \tag{2.10}$$

From (1.8) and (1.7),

$$E'(z, d) = (n-1) z (u'(z))^2 / (1 - z^2) \geq -2(n-1) E(z, d) / (1 - T^2) := -C E(z, d).$$

Integration on $[T, z]$ yields $E(z, d) \geq E(T, d) e^{-C}$, which together with (2.10) prove the lemma for $n > 3$.

If $n = 3$, then multiplying (1.7) by $(1 + z) u'$ and integrating results in

$$\begin{aligned} \Big(\frac{(u')^2}{2}\Big)(1-z)(1+z)^2 + F(u)(1+z)^2 &= \\ \frac{1}{2} \int_{-1}^{z} (u')^2 (1+t)^2 dt &+ 2 \int_{-1}^{z} (1+t) F(u) dt. \end{aligned} \tag{2.11}$$

Thus, for $d > D_2$ and $z \in [-1, t_0]$, $(1 - z) E(z, d) \geq F(kd)$. Since F is bounded from below, from (2.11), for $z \in [t_0, 0]$ we have

$$\liminf_{d \to +\infty} (1 - z) E(z, d) \geq \liminf_{d \to +\infty} F(kd) = +\infty,$$

which concludes the proof of the lemma. ∎

3 Phase Plane Analysis

Let $x(z, d) = u(z, d)$ and $y(z, d) = u'(z, d)$. From Lemma 2.3, there exists $D_3 > D_2$ such that if $d > D_3$, then $\rho(z, d) = \sqrt{x(z, d)^2 + y(z, d)^2} > 0$ for all $z \in [-1, 0]$. Hence, for $d > D_3$, there exists a differentiable function $\eta(z, d)$ that satisfies

$$\begin{cases} \eta(-1, d) = 0 \\ x(z, d) = \rho(z, d) \cos(\eta(z, d)) \\ y(z, d) = -\rho(z, d) \sin(\eta(z, d)) \end{cases} \tag{3.1}$$

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for $z \in [-1, 0]$. A straightforward calculation shows that

$$\eta'(z,d) = \sin^2 \eta(z,d) - \Big(\frac{(n-1)z}{1-z^2} y(z,d) - \frac{f(x(z,d))}{1-z} \Big) \frac{\cos \eta(z,d)}{\rho(z,d)}. \tag{3.2}$$

From (1.3) there exists a real number $M_1 > 0$ such that if $|v| > M_1$, then $v f(v) > 0$. Hence,

$$f\big(x(z,d)\big) \cos\big(x(z,d)\big) \geq \frac{\min\{v f(v); |v| \leq M_1\}}{\rho(z,d)} := \frac{M_2}{\rho(z,d)}. \tag{3.3}$$

Now we have enough information to prove the following lemma.

Lemma 3.1 *The η function satisfies*

$$\lim_{d \to +\infty} \eta(0,d) = +\infty.$$

Proof Let $D_4 > D_3$ be such that if $d > D_4$, then $\rho(z,d) \geq M_1$ for all $z \in [-1, 0]$. Therefore, if i is a non-negative integer and $\eta(z,d) = i\pi$, then $\eta'(z,d) > 0$. Hence, $\eta(\zeta,d) > i\pi$ for all $\zeta \in (z, 0]$. In particular, $\eta(\zeta,d) > 0$ for all $\zeta \in [-1,0]$. Let us see that given a positive integer j, there exists $d_j > 0$ such that if $d \geq d_j$, then $\eta(0,d) > j\pi$. Let j be given and $z \geq -\frac{3}{4}$. Let $\delta > 0$ be such that

$$\delta \leq \min\Big\{ \frac{\pi}{6}, \frac{1}{16jn} \Big\} \quad \text{and} \quad (1-\delta)^2 - \frac{15(n-1)}{16}\delta > \frac{3}{4}. \tag{3.4}$$

By (1.3) there exists $X_j > 0$ such that if $|x| \geq X_j$, then

$$\frac{f(x)}{x} \geq \frac{16j^2\pi + 3(n-1)}{2\cos^2(\frac{\pi}{2} - \delta)}. \tag{3.5}$$

Since $\lim_{d \to +\infty} \rho(z,d) = +\infty$, there exists a d_j such that if $d \geq d_j$, then

$$\rho(z,d) \geq \frac{X_j}{\cos(\delta)} + 4M_2^{1/2}.$$

Let k be a nonnegative odd integer. If $\eta(z,d) \in [\frac{k\pi}{2} - \delta, \frac{k\pi}{2} + \delta]$, then from (3.2), (3.4), and (3.3),

$$\eta'(z,d) \geq (1-\delta)^2 - \frac{15(n-1)}{16}\delta - \frac{4M_2}{\rho^2(z,d)} \geq \frac{1}{2}. \tag{3.6}$$

For $\eta(z) \in [\frac{k\pi}{2} + \delta, \frac{(k+2)\pi}{2} - \delta]$, from (3.1) we have $|x(z,d)| \geq \rho(z,d)|\cos(\delta)| > X_j$. Therefore, from (3.5), we have

$$\eta'(z,d) \geq 4j^2\pi. \tag{3.7}$$

Suppose $\eta(-3/4, d) \in [\frac{k\pi}{2} - \delta, \frac{k\pi}{2} + \delta]$ for some positive integer k. By (3.6), there exists $z_1 \in [-3/4, -3/4 + 2\delta]$ such that $\eta(z_1, d) = \delta + k\pi/2$. Similarly, by (3.7) there is a $z_2 \in (z_1, z_1 + \frac{\pi - 2\delta}{4j^2\pi})$, where

$$\eta(z_2, d) = \frac{(j+2)\pi}{2} - \delta,$$

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and, by (3.6), there exists $z_3 \in (z_2, z_2 + 4\delta)$, where $\theta(z_3, d) = \frac{(j+2)\pi}{2} + \delta$. Hence,

$$z_3 \le -3/4 + (4\delta + \frac{\pi - 2\delta}{4J^2\pi} + 4\delta).$$

Repeating this argument j times, we see that there exists

$$\widehat{z} \in \Big(-3/4, -3/4 + j\Big(8\delta + \frac{\pi - 2\delta}{4j^2\pi}\Big)\Big) \subset (-3/4, 0)$$

such that $\eta(\widehat{z}, d) \ge j\pi$. Similar arguments show that if $\theta(-3/4, d) \in [\frac{k\pi}{2} + \delta, \frac{(j+2)\pi}{2} - \delta]$ for some positive odd integer j, there is a

$$\widehat{z} \in \Big(-3/4, -3/4 + j\Big(4\delta + \frac{2(\pi - 2\delta)}{4J^2\pi}\Big)\Big), \quad \widehat{z} \le 0,$$

where $\eta(\widehat{z}, d) \ge j\pi$. Therefore, since $\widehat{z} \le 0$ and θ is increasing in z, we have that $\eta(0, d) \ge j\pi$ for $|d| \ge d_j$. This proves the lemma. ∎

Now we prove Theorem 1.1.

Proof of Theorem 1.1 By Lemma 3.1, there exists K_0 such that if $k \ge K_0$ is a positive integer then there exists $e_k > D_3$ such that $\eta(0, e_k) = 2k\pi$. Hence $u'(0, e_k) = 0$ for all $k \ge K_0$. That is, each

$$u_k(x_1, \dots, x_{n-1}, z) := u(z, e_k), \quad u_k(x_1, \dots, x_{n-1}, -z) := u_k(x_1, \dots, x_{n-1}, z)$$

defines a rotationally symmetric solution to the boundary value problem (1.5). This proves the theorem. ∎

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