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POLYNOMIAL-RATE CONVERGENCE TO THE STATIONARY STATE FOR THE CONTINUUM-TIME LIMIT OF THE MINORITY GAME

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Abstract

In this paper we show that the continuum-time version of the minority game satisfies the criteria for the application of a theorem on the existence of an invariant measure. We consider the special case of a game with a 'sufficiently' asymmetric initial condition, where the number of possible choices for each individual is $S = 2$ and $\Gamma < +\infty$. An upper bound for the asymptotic behavior, as the number of agents grows to infinity, of the waiting time for reaching the stationary state is then obtained.

Keywords: Continuum-time minority game; stationary state; polynomial mixing bounds

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1. Introduction

The minority game (MG) [4] is a simple model based on Arthur's 'El Farol' bar problem [1], which describes the behavior of a group of competing heterogeneous agents subject to the economic law of supply and demand. An application of the MG is, for example, the microscopic modeling of financial markets [3], [5], [6], [8].

In this paper the attention is focused on the continuum-time version of the MG (see, for example, [2] and [7]), where the number of possible choices for each individual is $S = 2$. In particular, we are interested in studying its long-time behavior, in terms of existence of an invariant measure for its dynamical variables and convergence of the dynamical variables distribution to it.

Let us fix some notation, consistent with the notation used in [9]. Consider the MG with N agents. Its dynamics are defined in terms of dynamical variables $U_{s,i}(t)$, $t = 0, 1, \dots$, in discrete time; these are scores corresponding to each of the possible agents' choices $s = +1, -1$. Each agent takes a decision $s_i(t)$ with

$$\Pr\{s_i(t) = s\} = \frac{\exp(\Gamma_i U_{s,i}(t))}{\sum_{s'} \exp(\Gamma_i U_{s',i}(t))},$$

where $\Gamma_i > 0$ and $s' \in \{-1, +1\}$. The original MG corresponds to $\Gamma_i = \infty$ and was generalized to $\Gamma_i = \Gamma < \infty$ [2]; here we consider the latter case.

The public information variable $\mu(t)$ is given to all agents; it belongs to the set of integers $\{1, \dots, P\}$ and can either be the binary encoding of the last M winning choices or drawn at random from a uniform distribution; here we consider the latter case.

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The action $a^{\mu(t)}_{s_i(t),i}$ of each agent depends on its choice $s_i(t)$ and on $\mu(t)$. The coefficients $a^{\mu}_{s,i}$, called strategies, are uniform random variables taking values ± 1 ($\Pr\{a^{\mu}_{s,i} = \pm 1\} = \frac{1}{2}$) independent of i, s, and μ.

Let us introduce the following random variables (to ease the notation, the choices $+1$ and -1 are shorted to $+$ and $-$, respectively):

$$\xi^{\mu}_i = \frac{a^{\mu}_{+,i} - a^{\mu}_{-,i}}{2}, \qquad \Theta^{\mu} = \sum_{i=1}^{N} \frac{a^{\mu}_{+,i} + a^{\mu}_{-,i}}{2},$$

and their averages

$$\overline{\xi_i \boldsymbol{\Theta}} = \frac{1}{P} \sum_{\mu=1}^{P} \xi^{\mu}_i \Theta^{\mu}, \qquad \overline{\xi_i \xi_j} = \frac{1}{P} \sum_{\mu=1}^{P} \xi^{\mu}_i \xi^{\mu}_j.$$

The only relevant quantity in the dynamics is the difference between the scores of the two strategies:

$$y_i(t) = \Gamma \frac{U_{+,i}(\tau) - U_{-,i}(\tau)}{2},$$

where $\tau = t/\Gamma$.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the probability space with respect to which all our random variables, $\boldsymbol{y} = (y_i)_{1 \le i \le N}$, $\boldsymbol{\Theta} = (\Theta^{\mu})_{1 \le \mu \le P}$, and $\boldsymbol{\xi} = (\xi_i)_{1 \le i \le N}$, are defined.

As shown in [9], if $P/N = \alpha \in \mathbb{R}_+$, $S = 2$, and $\Gamma_i = \Gamma > 0$ for all i, the dynamics of the continuum-time limit of the MG is given by the following N-dimensional stochastic differential equation:

$$\mathrm{d}y_i(t) = \biggl(-\overline{\xi_i \boldsymbol{\Theta}} - \sum_{j=1}^{N} \overline{\xi_i \xi_j} \tanh(y_j)\biggr) \mathrm{d}t + \boldsymbol{A}_i(\boldsymbol{y}, N, \Gamma, \boldsymbol{\xi})\, \mathrm{d}W(t), \qquad i = 1, \dots, N, \quad (1.1)$$

where $W(t)$ is an N-dimensional Wiener process and $\boldsymbol{A}_i$ is the ith row of the $N \times N$ matrix $\boldsymbol{A} = (A_{ij})$ such that

$$(\boldsymbol{A}\boldsymbol{A}^{\top})_{ij}(\boldsymbol{y}, N, \Gamma, \boldsymbol{\xi}) = \frac{\Gamma \sigma^2_{N,\Gamma}(\boldsymbol{y})}{\alpha N} \overline{\xi_i \xi_j}.$$

If $\alpha > \alpha_{\mathrm{c}}$, where $\alpha_{\mathrm{c}} = 0.3374\dots$, marks the transition point from a symmetric ($\alpha < \alpha_{\mathrm{c}}$) to an asymmetric phase ($\alpha > \alpha_{\mathrm{c}}$) characterized, respectively, by no predictability and predictability (in the asymmetric phase the choices $+1$ and -1 do not appear with equal probabilities for a given $\mu(t)$), then the function $\sigma^2_{N,\Gamma} \colon \mathbb{R}^N \to \mathbb{R}_+$ is continuous and

$$\lim_{N \to \infty} \sup_{y \in \mathbb{R}^N} \frac{\sigma^2_{N,\Gamma}(\boldsymbol{y})}{N} \le 1$$

(see [2] and [9]).

In [9] the authors derived the full stationary distribution of $\boldsymbol{y}$. Here, by proving that (1.1) satisfies the criteria for the application of Veretennikov's theorem (see [11] and Appendix A), we show, for the continuum-time version of the MG with $S = 2$, a 'sufficiently' asymmetric initial condition, and $\Gamma < +\infty$ and $\alpha > \alpha_{\mathrm{c}}$, that, for N sufficiently large, the distribution $\nu(t)$ of $\boldsymbol{y}$ converges almost surely (a.s.) with polynomial rate to an invariant measure ν_{inv} and that the waiting time for reaching the stationary state is at most $O(|y(0)|^2)$. For a finite asymmetric initial condition, this means that the waiting time is at most $O(N)$ or, since $P/N = \alpha > 0$, $O(P/\alpha)$.

The work is organized as follows. In Section 2 some preliminary computations useful to characterize the asymptotic behavior of the drift coefficient of (1.1) are performed. In Section 3 Veretennikov's theorem is applied to (1.1) with both finite and maximally ($|y(0)| \to \infty$) asymmetric initial conditions and an upper bound for the limiting behavior, as N grows to infinity, of the waiting time for reaching the stationary state is obtained. In Section 4 some conclusions are drawn, while Appendix A contains Veretennikov's theorem.

2. Preliminaries

Let us start by proving some preliminary results which show that, despite the fact that the coefficients present in the drift term of (1.1) are random variables assuming both negative and positive values, as N grows to infinity, the behavior of (1.1) becomes dissipative and, hence, suitable to the application of a stability theorem, such as Veretennikov's theorem. For the sake of notational simplicity, let us assume that $P = N$, i.e. $\alpha = 1$. The results obtained still hold for any $\alpha > 0$. From now on, when defining a probability event, we shall omit the explicit dependence of the random variables on the ωs, $\omega \in \Omega$.

Lemma 2.1. *For every $i = 1, 2, \dots,$*

$$\mathrm{P}\bigg\{\lim_{N\to\infty} \overline{\xi_i \mathbf{\Theta}} = 0 \wedge \lim_{N\to\infty} \overline{\xi_i^2} = a \wedge \lim_{N\to\infty} \sum_{j=1,\, j\neq i}^{N} \overline{\xi_i \xi_j} = 0\bigg\} = \mathrm{P}\bigg\{\lim_{N\to\infty} \overline{\xi_i^2} = a\bigg\},$$

where a is any constant.

Proof. Obviously, for every i,

$$\mathrm{P}\bigg\{\lim_{N\to\infty} \overline{\xi_i \mathbf{\Theta}} = 0 \wedge \lim_{N\to\infty} \overline{\xi_i^2} = a \wedge \lim_{N\to\infty} \sum_{j=1,\, j\neq i}^{N} \overline{\xi_i \xi_j} = 0\bigg\} \leq \mathrm{P}\bigg\{\lim_{N\to\infty} \overline{\xi_i^2} = a\bigg\}.$$

Hence, it remains to show that the inverse inequality also holds.

For every i,

$$\begin{aligned}
&\mathrm{P}\bigg\{\lim_{N\to\infty} \overline{\xi_i \mathbf{\Theta}} = 0 \wedge \lim_{N\to\infty} \overline{\xi_i^2} = a \wedge \lim_{N\to\infty} \sum_{j=1,\, j\neq i}^{N} \overline{\xi_i \xi_j} = 0\bigg\} \\
&\quad = \mathrm{P}\bigg\{\lim_{N\to\infty}\bigg(\frac{1}{N}\sum_{\mu=1}^{N}\bigg(\frac{(a_{+,i}^{\mu})^2 - (a_{-,i}^{\mu})^2}{4} + \xi_i^{\mu} \sum_{j=1,\, j\neq i}^{N} \frac{a_{+,j}^{\mu} + a_{-,j}^{\mu}}{2}\bigg)\bigg) \\
&\qquad = 0 \wedge \lim_{N\to\infty} \overline{\xi_i^2} = a \wedge \lim_{N\to\infty} \sum_{j=1,\, j\neq i}^{N} \frac{1}{N} \sum_{\mu=1}^{N} \xi_i^{\mu} \xi_j^{\mu} = 0\bigg\}. \qquad (2.1)
\end{aligned}$$

Since $\{\xi_i^{\mu}\}_{\mu,i}$ is a family of independent (on both μ and i) random variables, identically distributed over the set $\{-1, 0, 1\}$ and since $(a_{+,i}^{\mu})^2 - (a_{-,i}^{\mu})^2 = 0$, the right-hand side term of (2.1) is greater than or equal to

$$\begin{aligned}
&\mathrm{P}\bigg\{\lim_{N\to\infty}\frac{1}{N}\sum_{\mu=1}^{N}\sum_{j=1,\,j\neq i}^{N}\frac{a_{+,j}^{\mu}+a_{-,j}^{\mu}}{2}=0\wedge\lim_{N\to\infty}\overline{\xi_i^2}=a\wedge\lim_{N\to\infty}\sum_{j=1,\,j\neq i}^{N}\frac{1}{N}\sum_{\mu=1}^{N}\xi_j^{\mu}=0\bigg\}\\
&\quad=\mathrm{P}\Big\{\lim_{N\to\infty}\overline{\xi_i^2}=a\Big\}\\
&\qquad\times\mathrm{P}\bigg\{\lim_{N\to\infty}\frac{1}{N}\sum_{\mu=1}^{N}\sum_{j=1,\,j\neq i}^{N}\frac{a_{+,j}^{\mu}+a_{-,j}^{\mu}}{2}=0\wedge\lim_{N\to\infty}\sum_{j=1,\,j\neq i}^{N}\frac{1}{N}\sum_{\mu=1}^{N}\xi_j^{\mu}=0\bigg\}.
\end{aligned}$$

Let $\eta_j=\sum_{\mu=1}^{N}\xi_j^{\mu}$ and $\zeta_j=\sum_{\mu=1}^{N}(a_{+,j}^{\mu}+a_{-,j}^{\mu})/2$ (η_j and ζ_j both depend on N, but since, for every N, $\mathrm{E}[\eta_j]=\mathrm{E}[\zeta_j]=0$, we omit explicitly writing the N dependence); $\{\eta_j+\zeta_j=\sum_{\mu=1}^{N}a_{+,j}^{\mu}\}_j$ is a family of independent, identically distributed random variables with mean $\mathrm{E}[\eta_j+\zeta_j]=0$.

Since, by the law of large numbers (LLN),

$$\mathrm{P}\bigg\{\lim_{N\to\infty}\frac{1}{N}\sum_{j=1,\,j\neq i}^{N}(\eta_j+\zeta_j)=0\bigg\}=1,\qquad \mathrm{P}\bigg\{\lim_{N\to\infty}\frac{1}{N}\sum_{j=1,\,j\neq i}^{N}\eta_j=0\bigg\}=1,$$

it follows that

$$\mathrm{P}\bigg\{\lim_{N\to\infty}\frac{1}{N}\sum_{j=1,\,j\neq i}^{N}\zeta_j=0\bigg\}=1,$$

and the thesis follows.

Lemma 2.2. *For every* $i=1,2,\dots,$

$$\mathrm{P}\Big\{\lim_{N\to\infty}\overline{\xi_i^2}=\tfrac{1}{2}\Big\}=1.$$

Proof. Since $\mathrm{E}[(\xi_i^{\mu})^2]=\frac{1}{2}$, it is a consequence of the LLN.

Lemma 2.3. *For every* $i=1,2,\dots,$

$$\mathrm{P}\bigg\{\lim_{N\to\infty}\overline{\xi_i\boldsymbol{\Theta}}=0\wedge\lim_{N\to\infty}\overline{\xi_i^2}=\frac{1}{2}\wedge\lim_{N\to\infty}\sum_{j=1,\,j\neq i}^{N}\overline{\xi_i\xi_j}=0\bigg\}=1.$$

Proof. It is an immediate consequence of Lemma 2.1 and Lemma 2.2.

Lemma 2.4. *Let*

$$B_i=\bigg\{\lim_{N\to\infty}\overline{\xi_i^2}=\frac{1}{2}\bigwedge_{j=1,2,\dots,\,j\neq i}\lim_{N\to\infty}\overline{\xi_i\xi_j}=0\bigg\},\qquad i=1,2,\dots.$$

Then

$$\mathrm{P}\bigg\{\bigcap_{i=1,2,\dots}B_i\bigg\}=1.$$

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Proof. For every $i = 1, 2, \ldots,$

$$\begin{aligned} P(B_i) &\geq P\left\{\lim_{N\to\infty} \overline{\xi_i^2} = \frac{1}{2} \bigwedge_{j=1,2,\ldots,\, j\neq i} \lim_{N\to\infty} \frac{1}{N}\sum_{\mu=1}^{N} \xi_j^{\mu} = 0\right\} \\ &= P\left\{\lim_{N\to\infty} \overline{\xi_i^2} = \frac{1}{2}\right\} \prod_{j=1,2,\ldots,\, j\neq i} P\left\{\lim_{N\to\infty} \frac{1}{N}\sum_{\mu=1}^{N} \xi_j^{\mu} = 0\right\} \\ &= 1, \end{aligned} \tag{2.2}$$

where last equality is due to Lemma 2.2 and the LLN.

Let

$$D_1 = B_1, \qquad D_i = B_i \cap B_{i-1}, \qquad i = 2, 3, \ldots.$$

Obviously, $P\{D_1\} = P\{B_1\} = 1$. Under the inductive hypothesis, $P\{D_{i-1}\} = 1$; by (2.2),

$$P\{D_i\} = P\{D_{i-1} \cap B_i\} = P\{B_i \mid D_{i-1}\} P\{D_{i-1}\} = 1.$$

Since $\{D_i\}_{i\in\mathbb{N}}$ is a decreasing sequence,

$$P\left\{\bigcap_{i=1,2,\ldots} B_i\right\} = P\left\{\bigcap_{i=1,2,\ldots} D_i\right\} = \lim_{i\to\infty} P\{D_i\} = 1.$$

Let us define the following events:

$$A_1 = \left\{\lim_{N\to+\infty} \lim_{x_1\to\pm\infty} \frac{1}{N}\left(\overline{\xi_1 \boldsymbol{\Theta}} x_1 + \overline{\xi_1^2} \tanh(x_1) x_1 + \sum_{j=2}^{N} \overline{\xi_1 \xi_j} \tanh(x_j) x_1\right) = \frac{1}{2}\right\},$$

$$A_i = \left\{\lim_{N\to+\infty} \lim_{|x|\to\infty} \frac{1}{N}\left(\overline{\xi_i \boldsymbol{\Theta}} x_i + \overline{\xi_i^2} \tanh(x_i) x_i + \sum_{j=1,\, j\neq i}^{N} \overline{\xi_i \xi_j} \tanh(x_j) x_i\right) \in \left[0, \frac{1}{2}\right]\right\},$$

for $i = 2, 3 \ldots$ and

$$E_1 = A_1, \qquad E_i = A_i \cap E_{i-1}, \qquad i = 2, 3, \ldots.$$

Lemma 2.5. *For every $i = 1, 2, \ldots,$*

$$P\{A_i\} = 1.$$

Proof. Since

$$\lim_{N\to+\infty} \lim_{x_1\to\pm\infty} \frac{\tanh(x_1) x_1}{N} = 1 \quad \text{and} \quad \lim_{x_1\to\pm\infty,\, x_j\to\pm\infty} \frac{\tanh(x_1) x_1}{\tanh(x_j) x_1} = \pm 1,$$

by Lemma 2.3, $P\{A_1\} = 1$. For the A_i, $i = 2, 3, \ldots,$ if $x_i \to \pm\infty$, obviously, $P\{A_i\} = 1$. If $x_i \neq \pm\infty$, since tanh is a bounded odd function, by Lemma 2.3, $P\{A_i\} = 1$.

Lemma 2.6. *For every $i = 1, 2, \ldots,$*

$$P\{E_i\} = 1.$$

Proof. The proof is the same as Lemma 2.4 with E_i and A_i instead of D_i and B_i.

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3. Convergence to the invariant measure

In this section we show that, for sufficiently large N, the distribution of the random variable $\boldsymbol{y}$, the score differences vector whose dynamics are described by the stochastic differential equation (1.1), admits an invariant measure and we study the limiting behavior of the waiting time for reaching the stationary state.

For this purpose, we perform an opportune rescaling of the variable $\boldsymbol{y}$ and show that the dynamics of the rescaled variable $\boldsymbol{z}$ satisfy the criteria for the application of Veretennikov's theorem (see Theorem A.1 in Appendix A or [11]). As a second step, we extend the thesis of Veretennikov's theorem to the original random variable $\boldsymbol{y}$.

Veretennikov's theorem gives, under quite general regularity assumptions, a condition that suffices to ensure the existence of an invariant measure and the convergence to it for the distribution of a random variable satisfying a stochastic differential equation. The criteria for the application of such a theorem to a stochastic differential equation are based on the evaluation of the drift term on the initial condition that must be such that $|\boldsymbol{y}(0)| \neq 0$; this means that we must avoid a game where all the agents have a symmetric initial condition (i.e. $\boldsymbol{y}(0) = \boldsymbol{0}$). We focus our attention on two cases: a game where there exists at least one agent having a maximally asymmetric initial condition ($|y_i(0)| \to \infty$) irrespective of the other agents (they may or may not have symmetric initial conditions), and a game where $|\boldsymbol{y}(0)| < \infty$ and the number of agents with an asymmetric initial condition ($y_i(0) \neq 0$) is $O(N)$. The former case corresponds to a game where there is at least one agent who plays the same strategy from the beginning (a so called producer), while the latter case corresponds to a game where the number of agents who perceive a strategy to be more successful than another one from the beginning is $O(N)$.

Let $b_i^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{x})\colon (\{-1, 0, 1\}^{P\times N}, \mathbb{R}^P, \mathbb{R}^N) \to \mathbb{R}$ be defined as

$$b_i^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{x}) = \overline{\xi_i \boldsymbol{\Theta}} + \sum_{j=1}^{N} \overline{\xi_i \xi_j} \tanh(x_j), \qquad i = 1, 2, \dots, N,$$

and

$$\boldsymbol{b}^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{x}) = (b_i^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{x}))_{1\le i\le N}.$$

Equation (1.1) can be written in the form

$$\mathrm{d}\boldsymbol{y}(t) = -\boldsymbol{b}^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{y})\,\mathrm{d}t + \boldsymbol{A}(\boldsymbol{y}, N, \Gamma, \boldsymbol{\xi})\,\mathrm{d}W(t). \tag{3.1}$$

Obviously, $b^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{y})$ is a Borel-measurable, locally bounded function.

Let c be a bounded constant (that may depend on N) greater than 1 and $\boldsymbol{z} = c\boldsymbol{y}$; under this rescaling of the variable $\boldsymbol{y}$, (3.1) becomes

$$\mathrm{d}z(t) = -c\boldsymbol{b}^N\left(\boldsymbol{\xi}, \boldsymbol{\Theta}, \frac{z}{c}\right)\mathrm{d}t + c\boldsymbol{A}\left(\frac{z}{c}, N, \Gamma, \boldsymbol{\xi}\right)\mathrm{d}W(t). \tag{3.2}$$

Before going on to study the limiting behavior of (3.2) as N grows to infinity, let us prove the nondegeneracy of the the diffusion matrix $\boldsymbol{A}$, a necessary condition for the application of Veretennikov's theorem.

Proposition 3.1. *There exists $\hat{N} > 0$ such that, for every $N > \hat{N}$, the diffusion matrix $\boldsymbol{A}$ is a.s. nondegenerate.*

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Proof. Nondegenerancy is equivalent to the following condition:

$$\inf_{\boldsymbol{y}} \inf_{\boldsymbol{\xi}} \inf_{|\boldsymbol{x}|=1} \boldsymbol{x} \boldsymbol{A} \boldsymbol{A}^{\top}(\boldsymbol{y}, N, \Gamma, \boldsymbol{\xi}) \boldsymbol{x}^{\top} > 0,$$

where Γ is fixed and $\boldsymbol{x} \in \mathbb{R}^N$.

Since $\sigma^2_{N,\Gamma} > 0$, it is sufficient to show that

$$\mathrm{P}\left\{ \lim_{N\to\infty} \begin{pmatrix} \overline{\xi_1^2} & \overline{\xi_1\xi_2} & \cdots & \overline{\xi_1\xi_N} \\ \overline{\xi_2\xi_1} & \overline{\xi_2^2} & \cdots & \overline{\xi_2\xi_N} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\xi_N\xi_1} & \overline{\xi_N\xi_2} & \cdots & \overline{\xi_N^2} \end{pmatrix} = a\boldsymbol{I} \right\} = 1,$$

where a is a positive constant and $\boldsymbol{I}$ is the identity matrix.

This follows from Lemma 2.4 and, hence, we obtain the thesis.

3.1. Minority game with a maximally asymmetric initial condition

We consider a game where there exists at least one producer, that is, an agent i such that $|y_i(0)| \to \infty$. We show that, under an appropriate choice of the constant c, system (3.2) satisfies the criteria for the application of Veretennikov's theorem and then we extend the results of Veretennikov's theorem to the original system (3.1). To ease the notation, let $\boldsymbol{y}(0) = \boldsymbol{x}$ and let $\langle\cdot,\cdot\rangle$ denote the Euclidean scalar product in $\mathbb{R}^N$.

Proposition 3.2. *We have*

$$\mathrm{P}\left\{ \lim_{N\to\infty} \lim_{|\boldsymbol{x}|\to\infty} \frac{\langle \boldsymbol{b}^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{x}), \boldsymbol{x}\rangle}{N} \geq \frac{1}{2} \right\} = 1.$$

Proof. We have

$$\begin{aligned}
&\mathrm{P}\left\{ \lim_{N\to\infty} \lim_{|\boldsymbol{x}|\to+\infty} \frac{\langle \boldsymbol{b}^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{x}), \boldsymbol{x}\rangle}{N} \geq \frac{1}{2} \right\} \\
&\quad = \mathrm{P}\left\{ \lim_{N\to+\infty} \lim_{|\boldsymbol{x}|\to+\infty} \frac{1}{N} \sum_{i=1}^{N} \left(\overline{\xi_i \boldsymbol{\Theta}} x_i + \overline{\xi_i^2} \tanh(x_i) x_i + \sum_{j=1,\, j\neq i}^{N} \overline{\xi_i\xi_j} \tanh(x_j) x_i \right) \geq \frac{1}{2} \right\} \\
&\quad = \mathrm{P}\left\{ \sum_{i=1}^{\infty} \lim_{N\to+\infty} \lim_{|\boldsymbol{x}|\to+\infty} \frac{1}{N} \left(\overline{\xi_i \boldsymbol{\Theta}} x_i + \overline{\xi_i^2} \tanh(x_i) x_i + \sum_{j=1,\, j\neq i}^{N} \overline{\xi_i\xi_j} \tanh(x_j) x_i \right) \geq \frac{1}{2} \right\} \\
&\quad \geq \mathrm{P}\left\{ \bigcap_{i=1,2,\ldots} A_i \right\} \\
&\quad = \mathrm{P}\left\{ \bigcap_{i=1,2,\ldots} E_i \right\}.
\end{aligned}$$

Since $\{E_i\}_{i\in\mathbb{N}}$ is a decreasing sequence,

$$\mathrm{P}\left\{ \bigcap_{i=1,2,\ldots} E_i \right\} = \lim_{i\to+\infty} \mathrm{P}\{E_i\}.$$

By applying Lemma 2.6, the thesis follows.

Corollary 3.1. *There exist $\tilde{N} > 0$ and $M_0 > 0$ such that, for every $N > \tilde{N}$,*

$$\left\langle \boldsymbol{b}^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{x}), \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right\rangle > \left(\frac{N}{2} - 1\right)\frac{1}{|\boldsymbol{x}|}, \qquad |\boldsymbol{x}| \geq M_0 \text{ a.s.}$$

Proof. It is an immediate consequence of Proposition 3.2.

Proposition 3.3. *Equation (3.2) with $c > 1 + 2/(N/2 - 1)$ satisfies a.s. condition (A.1) in Appendix A with $r = c(N/2 - 1) > N/2 + 1$.*

Proof. It is an immediate consequence of Corollary 3.1 and Proposition 3.1.

In Proposition 3.4, below, Veretennikov's theorem is applied to (3.2), while the existence of an invariant measure for $\boldsymbol{y}$ and the rate of convergence to it is derived in Corollary 3.2, below.

Proposition 3.4. *Let $\nu(t)$ be the distribution of $\boldsymbol{z}(t) = (z_i(t))_{1\leq i\leq N}$ satisfying (3.2) with initial condition $\boldsymbol{z}(0)$ and $c > 1+2/(N/2 - 1)$. For sufficiently large N, there exists a.s. an invariant measure ν_{inv} for $\boldsymbol{z}$ such that*

$$|\nu(t) - \nu_{\text{inv}}| \leq m(1 + |\boldsymbol{z}(0)|^l)(1 + t)^{-(k+1)},$$

where m is a positive constant, $0 < k < c(N/2-1)-N/2-1$, $2k+2 < l < 2c(N/2-1)-N$, and $|\nu(t) - \nu_{\text{inv}}|$ is the total variation distance between $\nu(t)$ and ν_{inv}, i.e. $|\nu(t) - \nu_{\text{inv}}| := \sup_{\boldsymbol{A}\in\mathcal{B}_{\mathbb{R}^d}} |\nu(t)(\boldsymbol{A}) - \nu_{\text{inv}}(\boldsymbol{A})|$.

Proof. It is a consequence of Proposition 3.3 and Theorem A.1.

Corollary 3.2. *Let $\nu'(t)$ be the distribution of $\boldsymbol{y}(t) = (y_i(t))_{1\leq i\leq N}$ satisfying (3.1) with initial condition $\boldsymbol{y}(0)$, and let $c = 1 + 2/(N/2 - 1) + 2/(N/2 - 1)^2$. For sufficiently large N, there exists a.s. an invariant measure ν'_{inv} for $\boldsymbol{y}$ such that*

$$|\nu'(t) - \nu'_{\text{inv}}| \leq m'(1 + |c\boldsymbol{y}(0)|^l)(1 + t)^{-(k+1)}, \tag{3.3}$$

where m' is a positive constant, $0 < k < 2/(N/2 - 1)$, and $2k + 2 < l < 2 + 4/(N/2 - 1)$.

Proof. Since the limit, as N grows to infinity, of c is 1 and $\boldsymbol{z} = c\boldsymbol{y}$, for sufficiently large N, $|\nu'(t) - \nu'_{\text{inv}}| = m'|\nu(t) - \nu_{\text{inv}}|$, and the thesis follows from Proposition 3.4.

3.2. Minority game with a finite asymmetric initial condition

Now we move on to a game where $|\boldsymbol{y}(0)| < \infty$ and the number of agents with an asymmetric initial condition ($y_i(0) \neq 0$) is $O(N)$.

Let $\boldsymbol{y}(0) = \boldsymbol{x}$ and $N' = \#\{y_i : y_i(0) \neq 0\}$, and suppose that $N' = O(N)$, i.e. $\lim_{N\to\infty} N'/N = \gamma$, $0 < \gamma \leq 1$; moreover, let $\beta_i = \tanh(x_i)x_i$ and $\beta = \min\{\beta_i : \beta_i \neq 0\}$. To ease the notation, let us suppose that x_1 is such that $\beta = \tanh(x_1)x_1$.

Proposition 3.5. *We have*

$$\mathrm{P}\left\{\lim_{N\to\infty} \frac{\langle \boldsymbol{b}^N(\boldsymbol{\xi}, \boldsymbol{\Theta}, \boldsymbol{x}), \boldsymbol{x}\rangle}{N} \geq \frac{1}{2}\beta\gamma\right\} = 1.$$

Proof. We have

$$\begin{aligned}
&\mathrm{P}\left\{\lim_{N\to\infty}\frac{\langle \boldsymbol{b}^N(\boldsymbol{\xi},\boldsymbol{\Theta},\boldsymbol{x}),\boldsymbol{x}\rangle}{N}\geq\frac{1}{2}\beta\gamma\right\}\\
&\quad=\mathrm{P}\left\{\lim_{N'\to\infty}\frac{\gamma}{N'}\sum_{i=1}^{N'}\left(\overline{\xi_i\boldsymbol{\Theta}}x_i+\overline{\xi_i^2}\tanh(x_i)x_i+\sum_{j=1,\,j\neq i}^{N'}\overline{\xi_i\xi_j}\tanh(x_j)x_i\right)\geq\frac{1}{2}\beta\gamma\right\}\\
&\quad\geq\mathrm{P}\left\{\lim_{N'\to\infty}\left(\overline{\xi_1\boldsymbol{\Theta}}x_1+\overline{\xi_1^2}\tanh(x_1)x_1+\sum_{j=1,\,j\neq 1}^{N'}\overline{\xi_1\xi_j}\tanh(x_j)x_1\right)=\frac{1}{2}\beta\right\}\\
&\quad=1,
\end{aligned}$$

where the last equality is a consequence of Lemma 2.3.

Corollary 3.3. *There exist $\tilde{N}>0$ and $M_0>0$ such that, for every $N>\tilde{N}$,*

$$\left\langle \boldsymbol{b}^N(\boldsymbol{\xi},\boldsymbol{\Theta},\boldsymbol{x}),\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right\rangle>\beta\gamma\left(\frac{N}{2}-1\right)\frac{1}{|\boldsymbol{x}|},\qquad |\boldsymbol{x}|\geq M_0\ \text{a.s.}$$

Proof. It is an immediate consequence of Proposition 3.5.

Proposition 3.6. *If $0<\beta\gamma\leq 1$ then (3.2) with*

$$c>\frac{1}{\beta\gamma}\left(1+\frac{2}{N/2-1}\right)$$

satisfies a.s. condition (A.1) in Appendix A with $r=c(N/2-1)>N/2+1$.

Proof. It is an immediate consequence of Corollary 3.3 and Proposition 3.1.

In Proposition 3.7, below, Veretennikov's theorem is applied to (3.2), while the existence of an invariant measure for $\boldsymbol{y}$ and the rate of convergence to it is derived in Corollary 3.4, below.

Proposition 3.7. *Let $\nu(t)$ be the distribution of $\boldsymbol{z}(t)=(z_i(t))_{1\leq i\leq N}$ satisfying (3.2) with initial condition $\boldsymbol{z}(0)$ and*

$$c>\frac{1}{\beta\gamma}\left(1+\frac{2}{N/2-1}\right),$$

where $0<\beta\gamma\leq 1$. For sufficiently large N, there exists a.s. an invariant measure ν_{inv} for $\boldsymbol{z}$ such that

$$|\nu(t)-\nu_{\mathrm{inv}}|\leq m(1+|\boldsymbol{z}(0)|^l)(1+t)^{-(k+1)},$$

where m is a positive constant, $0<k<c(N/2-1)-N/2-1$, and $2k+2<l<2c(N/2-1)-N$.

Proof. It is a consequence of Proposition 3.6 and Theorem A.1 in Appendix A.

Corollary 3.4. *Let $\nu'(t)$ be the distribution of $\boldsymbol{y}(t)=(y_i(t))_{1\leq i\leq N}$ satisfying (3.1) with initial condition $\boldsymbol{y}(0)$, and let*

$$c=\frac{1}{\beta\gamma}\left(1+\frac{2}{N/2-1}+\frac{2}{(N/2-1)^2}\right)$$

with $0 < \beta\gamma \leq 1$*. For sufficiently large* N*, there exists a.s. an invariant measure* ν'_{inv} *for* $\boldsymbol{y}$ *such that*

$$|\nu'(t) - \nu'_{\mathrm{inv}}| \leq m'(1 + |c\boldsymbol{y}(0)|^l)(1+t)^{-(k+1)}, \tag{3.4}$$

where m' *is a positive constant,* $0 < k < (N/2+1)(1/\beta\gamma - 1) + (1/\beta\gamma)2/(N/2-1)$*, and* $2k+2 < l < N(1/\beta\gamma - 1) + 2/\beta\gamma + (2/\beta\gamma)2/(N/2-1)$.

Proof. Since the limit, as N grows to infinity, of c is $1/\beta\gamma$ and $\boldsymbol{z} = c\boldsymbol{y}$, for sufficiently large N, $|\nu'(t) - \nu'_{\mathrm{inv}}| = m'|\nu(t) - \nu_{\mathrm{inv}}|$, and the thesis follows from Proposition 3.7.

3.3. Waiting time for reaching the stationary state

Corollaries 3.2 and 3.4 provide a rate of convergence toward the invariant distribution for the score difference distribution; by making t explicit in (3.3) and (3.4), it is hence possible to obtain the limiting behavior, as N grows to infinity, of the waiting time for reaching the stationary state. Since the criteria for the application of Veretennikov's theorem are sufficient conditions for the existence of an invariant measure, the waiting time obtained may not be the smallest one and, hence, we have to consider the upper bound of the waiting time for reaching the stationary state.

Since the scores time is the $\boldsymbol{y}$s time rescaled by Γ, i.e.

$$y_i(t) = \Gamma\left(\frac{U_{+,i}(t/\Gamma) - U_{-,i}(t/\Gamma)}{2}\right),$$

in studying the waiting time for reaching a stationary state for the MG we have to refer to the time $\tau = t/\Gamma$, that is, the own time of the scores $U_{s,i}$ corresponding to each agents possible choices $s = +1, -1$.

In Proposition 3.8, below, the asymptotic behavior of the waiting time for reaching the stationary state both for an MG with maximally and finite asymmetric initial conditions is obtained.

Proposition 3.8. *For every* $\varepsilon > 0$*, let* T *be such that*

$$m'(1 + |c\boldsymbol{y}(0)|^l)(1 + T\Gamma)^{-(k+1)} = \varepsilon,$$

where l *and* k *are as in Corollaries 3.2 and 3.4 (for games with maximally and finite asymmetric initial conditions, respectively).*

It follows that

$$|\nu'(T\Gamma) - \nu'_{\mathrm{inv}}| \leq \varepsilon$$

and

(i) *if* $|\boldsymbol{y}(0)| \to \infty$,

$$\lim_{N\to\infty} \frac{T}{|\boldsymbol{y}(0)|^2} = \frac{m'}{\varepsilon\Gamma},$$

(ii) *if* $\lim_{N\to\infty} N'/N = \gamma$*, where* $0 < \gamma \leq 1$*,* $N' = \#\{y_i : y_i(0) \neq 0\}$*, and* $|\boldsymbol{y}(0)| < \infty$*,*

$$\frac{m''}{\varepsilon\Gamma} \leq \lim_{N\to\infty} \frac{T}{|\boldsymbol{y}(0)|^2} \leq \lim_{N\to\infty} \frac{m''}{\varepsilon\Gamma}|c\boldsymbol{y}(0)|^{N(1/\beta\gamma-1)+2/\beta\gamma+(2/\beta\gamma)2/(N/2-1)-2}, \tag{3.5}$$

(iii) *under the same conditions as (ii), with* $\beta\gamma = 1$,

$$\lim_{N\to\infty} \frac{T}{|\boldsymbol{y}(0)|^2} = \frac{m''}{\varepsilon\Gamma}, \tag{3.6}$$

where m' *and* m'' *are positive constants.*

Proof. (i) By Corollary 3.2, for sufficiently large N,

$$T = \frac{(m'/\varepsilon(1+|c\boldsymbol{y}(0)|^l))^{1/(k+1)} - 1}{\Gamma},$$

where m' is a positive constant, $0 < k < 2/(N/2-1)$, and $2k+2 < l < 2+4/(N/2-2)$. In the limit $N \to \infty$, we obtain $c = 1$, $k = 0$, and $l = 2$; it follows that

$$\lim_{N\to\infty} \frac{T}{|\boldsymbol{y}(0)|^2} = \frac{m'}{\varepsilon\Gamma}.$$

(ii) By Corollary 3.4,

$$\frac{(m''/\varepsilon(1+|c\boldsymbol{y}(0)|^{2k+2}))^{1/(k+1)} - 1}{\Gamma} < T < \frac{(m''/\varepsilon(1+|c\boldsymbol{y}(0)|^{N(1/\beta\gamma-1)+2/\beta\gamma+(2/\beta\gamma)2/(N/2-1)})) - 1}{\Gamma},$$

and (3.5) follows.

(iii) Since $0 < k < (N/2+1)(1/\beta\gamma - 1) + (1/\beta\gamma)2/(N/2-1)$ and $2k+2 < l < N(1/\beta\gamma - 1) + 2/\beta\gamma + (2/\beta\gamma)2/(N/2-1)$, in the limit $N \to \infty$, we obtain $c = 1$, $k = 0$, and $l = 2$, and (3.6) follows from (3.5).

Since, if the initial condition is finite, $|\boldsymbol{y}(0)|^2 = O(N)$, and since $P/N = \alpha > 0$, from Proposition 3.8(iii), it follows that $T = O(P/\alpha\Gamma)$.

4. Conclusions

By applying Veretennikov's theorem to the continuum-time version of the MG, we have obtained an upper bound for the asymptotic behavior, as the number of agents grows to infinity, of the waiting time for reaching the stationary state in the asymmetric phase ($\alpha > \alpha_c$).

Since Veretennikov's theorem gives a sufficient condition for the existence of an invariant measure and it applies only to stochastic differential equations with nonnull initial conditions, the waiting time obtained may not be the smallest one (it is an upper bound) and it holds only for an MG with an asymmetric initial condition. The fewer agents with initial asymmetry in evaluating their strategies, the stronger must be their asymmetry: if their number is $o(N)$ then at least one agent must have maximally initial asymmetry ($|y_i(0)| = \infty$), while if their number is $O(N)$, the initial asymmetry must be inversely proportional to their fraction with respect to the agents population size ($\beta\gamma = 1$). It follows that the single-agent's weakest initial asymmetry allowed by our result is x_i such that $x_i \tanh(x_i) = 1$ ($x_i \approx 1.2$), corresponding to a game where all the agents ($\gamma = 1$) have an asymmetric initial condition. Our result is simply not applicable to a game with initial asymmetry weaker than the previous game. It is worth noting that the limit we have derived agrees with the rule of thumb in performing numerical simulations to wait a number of time steps proportional to $P/\alpha\Gamma$ in order to reach the stationary state and that, being $T = O(P/\alpha\Gamma)$, the time to equilibrium increases as $\alpha > \alpha_c$ goes toward α_c.

Appendix A.

Consider the n-dimensional stochastic differential equation

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{b}(\boldsymbol{X}(t))\,\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}(t))\,\mathrm{d}W(t), \qquad \boldsymbol{X}(0) = \boldsymbol{x} \in \mathbb{R}^n.$$

Here $W(t)$ is an m-dimensional Wiener process with $m \geq n$, $\boldsymbol{b}$ is a locally bounded Borel function from $\mathbb{R}^n$ with values on $\mathbb{R}^n$, and $\boldsymbol{\sigma}$ a bounded continuous nondegenerate matrix $n \times m$-function. Suppose that the drift term satisfies the following condition: there exist constants $M_0 \geq 0$ and $r > 0$ such that

$$\left\langle \boldsymbol{b}(\boldsymbol{x}), \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right\rangle \leq -\frac{r}{|\boldsymbol{x}|}, \qquad |\boldsymbol{x}| \geq M_0. \tag{A.1}$$

Theorem A.1. (Veretennikov's theorem [11].) *Under assumption (A.1) with* $r > n/2 + 1$, *for any* $0 < k < r - n/2 - 1$ *with* $l \in (2k + 2, 2r - n)$,

$$|\mu_{\boldsymbol{x}}(t) - \mu_{\mathrm{inv}}| \leq c(1 + |\boldsymbol{x}|^l)(1 + t)^{-(k+1)},$$

where $|\mu_x(t) - \mu_{\mathrm{inv}}|$ *is the total variation distance between* $\mu_{\boldsymbol{x}}(t)$ *and* μ_{inv}, c *is a positive constant,* $\mu_{\boldsymbol{x}}(t)$ *is the distribution of* X_t, $\boldsymbol{x}$ *being the initial data, and* μ_{inv} *is the invariant measure for* X_t*; in particular,* μ_{inv} *does exists.*

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