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ON SYLOW GRAPHS

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Abstract

We characterize the classes of graphs of order n whose automorphism group either contains or coincides with the 2-Sylow subgroup of the symmetric group S_n.

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1. Introduction

Throughout this paper, all graphs will be on a finite set of vertices without loops, multiple edges or directed edges. Most of the graph theoretical terms may be found in Harary (1971). As a result we use $P_1 \times P_2$ for the direct sum of two permutation groups and $P_1[P_2]$ for the composition of P_1 around P_2.

If $G_i \in \{K_2, \bar{K}_2\}$, then by $[K_2^{\pm}]^r$ we mean the repeated (graph) composition $[G_1[G_2[\ldots[G_r]\ldots]]]$. We define the graph $H(2^r)$ to be the graph $[K_2^{\pm}]^r$ with $G_1 = K_2$ and $G_i \not\cong G_{i+1}$ for $i = 1, 2, \ldots, r-1$.

LEMMA 1.1. (Sabidussi (1959).) $\Gamma(G_1[G_2]) = \Gamma(G_1)[\Gamma(G_2)]$ *if and only if*

(i) *if there are distinct vertices in* G_1 *with the same open neighbourhood, then* G_2 *is connected,*

(ii) *if there are distinct vertices in* G_1 *with the same closed neighbourhood, then* $\bar{G}_2$ *is connected.*

COROLLARY. *For all* G_1 *and* G_2, $\Gamma(G_1)[\Gamma(G_2)] \leqslant \Gamma(G_1[G_2])$.

LEMMA 1.2. (Holton and Grant (1975).) *Let* G *be a vertex-transitive graph. If* $\Gamma(G)$ *contains a transposition, then for some* $m > 1$, $G = H[K_m]$ *or* $G = H[\bar{K}_m]$, *where* H *is a vertex-transitive graph. Conversely if* $G = H[K_m]$ *or* $G = H[\bar{K}_m]$ *for*

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some $m > 1$ *and* H *is vertex-transitive, then* G *is vertex-transitive and* $\Gamma(G)$ *contains a transposition.*

The X-join of two graphs is the generalization of composition introduced in Sabidussi (1961). If G is an X-join we write $G = X[Y_1, \ldots, Y_n]$ and refer to the graphs Y_j as *constituents* of G. The definition of externally related can be found in Hemminger (1968).

Suppose X, $Y \subseteq V(G)$. Let $X \circ Y = \{xy\colon xy \in EG, x \in X, y \in Y\}$. If

$$X \circ Y = \{xy\colon x \in X, y \in Y\}$$

we say that $X \circ Y$ is *full*. If X or Y is empty then $X \circ Y$ is full.

The graph G is a *weak 2-Sylow subgraph of* K_n, if G is of order n and $\Gamma(G)$ contains the 2-Sylow subgroup Π of $\mathscr{S}_n = \Gamma(K_n)$. We say that G is a *strong 2-Sylow subgraph of* K_n if G is of order n and $\Gamma(G) = \Pi$. We denote the weak 2-Sylow subgraphs of K_n by $W(n)$ and the strong 2-Sylow subgraphs by $W^+(n)$.

In the paper we show that $W^+(2^r) = \{\overline{H(2^r)}, H(2^r)\}$, $W(2^r) = \{[K_2^{\pm}]^r\}$ and characterize $W^+(n)$ and $W(n)$ in terms of $W^+(2^r)$ and $W(2^r)$, respectively.

2. The case $n = 2^r$

In this section we show that $W(2^r) = \{[K_2^{\pm}]^r\}$ and $W^+(2^r) = \{H(2^r), \overline{H(2^r)}\}$.

THEOREM 2.1. *Let* $r \geqslant 1$ *be an integer. Then* $W(2^r) = \{[K_2^{\pm}]^r\}$.

PROOF. Using the Corollary of Lemma 1.1 concerning the automorphism group of a composition and the fact that $|\Pi| = 2^{2^r-1}$, then it is straightforward to show that $\{[K_2^{\pm}]^r\} \subseteq W(2^r)$.

Let $G \in W(2^r)$. By the structure of Π, $\Gamma(G)$ is transitive and contains a transposition. Hence by Lemma 1.2, there exist graphs G_1 and H_1 with $G_1 \in \{K_{n_1}, \bar{K}_{n_1}\}$, H_1 vertex-transitive, and $G = H_1[G_1]$. Hence $n_1 = 2^{s_1}$ where $1 \leqslant s_1 \leqslant r$. Since $G_1 = [K_2]^{s_1}$ or $[\bar{K}_2]^{s_1}$, if $|V(H_1)| = 1$, then the result follows. We may therefore suppose $|V(H_1)| \neq 1$.

Suppose $\Gamma(H_1)$ does not contain a transposition. We know that $G = H_1[G_1]$; so $|V(H_1)| = 2^{r-s_1}$. Since $\Gamma(H_1)$ does not contain a transposition $\Gamma(G) = \Gamma(H_1)[\Gamma(G_1)]$, by Lemma 1.1. An order argument then shows that $|\Gamma(H_1)| = (2^{2^{(r-s_1)}-1})t$ where $2 \nmid t$. Hence $H_1 \in W(2^{r-s_1})$ or $|V(H_1)| = 1$. But $\Gamma(H_1)$ does not contain a transposition. So $|V(H_1)| = 1$.

Hence $\Gamma(H_1)$ contains a transposition. We may now proceed by induction till at stage k we have $G = H_k[G_k[G_{k-1}[\ldots[G_1]\ldots]]]$ with $|V(H_k)| = 1$. In each case $G_i \in \{[K_2]^{s_i}, [\bar{K}_2]^{s_i}\}$, where $\sum_{i=1}^{k} s_i = r$. Thus $G \in \{[K_2^{\pm}]^r\}$.

COROLLARY. $|W(2^r)| = 2^r$.

THEOREM 2.2. Let $r \geqslant 1$ be an integer. Then $W^+(2^r) = \{H(2^r), \overline{H(2^r)}\}$.

PROOF. Again it is clear that $\{H(2^r), \overline{H(2^r)}\} \subseteq W^+(2^r)$. If $G \in W^+(2^r)$ and $r = 1$, then the equality of the two sets is straightforward. We may therefore suppose that $r > 1$.

Let $G \in W^+(2^r)$. Since $W^+(2^r) \subseteq W(2^r)$ we have $G = G_1[G_2[G_3 \dots [G_r] \dots]]$, where $G_i \in \{K_2, \bar{K}_2\}$. If there exists a j, $1 \leqslant j < r$ such that $G_j \simeq G_{j+1}$, then $G_j[G_{j+1}] \in \{K_4, \bar{K}_4\}$. Hence $\Gamma(G_j[G_{j+1}]) = \mathscr{S}_4$ and so $\mathscr{S}_4 \leqslant \Gamma(G)$. But this is not possible since $\Gamma(G)$ is a 2-group. Hence $G_j \not\simeq G_{j+1}$, $1 \leqslant j < r$, and so $G \in \{H(2^r), \overline{H(2^r)}\}$.

We are now able to characterize $W(n)$. Suppose $n = \sum_{i=0}^{n} \varepsilon_i 2^i$, $\varepsilon_i = 0, 1$ and $M = \{i\colon \varepsilon_i = 1\}$ with $|M| = m$. Then let $A(n)$ be defined as follows. The graph $G \in A(n)$ if and only if

(i) G is an X-join,

(ii) $X = \{x_i\colon i \in M\}$, and

(iii) $Y_{x_i} \in W(2^i)$ for some $i \in M$.

THEOREM 2.3. $W(n) = A(n)$.

PROOF. If $G \in A(n)$, then clearly $\Pi \leqslant \Gamma(G)$ and so $G \in W(n)$.

If $G \in W(n)$, then $\Pi = \times_{i=0}^{n} \varepsilon_i \Gamma(2^i) \leqslant \Gamma(G)$, where $\Gamma(2^i) = \Gamma(H(2^i)) = \Gamma(\overline{H(2^i)})$. Further, $V(G) = \bigcup_{i=0}^{n} \varepsilon_i \Delta_i$, where Δ_i is the orbit of Π induced by $\Gamma(2^i)$. If $\varepsilon_i = 1$, then $\Gamma(2^i) \leqslant \Gamma(\langle \Delta_i \rangle)$ and so $\langle \Delta_i \rangle \in W(2^i)$. By the definition of the direct product, $\varepsilon_i \Delta_i \circ \varepsilon_j \Delta_j$ is either empty or full. Hence $G \in A(n)$.

3. The case for general n: $W(n)$

The main result of this section is Theorem 3.8. This theorem characterizes $W(n)$. We also include some results which will be of value in the next section.

For the balance of the paper, we assume that n is an integer larger than 1 and that its binary decomposition is $\sum_{i=0}^{n} \varepsilon_i 2^i$ where $\varepsilon_i = 0, 1$.

We first need some number theoretical results.

LEMMA 3.1. *Let* $n = 2^\delta + s$, $0 \leqslant s < 2^\delta$, $(2^\delta)! = 2^\alpha t$, $2 \nmid t$. *If*

$$n! = 2^\gamma u, \quad s! = 2^\beta v \quad (2 \nmid u, 2 \nmid v)$$

then $\gamma = \alpha + \beta$.

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LEMMA 3.2. *If* $n! = 2^\gamma u\ (2 \nmid u)$, *then* $\gamma = \sum_{i=0}^{n} \varepsilon_i(2^i - 1)$.

THEOREM 3.3. *Let* $H(n) = \dot{\bigcup}_{i=0}^{n} \varepsilon_i H(2^i)$ *where* $H(2^0)$ *is the graph consisting of a single vertex and if* $\varepsilon_i = 0$, $\varepsilon_i H(2^i) = \Omega$ (*the empty graph*). *Then*

$$\Gamma(H(n)) = \mathop{\times}_{i=0}^{n} \varepsilon_i \Gamma(2^i) \quad \text{and} \quad H(n) \in W^+(n).$$

PROOF. If $r \geqslant 1$, $H(2^r)$ is connected. Furthermore, if $i,j \geqslant 1$, $i \neq j$ then $H(2^i) \not\cong H(2^j)$. By definition, $H(2^0)$ is connected and $H(2^0) \not\cong H(2^i)$, $i \geqslant 1$. Now it is readily seen that $\Gamma(H(n)) = \times_{i=0}^{n} \varepsilon_i \Gamma(2^i)$ and so $|\Gamma(H(n))| = \prod_{i=0,\varepsilon_i \neq 0}^{n} |\Gamma(2^i)|$. Therefore $|\Gamma(H(n))| = \prod_{i=0,\varepsilon_i \neq 0}^{n} 2^{2^i-1}$. Thus $|\Gamma(H(n))| = 2^\gamma$ where $\gamma = \sum_{i=1}^{n} \varepsilon_i(2^i - 1)$. Hence, by Lemma 3.2, $\Gamma(H(n))$ is a 2-Sylow subgroup of $\mathscr{S}_n$ and so $H(n) \in W^+(n)$.

LEMMA 3.4. *If* $G \in W(n)$, *then* $\times_{i=0}^{n} \varepsilon_i \Gamma(2^i) \leqslant \Gamma(G)$. *If* $G \in W^+(n)$, *then* $\Gamma(G) = \times_{i=0}^{n} \varepsilon_i \Gamma(2^i) = \Pi$.

PROOF. Let $G \in W(n)$. Then $\Gamma(G)$ contains a 2-Sylow subgroup of $\mathscr{S}_n$. By Theorem 3.3, $H(n) \in W^+(n)$. Hence $\Gamma(H(n)) = \times_{i=0}^{n} \varepsilon_i \Gamma(2^i)$ is a 2-Sylow subgroup of $\mathscr{S}_n$. Thus $\times_{i=0}^{n} \varepsilon_i \Gamma(2^i) \leqslant \Gamma(G)$. If $G \in W^+(n)$ it is clear from $|\Gamma(G)|$ that $\times_{i=0}^{n} \varepsilon_i \Gamma(2^i) = \Gamma(G) = \Pi$.

LEMMA 3.5. *Let* G *be a graph with* $V(G) = X \dot{\cup} Y$. *Let* P *and* Q *be permutation groups acting on* X *and* Y *respectively and let* $\Gamma(G) = P \times Q$. *If* P *acts transitively on* X, *then* $\Gamma(\langle X \rangle) = P$, *where* $\langle X \rangle$ *is the induced graph on the set* X *of vertices.*

PROOF. Clearly $P \leqslant \Gamma(\langle X \rangle)$.

Since P is transitive, if there exists an edge $x \sim y$ in G with $x \in X$, $y \in Y$, then $x' \sim y$ in G, for all $x' \in X$. Similarly if $x \nsim y$ in G, then $x' \nsim y$ in G for all $x' \in X$.

Suppose $\sigma \in \Gamma(\langle X \rangle)$, then consider $\sigma' = \sigma \times 1_Q$ where 1_Q is the identity element of Q. Certainly σ' preserves adjacencies in $\langle X \rangle$ and $\langle Y \rangle$. But $(x \sim y)^{\sigma'} = x^\sigma \sim y = x' \sim y$ for some $x' \in X$ and since we know that when $x \sim y$ in G then $x' \sim y$ in G (and similarly for $x \nsim y$) then $\sigma' \in \Gamma(G)$. Hence $\Gamma(\langle X \rangle) \leqslant P$.

LEMMA 3.6. *Let* $V(G) = \dot{\bigcup}_{i=1}^{s} X_i$ *and let* P_i *be a permutation group acting transitively on* X_i, $i = 1, 2, \ldots, s$. *If* $\Gamma(G) = \times_{i=1}^{s} P_i$, *then* $P_i = \Gamma(\langle X_i \rangle)$. *Furthermore, if* $1 \leqslant i < j \leqslant s$, *then* $X_i \circ X_j$ *is either empty or full. If* $\Gamma(G) = \times_{i=1}^{s} P_i$, *then* $P_i = \Gamma(\langle X_i \rangle)$.

PROOF. If $\Gamma(G) = \times_{i=1}^{s} P_i$, then clearly $P_i \leqslant \Gamma(\langle X_i \rangle)$. Let $1 \leqslant i < j \leqslant s$. Then since P_i and P_j are both transitive $X_i \circ X_j$ is either empty or full by the argument used in the proof of Lemma 3.5.

If $\Gamma(G) = \times_{i=1}^{s} P_i$, then $\Gamma(\langle X_i \rangle) = P_i$ by Lemma 3.5 and induction.

Recall that $n = \sum_{i=0}^{n} \varepsilon_i 2^i$, ($\varepsilon_i \in \{0, 1\}$) is the binary decomposition of n. Let $X(i)$ be a set such that $|X(i)| = 2^i$ and $X(n) = \dot{\bigcup}_{i=0}^{n} \varepsilon_i X(i)$ where

$$\varepsilon_i X(i) = X(i) \text{ for } \varepsilon_i = 1 \quad \text{and} \quad \varepsilon_i X(i) = \varnothing \text{ for } \varepsilon_i = 0.$$

Notice that $|X(n)| = n$. Henceforward we will assume that $V(H(2^i)) = X(i)$. Let $H(n)$ be as in Theorem 3.3. Then

$$V(H(n)) = \dot{\bigcup_{i=0}^{n}} \varepsilon_i V(H(2^i)) = \dot{\bigcup_{i=0}^{n}} \varepsilon_i X(i) = X(n).$$

LEMMA 3.7. *Let $G \in W(n)$ and let $V(G) = X(n)$. Then the vertices of G can be ordered so that $\langle \varepsilon_i X(i) \rangle \in W(2^i)$ and $\varepsilon_i X(i) \circ \varepsilon_j X(j)$ is either empty or full, $1 \leqslant i < j \leqslant s$. If $G \in W^+(n)$ then $\langle \varepsilon_i X(i) \rangle \in \{H(2^i), \overline{H(2^i)}\}$.*

PROOF. Let $G \in W(n)$. By Lemma 3.4, $\times_{i=0}^{n} \varepsilon_i \Gamma(2^i) \leqslant \Gamma(G)$, where $\Gamma(2^i)$ acts transitively on $\varepsilon_i X(i)$, $i = 1, 2, \ldots, n$. Hence, by Lemma 3.6, if $\varepsilon_i = 1$, $\Gamma(2^i) \leqslant \Gamma(\langle X(i) \rangle)$ and so $\langle X(i) \rangle \in W(2^i)$. If $1 \leqslant i < j \leqslant s$ then, by Lemma 3.6, $\varepsilon_i X_i \circ \varepsilon_j X_j$ is either empty or full.

If $G \in W^+(n)$ then, by Lemma 3.4, $\Gamma(G) = \times_{i=0}^{n} \varepsilon_i \Gamma(2^i)$, where $\Gamma(2^i)$ acts transitively on $\varepsilon_i X(i)$, $i = 1, 2, \ldots, n$. Hence by Lemma 3.6, if $\varepsilon_i = 1$, $\Gamma(\langle X_i \rangle) = \Gamma(2^i)$ and so $\langle X_i \rangle \in W^+(2^i)$. Thus, $\langle X_i \rangle \in \{H(2^i), \overline{H(2^i)}\}$, which gives

$$\langle \varepsilon_i X(i) \rangle \in \{H(2^i), \overline{H(2^i)}\}.$$

Now if $n = \sum_{i=0}^{n} \varepsilon_i 2^i$ and $M = \{i \colon \varepsilon_i \neq 0\}$ with $|M| = m$, then we define the set of graphs $A(n)$ in the following way. The graph G is in $A(n)$, if and only if

(i) G is an X-join;
(ii) $|V(X)| = m$;
(iii) $Y_{x_i} \in W(2^i)$ for some $i \in M$, where $X = \{x_1, x_2, \ldots, x_m\}$.

THEOREM 3.8. $W(n) = A(n)$.

PROOF. If $G \in W(n)$, then $G \in A(n)$ from Lemma 3.7, with $\langle \varepsilon_i X(i) \rangle = Y_x$ for suitable $x \in X$.

If $G \in A(n)$, then clearly $\Pi \leqslant \Gamma(G)$ and so $G \in W(n)$ by the definition of $W(n)$.

4. The case for general n: $W^+(n)$

In this section we characterize the set $W^+(n)$. We define $A^+(n)$ in the following way. The graph G is in $A^+(n)$ if and only if

(i) G is an X-join;
(ii) $X = \{x_i \colon i \in M\}$;

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(iii) $Y_{x_i} \in W^+(2^i)$ for some $i \in M$;
(iv) if $N(x) = N(x')$, then Y_x and $Y_{x'}$ do not have a common component;
(v) if $\overline{N(x)} = \overline{N(x')}$, then $\bar{Y}_x$ and $\bar{Y}_{x'}$ do not have a common component.
Our aim is now to show that $W^+(n) = A^+(n)$.

Lemma 4.1. $W^+(n) \subseteq A^+(n)$.

Proof. If $G \in W^+(n)$, then $\Gamma(G) = \Pi$. From Lemma 3.7 we know that G is the X-join of $\{Y_x\}$ where $Y_x \in W^+(2^i)$ and conditions (i) through (iii) are satisfied. If condition (iv) or (v) is not satisfied by G, then clearly $\Pi > \Gamma(G)$ and we have a contradiction.

To obtain $A^+(n) \subseteq W^+(n)$ we must work a little harder. We need some preliminary results.

Lemma 4.2. *$H(2^i) = T + T'$ if and only if $T = H(2^i)$, $T' = H(2^i)$ or*

$$T \cong T' \cong \overline{H(2^{i-1})}.$$

Proof. If $T = H(2^i)$, $T' = H(2^i)$ or $T \cong T' \cong \overline{H(2^{i-1})}$, then the result follows trivially.

Suppose $H(2^i) = T + T'$ and T is nonempty. Now

$$H(2^i) = [H(2^{i-2}) \dot{\cup} H(2^{i-2})] + [H(2^{i-2}) \dot{\cup} H(2^{i-2})].$$

Let P be one of the copies of $H(2^{i-2}) \dot{\cup} H(2^{i-2})$. If T is a proper subgraph of P, then there exists $v \in V(P) - V(T)$ and $t \in V(T)$ such that $v \nsim t$. Hence $H(2^i) \neq T + T'$ for any T'. So we must have $P \subseteq T$. If $P = T$, then we are done. If P is a proper subgraph of T, then again $H(2^i) \neq T + T'$ for any T'.

We now consider a graph G from the set $A^+(n)$, and the image of one of the graphs Y_x under an automorphism of g. By considering all possibilities we show that Y_x is fixed under $\Gamma(G)$ and hence $\Gamma(G) = \Pi$ and so $A^+(n) \subseteq W^+(n)$. This will be accomplished in a series of lemmas, with the proof completed in Theorem 4.7. Throughout, we assume that $Y_x^g \cap Y_y \neq \varnothing$ implies $Y_y \nsubseteq Y_x^g$. If it does, the arguments follow by using g^{-1} instead of g.

Lemma 4.3. *If $Y_x = H(2^i)$, $Y_y = H(2^j)$, $g \in \Gamma(G)$ and $Y_x^g \cap Y_y = T \neq \varnothing$, then $H(2^{i+2}) = Y_y$.*

PROOF. Suppose that $Y_x^g \nsubseteq Y_y$. Clearly Y_x^g is externally related. Since Y_y is connected, there exists $v \in V(Y_y) - V(T)$ and $t \in V(T)$ such that $v \sim t$. Hence t is adjacent to every vertex in $V(Y_y) - V(T)$ and $Y_y = T + T'$. Similarly

$$Y_x^g \cong Y_x \cong T + T''.$$

If T' is empty, $T = Y_y$ and if T'' is empty, $T = Y_x$. This is not possible since this implies $i = j$ and contradicts the choice of G from $A^+(n)$.

If T' is empty and T'' is not empty, then $Y_x \cong T + T''$. By Lemma 4.2, we have $T \cong T'' \cong \overline{H(2^{i-1})}$. Hence $Y_y \cong \overline{H(2^{i-1})}$. This is clearly a contradiction.

If T' is not empty and T'' is empty, we get a similar contradiction to the last paragraph.

If both T' and T'' are non-empty, then, by Lemma 4.2, we have $i = j$, which is not possible.

Suppose then, that $Y_x^g \subset Y_y$. Let $S \subseteq V(Y_y)$ and $R \subseteq V(Y_y)$ be the vertices which are adjacent to every vertex of $V(Y_x^g)$, and no vertex of $V(Y_x^g)$, respectively. Clearly $R \neq \emptyset$ by Lemma 4.2. Since $Y_y = H(2^{j-1}) + H(2^{j-1})$, we have $Y_x^g \mathbin{\dot\cup} \langle R \rangle$ as a subgraph of one of the copies of $\overline{H(2^{j-1})}$ in Y_y, and hence $Y_x^g \subseteq H(2^{j-2})$. But, because of the symmetry of the situation, there must also be a copy of $H(2^i)$ in R and a copy of $H(2^i) \mathbin{\dot\cup} H(2^i)$ in both copies of $\overline{H(2^{j-1})}$. Hence

$$[H(2^i) \mathbin{\dot\cup} H(2^i)] + [H(2^i) \mathbin{\dot\cup} H(2^i)] \subseteq H(2^j)$$

and so $H(2^{i+2}) \subseteq H(2^j)$.

LEMMA 4.4. *If* $Y_x = \overline{H(2^i)}$, $Y_y = H(2^j)$, $g \in \Gamma(G)$ *and* $Y_x^g \cap Y_y = T \neq \emptyset$ *then* $i - 1 = j$ *or* $H(2^{i+1}) \subseteq Y_y$.

PROOF. Now $Y_x = \overline{H(2^i)} = H(2^{i-1}) \mathbin{\dot\cup} H(2^{i-1})$. Let one copy of $H(2^{i-1})$ in Y_x^g be A and the other be B. If $A \cap Y_y = \emptyset$, then $B \cap Y_y \neq \emptyset$ by hypothesis. Now Y_y is connected and so there exists $v \in VB$ and $w \in V(Y_y) - V(B)$ with $v \sim w$. But Y_x^g is externally related and so for every $v' \in Y_x^g$ and $w' \in Y_y$ we have $v' \sim w'$. Since $B \cap Y_y \neq \emptyset$, there exists $a \in V(A)$ and $b \in V(B)$ such that $a \sim b$, which gives a contradiction, unless $B \cap Y_y = B$, when $i - 1 = j$.

If $A \cap Y_y \neq \emptyset$ and $B \cap Y_y \neq \emptyset$, it is clear from the arguments above, that $Y_x^g \cap Y_y = Y_x^g$. Let S (R) be the subset of $V(Y_y)$ such that every vertex in $V(Y_x^g)$ is adjacent to every (no) vertex of S (R). If $R = \emptyset$, then $Y_y = H(2^{i+1})$ and the result follows. If $R \neq \emptyset$, then we can proceed by a similar argument to that used in the latter half of Lemma 4.3, to obtain $H(2^{i+1}) \subset Y_y$.

LEMMA 4.5. *If* $Y_x = H(2^i)$, $Y_y = \overline{H(2^j)}$, $g \in \Gamma(G)$ *and* $Y_x^g \cap Y_y = T \neq \emptyset$, *then* $i = j - 1$ *or* $H(2^{i+2}) \subseteq Y_y$.

PROOF. Since Y_x^g is connected and Y_y is not, then Y_x^g only intersects one component of Y_y and this component is isomorphic to $H(2^{j-1})$. The result then follows from the proof of Lemma 4.3.

LEMMA 4.6. *If* $Y_x = \overline{H(2^i)}$, $Y_y = \overline{H(2^j)}$, $g \in \Gamma(G)$ *and* $Y_x^g \cap Y_y = T \neq \emptyset$, *then* $H(2^{i+1}) \subset Y_y$.

PROOF. Let A and B be the two components of Y_x^g which are isomorphic to $H(2^{i-1})$ and C and D be the two components of Y_y which are isomorphic to $H(2^{j-1})$. It is clear that we cannot have $A \cap C \neq \emptyset$ and $A \cap D \neq \emptyset$ simultaneously, nor can we have both $B \cap C \neq \emptyset$ and $B \cap D \neq \emptyset$.

Suppose $A \subseteq C$ and $B \nsubseteq Y_y$. Then an argument using the externally related property of Y_y, gives the contradiction that for all $a \in V(A)$ and for all $b \in V(B)$, $a \sim b$, unless $B \cap Y_y = \emptyset$. In this case, we have $H(2^{i-1}) \subseteq H(2^{j-1}) \subset Y_y$ and by Lemma 4.3, we have $H(2^{i+1}) \subseteq H(2^{j-1}) \subset Y_y$. The same argument applies to $A \subseteq C$ and $B \subseteq D$.

Suppose $A \subseteq C$ and $B \subseteq C$. Here we can use Lemma 4.4, to obtain

$$H(2^{i+1}) \subseteq H(2^{j-1}) \subset Y_y.$$

Suppose $A \cap C \neq \emptyset$ and $A \cap C \neq C$. Then an argument using the externally related properties of Y_x^g and Y_y gives the contradiction that Y_x^g is connected.

We are now in a position to prove the main result of this section.

THEOREM 4.7. $W^+(n) = A^+(n)$.

PROOF. We already know that $W^+(n) \subseteq A^+(n)$, by Lemma 4.1.

Suppose $G \in A^+(n)$. Clearly $\Pi \leqslant \Gamma G)$. From among all $g \in \Gamma(G) - \Pi$, for a fixed j, choose $Y_x \in W^+(2^i)$ so that i is a maximum among all $w \in V(X)$ such that $Y_w^g \cap Y_y \neq \emptyset$, where $Y_y \in W^+(2^j)$. Let $u \in Y_x$.

Case 1. Assume $u \nsim u^g$.

1.1. Suppose $Y_x = H(2^i)$ and $Y_y = H(2^j)$. By Lemma 4.3, $H(2^{i+2}) \subseteq Y_y$. Let M be a copy of $H(2^i)$ in Y_y such that $Y_x^g + M \subseteq Y_y$. Then $Y_x + M^{g^{-1}}$ is a subgraph of G and $M^{g^{-1}} \nsubseteq Y_y$, since $u \nsim u^g$.

1.1.1. If $M^{g^{-1}} = \dot{\bigcup} Y_z$, the disjoint union being over some subset of $V(X)$, then since $|V(M^{g^{-1}})| = 2^i$, a number theoretic argument shows that $\dot{\bigcup} Y_z = Y_v$ for some $v \in V(X)$. But then $Y_v \cong Y_x$, which contradicts the construction of G.

1.1.2. If $M^{g^{-1}} \subset Y_v$ for some v, then $Y_v \in W^+(2^k)$ for $k > i$, and the choice of g is contradicted.

1.1.3. If $M^{g^{-1}} \subseteq \dot{\bigcup} Y_z$, where the union is over some subset Q (with $|Q| \geqslant 2$) of of $V(X)$, then since $M^{g^{-1}}$ is connected, then Q is a clique in X. Suppose $q \in Q$ and without loss of generality, $Y_q - M \neq \emptyset$. Let $M_q = M^{g^{-1}} \cap Y_q$. Since Q is a clique, every vertex of M_q is adjacent to every other vertex of $M^{g^{-1}}$ and so $M^{g^{-1}} = M_q + M'$ for some non-empty M'. Hence $M_q = \overline{H(2^{i-1})}$. The fact that Q is a clique and that $M^{g^{-1}}$ is an externally related set, forces Y_q to be connected. Then the externally related property of $M^{g^{-1}}$ gives $Y_q = M_q + Y'_q$ with Y'_q non-empty. Hence $Y_q \cong Y_x$, a contradiction.

1.2. Suppose $Y_x = \overline{H(2^i)}$ and $Y_y = H(2^j)$. By Lemma 4.3, $i-1=j$ or $H(2^{i+1}) \subseteq Y_y$. Let M be a copy of $\overline{H(2^i)}$ in Y_y such that $Y_x^g + M \subseteq Y_y$. Then $Y_x + M^{g^{-1}}$ is a subgraph of G and $M^{g^{-1}} \nsubseteq Y_y$, since $u \nsim u^g$.

1.2.1. If $M^{g^{-1}} = \dot{\bigcup} Y_z$, the argument of case 1.1.1 applies.

1.2.2. If $M^{g^{-1}} \subset Y_v$, for some v, the argument of case 1.1.2 applies.

1.2.3. If $M^{g^{-1}} \subseteq \dot{\bigcup} Y_z$, where the union is over some subset Q of $V(X)$, then suppose $M^{g^{-1}} = M_1 \cup M_2$ where $M_1 \cong M_2 \cong H(2^{i-1})$. Choose $q \in Q$ such that $Y_q \cap M_1 \neq \emptyset$. If there exists a vertex $v \in V(Y_q) - V(M_1)$ and a vertex $w \in V(M_1)$ such that $v \sim w$, then the externally related properties of M and Y_q, show that M_1 and M_2 are connected by an edge. Hence Y_q is connected and $Y_q = M_1$ or Y_q is disconnected and $Y_q = M_1 \cup Y_q^*$. In the former case there must exist $r \in Q$ with Y_r connected and $Y_q = M_2$ or Y_r disconnected and $Y_r = M_2 \cup Y_r^*$. By the construction of G, one of Y_q, Y_r must be disconnected, in which case it is isomorphic to Y_x and we obtain a contradiction.

1.2.4. If $i-1=j$, then we are able to show that condition (iv) is contradicted. The proof here is essentially that of case 1.3.2 which we give in full.

1.3. Suppose $Y_x = H(2^i)$ and $Y_y = \overline{H(2^j)}$. By Lemma 4.5, we know that either $i = j-1$ or $H(2^{i+2}) \subseteq Y_y$.

1.3.1. If $H(2^{i+2}) \subseteq Y_y$, then $H(2^{i+2})$ is contained in a copy of $H(2^{j-1})$ in Y_y. The arguments of case 1.1 can then be applied to obtain contradictions.

1.3.2. If $i = j-1$, then $Y_y = Y_x^g \cup K$, where K is isomorphic to $H(2^i)$. We consider the image of K under g. Now $K^g \neq Y_v$ for some $v \in V(X)$ unless $v = x$. If $K^g \subset Y_v$ for some $v \in V(X)$, then $Y_v = K^g + Y_v^*$ by the externally related property of K^g, if Y_v is connected, which is clearly a contradiction. If $K^g \subset Y_v$ and Y_v is disconnected, then by the previous argument, K^g must be a component of Y_v and hence $Y_v \cong Y_y$. This is not possible unless $v = y$ and $K^g = K$. By an order argument, $K^g \neq \dot{\bigcup} Y_z$, so we must have $K^g \subset \dot{\bigcup} Y_z$, for some subset Q of $V(X)$ and we may assume that for $q \in Q$, $Y_q - K^g \neq \emptyset$. If Y_q is connected, then externally related-type arguments show that $Y_q \cong Y_x$. If Y_q is not connected, we must have $Y_q \cong Y_y$. Neither situation is tenable.

Hence we see that $K^g = Y_x$ or $K^g = K$. Similar arguments show that $(Y_x^g)^g = K$ or Y_x and so $N(x) = N(y)$. Thus condition (iv) in the construction of G is violated.

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1.4. Suppose $Y_x = \overline{H(2^i)}$ and $Y_y = \overline{H(2^j)}$. By Lemma 4.6, we know that $H(2^{i+1}) \subset Y_y$. Let L be such that $Y_x^g + L = H(2^{i+1})$.

1.4.1. If $L^{g^{-1}} = \dot{\bigcup} Y_z$, then we repeat the argument of case 1.1.1.

1.4.2. If $L^{g^{-1}} \subset Y_v$, for some v, then we repeat the argument of case 1.1.2.

1.4.3. Let $L^{g^{-1}} \subseteq \dot{\bigcup} Y_z$, where the union is over some subset Q of $V(X)$, we repeat the argument of case 1.2.3.

Case 2. Assume $u \sim u^g$.

In this case we consider $\bar{G}$. By construction of G, $\bar{G} \in A^+(n)$ and now $u \nsim u^g$. By case 1, the theorem holds unless $\bar{G}$ does not satisfy condition (iv). But if $\bar{G}$ does not satisfy condition (iv), then G does not satisfy condition (v).

Hence in both cases we see that $\Gamma(G) - \Pi = \emptyset$ and so $A^+(n) \subseteq W^+(n)$.

5. Graphical 2-groups

Given a permutation group P which is a subgroup of the 2-Sylow subgroup of $\mathscr{S}_n$, the question now arises as to whether or not there is a graph on n vertices whose automorphism group is isomorphic, as a permutation group, to P.

This seems to be a non-trivial question. We content ourselves here with proving that there is some permutation 2-subgroup of the 2-Sylow subgroup of $\mathscr{S}_n$ which is graphical, for every possible order of such 2-subgroups.

The pattern of proof continues as in the earlier part of the paper. First we establish the result for $n = 2^r$ and then we consider the general case.

LEMMA 5.1. *Let $n = 2^r$ $(r \geqslant 1)$ and $n! = 2^\alpha s$ $(2 \nmid s)$. If $1 \leqslant \beta \leqslant \alpha$, then there exists a spanning subgraph H of K_n such that $|\Gamma(H)| = 2^\beta$.*

PROOF. Let $n = 2^r$ $(r \geqslant 1)$ and $n! = 2^\alpha s$ $(2 \nmid s)$. We note that $\alpha = 2^r - 1$. We proceed by induction on r. Thus $P(r)$ is the statement that if $1 \leqslant \beta \leqslant 2^r - 1$, then there exists a spanning subgraph H of K_n such that $|\Gamma(H)| = 2^\beta$. Clearly $P(1)$ is true. Now assume $P(k)$ is true for $1 \leqslant k < r$ and consider $P(r)$. If $\beta = 2^r - 1$, then there exists a spanning subgraph H of K_n (for example $H(2^r)$) such that $|\Gamma(H)| = 2^{2^r - 1}$. Therefore suppose $1 \leqslant \beta < 2^r - 1$. Choose, if possible, integers β_1 and β_2 such that $1 \leqslant \beta < \beta_2 \leqslant 2^{r-1} - 1$ and $\beta = \beta_1 + \beta_2$. Then by the inductive hypothesis we may choose connected graphs G_1 and G_2 so that $|V(G_1)| = |V(G_2)| = 2^{r-1}$ and $|\Gamma(G_1)| = 2^{\beta_1}$, $|\Gamma(G_2)| = 2^{\beta_2}$. Let $G = G_1 \cup G_2$. Then since $\Gamma(G_1) \not\cong \Gamma(G_2)$, $G_1 \not\cong G_2$. Hence, $|\Gamma(G)| = |\Gamma(G_1)|\,|\Gamma(G_2)| = 2^{\beta_1 + \beta_2} = 2^\beta$. Now clearly if $2 < \beta \leqslant 2^r - 3$ we may always choose β_1 and β_2 in this way. Now suppose $\beta = 2^r - 2$. Let $G = H(2^{r-1}) \cup \overline{H(2^{r-1})}$. Then $|V(G)| = 2^r$ and $|\Gamma(G)| = 2^{2(2^{r-1} - 1)} = 2^{2^r - 2} = 2^\beta$.

Suppose $\beta=2$. Let P_l denote the path of length l. Then if $r\neq 3$ choose $G=P_{2^r-1}\cup \bar{P}_{2^r-1}$. Clearly $|V(G)|=2^r$ and $|\Gamma(G)|=2^2=2^\beta$. When $r=3$ we may choose the graph G illustrated in Fig. 1. Clearly

$$|V(G)|=8=2^r \quad \text{and} \quad |\Gamma(G)|=2^2=2^\beta.$$

Finally, if $\beta=1$, choose $G=P_{2^r}$. Thus $|V(G)|=2^r$ and $|\Gamma(G)|=2=2^\beta$.

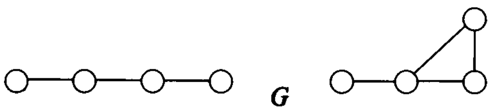

FIGURE 1

We now extend this result to the case of n any integer greater than 1.

THEOREM 5.2. *Let $n\geqslant 2$ be an integer and let $n!=2^\alpha s\,(2\nmid s)$. Then if $1\leqslant\beta\leqslant\alpha$ there exists a spanning subgraph H of K_n such that $|\Gamma(H)|=2^\beta$.*

PROOF. Let $n=\sum_{i=0}^{n}\varepsilon_i 2^i\ (\varepsilon_i\in\{0,1\})$ be the binary decomposition of n. By Lemma 5.1 we may choose a connected graph H_i such that $|V(H_i)|=2^i$ and $|\Gamma(H_i)|=2^{\beta_i}$, $1\leqslant\beta_i\leqslant 2^i-1$. Let $G=\dot{\bigcup}_{i=0}^{n}\varepsilon_i H_i$. Then $|V(G)|=\sum_{i=1}^{n}\varepsilon_i 2^i=n$ and $|\Gamma(G)|=\prod_{i=0}^{n}\varepsilon_i|\Gamma(H_i)|=2^\beta$ when $\beta=\sum_{i=0}^{n}\varepsilon_i\beta_i$. Now by Lemma 3.2, $\alpha=\sum_{i=0}^{n}\varepsilon_i(2^i-1)$. Hence $\sum_{i=0}^{n}\varepsilon_i\leqslant\beta\leqslant\alpha$. Clearly therefore if $\sum_{i=0}^{n}\varepsilon_i\leqslant\beta\leqslant\alpha$, then we can construct G so that $|V(G)|=n$ and $|\Gamma(G)=2^\beta$. By combining paths, asymmetric graphs and unicyclic graphs of the type which is a component of the graph of Fig. 1, we may construct graphs whose automorphism groups have order 2^β for $1\leqslant\beta<\sum_{i=1}^{n}\varepsilon_i$.

As we have already stated it would be of interest to determine all graphical permutation 2-groups. It is clear that not all permutation 2-groups are graphical since C_{2^r}, the cyclic group generated by a cycle of length 2^r, comes into this class.

The results we have obtained in this area are not very deep and are obviously much weaker than the results on 2-Sylow subgroups where we can actually characterize the graphs involved. It would be nice to have a characterization of the graphs (and groups) in the more general situation.

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