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PRIME AND SPECIAL IDEALS IN STRUCTURAL MATRIX RINGS OVER A RING WITHOUT UNITY

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Abstract

A. D. Sands showed that there is a 1-1 correspondence between the prime ideals of an arbitrary associative ring R and the complete matrix ring $\mathsf{M}_n(R)$ via $P \to \mathsf{M}_n(P)$. A structural matrix ring $\mathsf{M}(B,R)$ is the ring of all $n \times n$ matrices over R with 0 in the positions where the $n \times n$ Boolean matrix B, B a quasi-order, has 0. The author characterized the special ideals of $\mathsf{M}(B,R')$, in case R' has unity, for certain special classes of rings. In this note results of Sands and the author are generalized to structural matrix rings over rings without unity. It turns out that, although the class of prime simple rings is not a special class, Nagata's $\mathcal{M}$-radical has the same form in structural matrix rings as the special radicals studied by the author.

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1. Introduction

We work entirely in the category of associative rings, and presuppose a familiarity with the basic results of radical theory, most of which can be found in [6], and of special classes of rings and special radicals (see [1]). R will be a generic symbol for a ring, and "ideal" will mean "two-sided ideal".

$B = [b_{ij}]$ will be a reflexive and transitive $n \times n$ Boolean matrix, that is, an $n \times n$ quasi-order, and E_{st} will be the $n \times n$ Boolean matrix with 1 in position (s,t) and 0 elsewhere. For $\varnothing \neq V \subseteq R$ we set

$$\mathsf{M}(B,V) := \{X = [x_{ij}] \in \mathsf{M}_n(V) \colon b_{ij} = 0 \Rightarrow x_{ij} = 0\},$$

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and we call $\mathbf{M}(B,R)$ a *structural matrix ring.* B determines and is determined by the binary relation $\leq_B$ on $\mathbf{n} := \{1,2,\ldots,n\}$ defined by $i \leq_B j :\Leftrightarrow b_{ij} = 1$. This quasi-order relation naturally gives rise to the equivalence relation $\sim_B$ on $\mathbf{n}$ defined by $i \sim_B j :\Leftrightarrow i \leq_B j$ and $j \leq_B i$. Let β be the number of equivalence classes induced by $\sim_B$ on $\mathbf{n}$, and let $z_1, z_2, \ldots, z_\beta$ be their representatives. We use n_μ to denote the cardinality of z_μ's equivalence class, $\mu \in \boldsymbol{\beta}$, and $y_{1,\mu}, y_{2,\mu}, \ldots, y_{n_\mu,\mu}$ to denote its elements, with $y_{i,\mu} < y_{i+1,\mu}$ for $i = 1,2,\ldots,n_\mu - 1$. Consider the ring epimorphism

$$f_\mu \colon \mathbf{M}(B,R) \to \mathbf{M}_{n_\mu}(R) \text{ defined by } f_\mu([c_{qr}]) = [d_{st}],$$

where $d_{st} = c_{y_{s,\mu} y_{t,\mu}}; 1 \leq s,t \leq n_\mu$. Then

$$\mathcal{K}_\mu := \{X = [x_{ij}] \in \mathbf{M}(B,R) \colon x_{ij} = 0 \text{ if } i \sim_B z_\mu \text{ and } j \sim_B z_\mu\}$$

is the kernel of f_μ.

Theorems 1 and 2 of [5] characterize the prime (resp. prime maximal) ideals of the complete matrix ring $\mathbf{M}_n(R)$ as the sets $\mathbf{M}_n(P)$ corresponding to the prime (resp. prime maximal) ideals P of R, and in [7] the prime (resp. maximal) ideals of $\mathbf{M}(B,R')$, R' a ring with unity, are characterized as the sets $\mathbf{M}(B,P) + \mathcal{K}_\mu$ corresponding to the prime (resp. maximal) ideals P of R' and the ideals $\mathcal{K}_\mu, \mu \in \boldsymbol{\beta}$, of $\mathbf{M}(B,R')$. In fact, if $\mathcal{C}$ is a special class of rings such that $S \in \mathcal{C}$ if and only if $\mathbf{M}_n(S) \in \mathcal{C}$ whenever S has unity, then proposition 2.6 of [7] states that the $\mathcal{C}$-special ideals of $\mathbf{M}(B,R')$ are the sets $\mathbf{M}(B,P') + \mathcal{K}_\mu$ corresponding to the $\mathcal{C}$-special ideals P' of R' and the ideals $\mathcal{K}_\mu, \mu \in \boldsymbol{\beta}$, of $\mathbf{M}(B,R')$. We generalize these results to structural matrix rings over a ring without unity and obtain similarities to the case of a ring with unity as far as prime ideals are concerned, but also a striking difference in the case of maximal ideals.

Theorem 2.7 of [7] shows that for the upper radical class $\mathcal{R}$ determined by a special class $\mathcal{C}$ of rings satisfying the mentioned condition, the equality $\mathcal{R}(\mathbf{M}(B,R)) = \mathbf{M}(B,\mathcal{R}(R)) + \bigcap_{\mu\in\boldsymbol{\beta}} \mathcal{K}_\mu$ holds for every ring R. The ideal $\bigcap_{\mu\in\boldsymbol{\beta}} \mathcal{K}_\mu$ is called the antisymmetric radical of $\mathbf{M}(B,R)$ in [7]. The $\mathcal{M}$-radical of R is the intersection of the prime ideals P of R such that R/P is a simple ring (see [4]), that is, the intersection of the prime maximal ideals of R. We show that, although the class of prime simple rings is not a special class (see Corollary 1 and Theorem 5 of [3]), the equality $\mathcal{M}(\mathbf{M}(B,R)) = \mathbf{M}(B,\mathcal{M}(R)) + \bigcap_{\mu\in\boldsymbol{\beta}} \mathcal{K}_\mu$ holds for the $\mathcal{M}$-radical of a structural matrix ring.

We first state the following two easily-proved generalizations of Lemmas 1.6 and 1.7 of [7].

LEMMA 1.1. *Let* $0 \in V \subseteq R$. *Then*

$$\mathbf{M}(B,V) + \bigcap_{\mu\in\boldsymbol{\beta}} \mathcal{K}_\mu = \bigcap_{\mu\in\boldsymbol{\beta}} (\mathbf{M}(B,V) + \mathcal{K}_\mu).$$

LEMMA 1.2. *Let* $0 \in V \subseteq R$. *Then* $f_\mu^{-1}(\mathbf{M}_{n_\mu}(V)) = \mathbf{M}(B,V) + \mathcal{K}_\mu$.

We denote the set of prime ideals of R by $\operatorname{spec}(R)$.

2. Prime and special ideals

For all $\mu, \xi \in \boldsymbol{\beta}$ such that $z_\mu \leq_B z_\xi$, we set

$$\Lambda_{\mu\xi} := \{\nu \in \boldsymbol{\beta} \colon z_\mu \leq_B z_\nu \text{ and } z_\nu \leq_B z_\xi\}.$$

For a ring R' with unity, a standard argument using the matrix units shows that every ideal of $\mathbf{M}(B,R')$ has the form

$$\mathcal{A}_\theta := \{X = [x_{ij}] \in \mathbf{M}(B,R') \colon x_{ij} \in \theta(\Lambda_{\mu\xi}) \text{ if } i \sim_B z_\mu, j \sim_B z_\xi \text{ and } z_\mu \leq_B z_\xi\}$$

for some set-inclusion preserving function

$$\theta \colon \{\Lambda_{\mu\xi} \colon \mu, \xi \in \boldsymbol{\beta} \text{ and } z_\mu \leq_B z_\xi\} \to \{A' \colon A' \text{ is an ideal in } R'\},$$

that is, a function θ such that $\Lambda_{\mu\xi} \subseteq \Lambda_{\eta\sigma}$ implies $\theta(\Lambda_{\mu\xi}) \subseteq \theta(\Lambda_{\eta\sigma})$.

PROPOSITION 2.1. *The prime ideals of* $\mathbf{M}(B,R)$ *are the sets* $\mathbf{M}(B,P) + \mathcal{K}_\mu$ *corresponding to the prime ideals* P *of* R *and the ideals* $\mathcal{K}_\mu, \mu \in \boldsymbol{\beta}$, *of* $\mathbf{M}(B,R)$.

PROOF. Let $\beta > 1$, otherwise the statement is just Theorem 1 of [5]. If $P \in \operatorname{spec}(R)$ and $\mu \in \boldsymbol{\beta}$, then, by the same theorem, $\mathbf{M}_{n_\mu}(P) \in \operatorname{spec}(\mathbf{M}_{n_\mu}(R))$, and so, by Lemma 1.2, $\mathbf{M}(B,P) + \mathcal{K}_\mu \in \operatorname{spec}(\mathbf{M}(B,R))$.

Conversely, let $\mathcal{P} \in \operatorname{spec}(\mathbf{M}(B,R))$, and imbed R in a ring R' with unity such that R is an ideal in R'. It is trivially seen that $\mathbf{M}(B,R)$ is an ideal in $\mathbf{M}(B,R')$, which implies that $\mathcal{P}$ is an ideal in $\mathbf{M}(B,R')$ (see Remark 2 (page 333) of [4]). Hence, by the remark preceding this proposition, $\mathcal{P} = \mathcal{A}_\theta$ for some set-inclusion preserving function $\theta \colon \{\Lambda_{\mu\xi} \colon \mu, \xi \in \boldsymbol{\beta} \text{ and } z_\mu \leq_B z_\xi\} \to \{A' \colon A' \text{ is an ideal in } R'\}$. Let $\mu, \xi \in \boldsymbol{\beta}$ such that $z_\mu \leq_B z_\xi$, and set $A_{\mu\xi} := \{a \in R \colon a = x_{ij} \text{ for some } i \sim_B z_\mu, j \sim_B z_\xi \text{ and } X = [x_{ij}] \in \mathcal{P}\}$. Then it follows from the definition of $\mathcal{A}_\theta$ that $A_{\mu\xi} = \theta(\Lambda_{\mu\xi})$, and so $\theta(\Lambda_{\mu\xi})$ is an ideal in R. Furthermore, as mentioned in the introduction, $\mathcal{R}(\mathbf{M}(B,R)) = \mathbf{M}(B,\mathcal{R}(R)) + \bigcap_{\mu\in\beta} \mathcal{K}_\mu$ for every special radical $\mathcal{R}$ determined by a special class $\mathcal{C}$ of rings such that $S \in \mathcal{C}$ if and only if $\mathbf{M}_n(S) \in \mathcal{C}$ whenever S has unity, and so in particular for the prime radical $\mathcal{B}$ (that is, Baer's lower radical), $\bigcap_{\mu\in\beta} \mathcal{K}_\mu \subseteq \mathcal{B}(\mathbf{M}(B,R)) \subseteq \mathcal{P}$. Therefore, $A_{\mu\xi} = R$ if $\mu \neq \xi$. However $A_{\mu\mu} \neq R$ for some $\mu \in \boldsymbol{\beta}$, otherwise $\mathcal{P} = \mathbf{M}(B,R)$. We assert now that $\{\nu \in \boldsymbol{\beta} \colon A_{\nu\nu} \neq R\} = \{\mu\}$, for, if further $A_{\eta\eta} \neq R$ for some $\eta \neq \mu$, then we consider the ideals $\mathcal{A}_{\theta'}$ and $\mathcal{A}_{\theta''}$ of $\mathbf{M}(B,R)$ defined by

$$\theta'(\Lambda_{\rho\sigma}) = \begin{cases} A_{\rho\rho}, & \text{if } \rho = \sigma \neq \eta, \\ R, & \text{otherwise,} \end{cases}$$

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and

$$\theta''(\Lambda_{\rho\sigma}) = \begin{cases} A_{\rho\rho}, & \text{if } \rho = \sigma \neq \mu, \\ R, & \text{otherwise.} \end{cases}$$

Then $\mathcal{A}_{\theta'}\mathcal{A}_{\theta''} \subseteq \mathcal{P}$, but $\mathcal{A}_{\theta'} \not\subseteq \mathcal{P}$ and $\mathcal{A}_{\theta''} \not\subseteq \mathcal{P}$, which contradicts the primeness of $\mathcal{P}$, and so our assertion is valid. It follows now from the form of $\mathcal{A}_\theta (= \mathcal{P})$ that $\mathcal{P} = \mathbf{M}(B, A_{\mu\mu}) + \mathcal{K}_\mu$, and so $\mathcal{P}$ contains the kernel of f_μ, which implies that $\mathbf{M}_{n_\mu}(P) = f_\mu(\mathbf{M}(B, A_{\mu\mu}) + \mathcal{K}_\mu) \in \operatorname{spec}(\mathbf{M}_{n_\mu}(R))$. This completes the proof in the light of Theorem 1 of [5].

EXAMPLE 2.2. If

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

then the prime ideals of $\mathbf{M}(B, R)$ are

$$\begin{bmatrix} P & P & R & 0 \\ P & P & R & 0 \\ 0 & 0 & R & 0 \\ R & R & R & R \end{bmatrix}, \begin{bmatrix} R & R & R & 0 \\ R & R & R & 0 \\ 0 & 0 & P & 0 \\ R & R & R & R \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} R & R & R & 0 \\ R & R & R & 0 \\ 0 & 0 & R & 0 \\ R & R & R & P \end{bmatrix}$$

for $P \in \operatorname{spec}(R)$.

COROLLARY 2.3. *The semi-prime ideals of* $\mathbf{M}(B, R)$ *are the sets* $\mathcal{A}_\theta$ *corresponding to the set-inclusion preserving functions* θ *mapping all the* $\Lambda_{\nu\nu}$*'s onto semi-prime ideals of* R *and all the* $\Lambda_{\rho\sigma}$*'s,* $\rho \neq \sigma$*, onto* R.

Let $\mathcal{C}$ be a special class of rings such that $T \in \mathcal{C}$ if and only if $\mathbf{M}_n(T) \in \mathcal{C}$. Examples of such rings are the classes of simple rings with unity, strongly prime rings (see [2]) and, of course, prime rings. Note, however, that the latter condition is stronger than condition I in [1] and in Proposition 2.6 of [7]:

I. If R' is a ring with unity, then $R' \in \mathcal{C}$ if and only if $\mathbf{M}_n(R') \in \mathcal{C}$. Since every $\mathcal{C}$-special ideal is prime, we can now generalize Proposition 2.1.

THEOREM 2.4. *Let* $\mathcal{C}$ *be a special class of rings such that* $T \in \mathcal{C}$ *if and only if* $\mathbf{M}_n(T) \in \mathcal{C}$. *The* $\mathcal{C}$*-special ideals of* $\mathbf{M}(B, R)$ *are the sets* $\mathbf{M}(B, P) + \mathcal{K}_\mu$ *corresponding to the* $\mathcal{C}$*-special ideals* P *of* R *and the ideals* $\mathcal{K}_\mu$*,* $\mu \in \boldsymbol{\beta}$*, of* $\mathbf{M}(B, R)$.

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PROOF. See the proof of Proposition 2.6 of [7].

3. Maximal and prime maximal ideals

It follows from Theorem 2.4 that the maximal ideals $\mathcal{N}$ of $\mathbf{M}(B,R)$ such that $\mathbf{M}(B,R)/\mathcal{N}$ has unity, are the sets $\mathbf{M}(B,N)+\mathcal{K}_\mu$ corresponding to the maximal ideals N of R such that R/N has unity and the ideals $\mathcal{K}_\mu, \mu \in \boldsymbol{\beta}$, of $\mathbf{M}(B,R)$. Since a characterization of the maximal ideals of $\mathbf{M}_n(R)$, the "simplest" structural matrix ring, has not yet been obtained, one cannot expect the maximal ideals of $\mathbf{M}(B,R)$, in general, to have the form in Theorem 2.4. What is surprising, however, is that not even the antisymmetric radical $\bigcap_{\mu\in\boldsymbol{\beta}} \mathcal{K}_\mu$ of $\mathbf{M}(B,R)$ is, in general, contained in the maximal ideals of $\mathbf{M}(B,R)$, that is, the form of the ideals in Theorem 2.4 breaks down:

EXAMPLE 3.1. Consider the ideal $\{0,2,4,6\}$ of $\mathbf{Z}_8$ as the ring R, and let $N := \{0,4\}$. Then N is maximal, but not prime, in R, and R/N is without unity. If

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{then} \quad \begin{bmatrix} R & 0 \\ N & R \end{bmatrix}$$

is a maximal ideal in $\mathbf{M}(B,R)$. Note that this does not correspond to a set-inclusion preserving function θ.

The prime maximal ideals of a structural matrix ring still have the form in Theorem 2.4.

THEOREM 3.2. *The prime maximal ideals of* $\mathbf{M}(B,R)$ *are the sets* $\mathbf{M}(B,P)+\mathcal{K}_\mu$ *corresponding to the prime maximal ideals* P *of* R *and the ideals* $\mathcal{K}_\mu$, $\mu\in\boldsymbol{\beta}$, *of* $\mathbf{M}(B,R)$.

PROOF. We first show that every prime maximal ideal $\mathcal{P}$ of $\mathbf{M}(B,R)$ has the asserted form. By Proposition 2.1, $\mathcal{P} = \mathbf{M}(B,P)+\mathcal{K}_\mu$ for some $P \in \operatorname{spec}(R)$, $\mu \in \boldsymbol{\beta}$. If P is not maximal in R, then $\mathbf{M}(B,P)+\mathcal{K}_\mu$ is strictly contained in $\mathbf{M}(B,A)+\mathcal{K}_\mu \neq \mathbf{M}(B,R)$ for some proper ideal A of R strictly containing P, which contradicts the maximality of $\mathbf{M}(B,P)+\mathcal{K}_\mu$. Hence, P is a prime maximal ideal of R.

Conversely, let P be a prime maximal ideal of R, and let $\mu \in \boldsymbol{\beta}$. Then, by Proposition 2.1, $\mathbf{M}(B,P)+\mathcal{K}_\mu \in \operatorname{spec}(\mathbf{M}(B,R))$. Let $\mathcal{N}$ be an ideal of $\mathbf{M}(B,R)$ strictly containing $\mathbf{M}(B,P)+\mathcal{K}_\mu$. Then $f_\mu(\mathcal{N})$ strictly contains $f_\mu(\mathbf{M}(B,P)+\mathcal{K}_\mu) = \mathbf{M}_{n_\mu}(P)$, and so, by Theorem 2 of [5], $f_\mu(\mathcal{N}) = \mathbf{M}_{n_\mu}(R)$. Keeping in mind that $\mathcal{N}$ contains the kernel of f_μ, it follows from Lemma 1.2 that $\mathcal{N} = \mathbf{M}(B,R)$, which completes the proof.

We conclude this section by showing that the maximal ideals of $\mathbf{M}(B,R)$ which have the form $\mathbf{M}(B,P)+\mathcal{K}_\mu$ for some (maximal, of course) ideal P of R, are prime.

PROPOSITION 3.3. *Let P be a maximal ideal of R, and let $\mu \in \beta$. Then $\mathbf{M}(B,P)+\mathcal{K}_\mu$ is maximal in $\mathbf{M}(B,R)$ if and only if P is prime in R.*

PROOF. An easy argument using Theorem 4 of [5] and Theorem 3.2 yields the result.

4. Nagata's $\mathcal{M}$-radical

THEOREM 4.1. $\mathcal{M}(\mathbf{M}(B,R)) = \mathbf{M}(B,\mathcal{M}(R)) + \bigcap_{\mu\in\beta} \mathcal{K}_\mu$.

PROOF. Let $\mu \in \beta$, and set $\mathcal{L}_\mu := \bigcap\{\mathbf{M}(B,P)+\mathcal{K}_\mu : P \text{ is a prime maximal ideal of } R\}$. Then, by Theorem 3.2, $\mathcal{M}(\mathbf{M}(B,R)) = \bigcap_{\mu\in\beta}\mathcal{L}_\mu$. But $\mathcal{K}_\mu$, the kernel of f_μ, is contained in $\mathcal{L}_\mu$, and so

$$\begin{aligned}
\mathcal{M}(\mathbf{M}(B,R)) &= \bigcap_{\mu\in\beta} f_\mu^{-1}(f_\mu(\mathcal{L}_\mu)) \\
&= \bigcap_{\mu\in\beta} f_\mu^{-1}\left(\bigcap\{f_\mu(\mathbf{M}(B,P)+\mathcal{K}_\mu) : P \text{ is a prime maximal ideal of } R\}\right) \\
&= \bigcap_{\mu\in\beta} f_\mu^{-1}\left(\bigcap\{\mathbf{M}_{n_\mu}(P) : P \text{ is a prime maximal ideal of } R\}\right) \\
&= \bigcap_{\mu\in\beta} f_\mu^{-1}(\mathbf{M}_{n_\mu}(\mathcal{M}(R))).
\end{aligned}$$

The assertion follows now from Lemmas 1.1 and 1.2.

As in Corollary 2.9 of [7] we see that the $\mathcal{M}$-radical carries over from R to $\mathbf{M}(B,R)$ in the simplest possible way, that is, $\mathcal{M}(\mathbf{M}(B,R)) = \mathbf{M}(B,\mathcal{M}(R))$ if and only if $\leq_B$ is an equivalence relation.

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