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AN UNCERTAINTY PRINCIPLE FOR THE DUNKL TRANSFORM

MARGIT RÖSLER

This note presents an analogue of the classical Heisenberg-Weyl uncertainty principle for the Dunkl transform on $\mathbb{R}^N$. Its proof is based on expansions with respect to generalised Hermite functions.

1. INTRODUCTION

The Dunkl transform is an integral transform on $\mathbb{R}^N$ which generalises the classical Fourier transform. On suitable function spaces, it establishes a natural correspondence between the action of multiplication operators on one hand and so-called Dunkl operators on the other. These are differential-difference operators, generalising the usual partial derivatives, which are associated with a finite reflection group on some Euclidean space. They play, for example, a useful role in the algebraic description of exactly solvable quantum many body systems of Calogero–Moser–Sutherland type; among the broad literature in this context, we refer to [**1**], [**9**], and [**11**]. In his paper [**8**], de Jeu proved a quite general uncertainty principle for integral operators with bounded kernel which applies to the Dunkl transform; this result has the form of an $\varepsilon-\delta$-concentration principle as first stated in [4] for the Fourier transform. Analogues of the classical variance-based Weyl–Heisenberg uncertainty principle for the Dunkl transform have up to now only been given in the one-dimensional case ([**14**] and [**15**]). It is the aim of this note to present an extension to general Dunkl transforms in arbitrary dimensions. Our setting, which is described in more detail in section 2, is as follows: Let R be a finite (reduced) root system on $\mathbb{R}^N$ and $k : R \to [0,\infty]$ a nonnegative multiplicity function on R. Let w_k denote the weight function

$$w_k(x) = \prod_{\alpha\in R} |\langle \alpha, x\rangle|^{k(\alpha)}$$

on $\mathbb{R}^N$, where $\langle .,. \rangle$ is the Euclidean scalar product on $\mathbb{R}^N$, and put $\gamma := \sum_{\alpha\in R} k(\alpha)/2$. We shall prove the following uncertainty principle for the associated Dunkl transform $f \mapsto \widehat{f}^k$ on $L^2(\mathbb{R}^N, w_k(x)dx)$:

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I am indebted to Richard Askey, who stimulated this work by bringing C. Roosenraad's thesis [15] to my attention.

Theorem 1.1. *Let $f \in L^2(\mathbb{R}^N, w_k(x)dx)$. Then*

$$\| |x|f\|_{2,w_k} \cdot \| |\xi|\widehat{f}^k\|_{2,w_k} \geqslant (\gamma + N/2) \cdot \|f\|_{2,w_k}. \tag{1.1}$$

Moreover, equality holds if and only if $f(x) = c\,e^{-d|x|^2}$ for some constants $c \in \mathbb{C}$ and $d > 0$.

If the multiplicity function k is identically 0, then the corresponding Dunkl transform coincides with the usual Fourier transform (independently of the underlying root system), and the above result coincides with the classical Weyl-Heisenberg inequality on $L^2(\mathbb{R}^N)$.

Our proof of Theorem 1.1 is based on expansions in terms of generalised Hermite functions, which were introduced in [**12**]. This generalises a well-known method for the (one-dimensional) classical situation, see for example, [**2**]. Our method needs not much more effort than in the classical case, but requires a zero-centred situation. This restriction cannot easily be removed. For the one-dimensional case, the result of Theorem 1.1 was already obtained in [**15**], by a very similar method. In contrast, the version given in [**14**] is uncentred. It is based on commutator methods which become difficult to handle in higher dimensions. However, the lower bound in [**14**] is not uniform, and coincides with the one above for even functions only.

2. Dunkl operators and the Dunkl transform

In this section, we collect some basic facts from Dunkl's theory which will be needed later on. General references here are [**7, 5, 6**].

For $\alpha \in \mathbb{R}^N \setminus \{0\}$ we denote by σ_α the reflection in the hyperplane orthogonal to α, given by $\sigma_\alpha(x) = x - \big(2\langle \alpha, x\rangle/|\alpha|^2\big)\,\alpha$. Let R be a (reduced) root system in $\mathbb{R}^N$, that is, a finite subset of $\mathbb{R}^N \setminus \{0\}$ with $R \cap \mathbb{R}\cdot\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. We assume that the root system R is normalised, that is, $|\alpha|^2 = 2$ for all $\alpha \in R$. The reflections σ_α, $\alpha \in R$ generate a finite group G, the reflection group associated with R. A function $k : R \to \mathbb{C}$ is called a multiplicity function on R if it is invariant under the natural action of G on R. Now fix a reflection group G on $\mathbb{R}^N$ and a multiplicity function $k \geqslant 0$ on its root system R. The Dunkl operators T_i $(i = 1, \dots, N)$ on $\mathbb{R}^N$ associated with G and k are defined by

$$T_i f(x) := \partial_i f(x) + \frac{1}{2}\sum_{\alpha\in R} k(\alpha)\,\alpha_i \cdot \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^N);$$

here ∂_i denotes the i-th partial derivative. In the case $k = 0$, the T_i reduce to the usual partial derivatives. In this paper, we assume that all values of k are nonnegative, for short, $k \geqslant 0$. The most important basic properties of the operators T_i are as follows: Let $\mathcal{P} = \mathbb{C}[x_1, \dots, x_N]$ denote the algebra of polynomial functions on $\mathbb{R}^N$ and $\mathcal{P}_n$ $\big(n \in \mathbb{Z}_+ = \{0, 1, \dots\}\big)$ the subspace of homogeneous polynomials of degree n. Then

(1.1) Each T_i is homogeneous of degree -1 on $\mathcal{P}$, that is, $T_i p \in \mathcal{P}_{n-1}$ for $p \in \mathcal{P}_n$.

(1.2) The set $\{T_i, i = 1, \dots, N\}$ generates a commutative algebra of differential-difference operators on $\mathcal{P}$.

For a polynomial $p \in \mathcal{P}$, we denote by $p(T)$ the linear operator derived from $p(x)$ by replacing x_i by T_i. In particular, the generalised Laplacian is defined by $\Delta_k := p(T)$ with $p(x) = |x|^2$. Note that Δ_k is homogeneous of degree -2, and hence for each $c \in \mathbb{C}$, the exponential $e^{c\Delta_k}$ is a well-defined linear operator on $\mathcal{P}$ with inverse $e^{-c\Delta_k}$.

The solution of the joint eigenfunction problem for the Dunkl operators $\{T_i, i = 1, \dots, N\}$ is given by the Dunkl kernel K_G on $\mathbb{R}^N \times \mathbb{R}^N$: for each fixed $y \in \mathbb{R}^N$, the function $x \mapsto K_G(x, y)$ is characterised as the unique solution of the system $T_i f = y_i f$ $(i = 1, \dots, N)$ with $f(0) = 1$; see [10]. The kernel $K_G(x, y)$ is symmetric in its arguments and has a unique holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$. It satisfies $K_G(z, 0) = 1$ and $K_G(\lambda z, w) = K_G(z, \lambda w)$ for all $z, w \in \mathbb{C}^N$ and all $\lambda \in \mathbb{C}$. Moreover, the function $x \mapsto K_G(ix, y)$. $(y \in \mathbb{R}^N$ fixed) is positive definite on $\mathbb{R}^N$. See [13]. In particular, $|K_G(ix, y)| \leqslant 1$ for all $x, y \in \mathbb{R}^N$.

The Dunkl transform associated with G and k is given by

$$\widehat{\cdot}^{k} \;:\; L^1(\mathbb{R}^N, w_k(x)dx) \;\to\; C_b(\mathbb{R}^N);$$
$$\widehat{f}^k(\xi) := 2^{-\gamma - N/2} c_k \int_{\mathbb{R}^N} f(x) K_G(-i\xi, x)\, w_k(x) dx \quad (\xi \in \mathbb{R}^N),$$

with the Mehta-type constant

$$c_k := \Big(\int_{\mathbb{R}^N} e^{-|x|^2} w_k(x) dx \Big)^{-1}.$$

This transformation has many properties analogous to the Fourier transform on $\mathbb{R}^N$, among which we shall in particular need the following:

PROPOSITION 2.1. [7]

(1) *The Dunkl transform $f \to \widehat{f}^k$ is a homeomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing functions on $\mathbb{R}^N$.*

(2) $\widehat{T_j f}^k(\xi) = i\xi_j \widehat{f}^k$ *for all $f \in \mathcal{S}(\mathbb{R}^N)$ and $j = 1, \dots, N$.*

(3) *(Plancherel theorem) The Dunkl transform has a unique extension to an isometric isomorphism of $L^2(\mathbb{R}^N, w_k(x)dx)$, which is again denoted by $f \to \widehat{f}^k$.*

EXAMPLES 2.2. (1) If $k = 0$, then $K_G(z, w) = e^{\langle z, w\rangle}$ for all $z, w \in \mathbb{C}^N$. Here the Dunkl transform is the usual Fourier transform on $\mathbb{R}^N$.

(2) If $N = 1$ and $G = \mathbb{Z}_2$, sending $x \in \mathbb{R}$ to $-x$, then the multiplicity function is a single parameter $k \geqslant 0$, and the Dunkl kernel is given by

$$K_{\mathbb{Z}_2}(z, w) \;=\; j_{k-1/2}(izw) + \frac{zw}{2k+1} j_{k+1/2}(izw) \quad (z, w \in \mathbb{C}),$$

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where for $\alpha \geqslant -1/2$, j_α is the normalised spherical Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}.$$

The corresponding Dunkl transform coincides with the Fourier transform on a certain (signed) hypergroup structure on $\mathbb{R}$; for details see **[14]** and the literature cited there.

3. Generalised Hermite functions

Let G be a finite reflection group on $\mathbb{R}^N$ and $k \geqslant 0$ a fixed multiplicity function on its root system R. In **[12]** we introduced complete systems of orthogonal polynomials with respect to the weight function $w_k(x)\,e^{-|x|^2}$ on $\mathbb{R}^N$, called generalised Hermite polynomials. The key to their definition is the following bilinear form on $\mathcal{P}$, which was introduced in **[6]**:

$$[p,q]_k := \big(p(T)q\big)(0) \quad \text{for } p,q \in \mathcal{P}.$$

The homogeneity of the Dunkl operators implies that $\mathcal{P}_n \perp \mathcal{P}_m$ for $n \neq m$. Moreover, if $p,q \in \mathcal{P}_n$, then

$$(3.1) \qquad [p,q]_k = 2^n c_k \int_{\mathbb{R}^N} e^{-\Delta_k/4}p(x)\, e^{-\Delta_k/4}q(x)\, e^{-|x|^2}w_k(x)dx.$$

This is obtained from Theorem 3.10 of **[6]** by rescaling, see **[12**, Lemma 2.1**]**. So in particular, $[.,.]_k$ is a scalar product on the vector space $\mathcal{P}_\mathbb{R} = \mathbb{R}[x_1,\dots,x_N]$.

Now let $\{\varphi_\nu, \nu \in \mathbb{Z}_+^N\}$ be an (arbitrary) orthonormal basis of $\mathcal{P}_\mathbb{R}$ with respect to $[.,.]_k$ such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$. (For details concerning the construction and canonical choices of such a basis, we refer to **[12]**). Then the generalised Hermite polynomials $\{H_\nu, \nu \in \mathbb{Z}_+^N\}$ and the (normalised) generalised Hermite functions $\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ associated with G, k and $\{\varphi_\nu\}$ are defined by

$$H_\nu(x) := 2^{|\nu|}e^{-\Delta_k/4}\varphi_\nu(x) \quad \text{and} \quad h_\nu(x) := \sqrt{c_k}\,2^{-|\nu|/2}e^{-|x|^2/2}H_\nu(x) \quad (x \in \mathbb{R}^N).$$

Note that H_ν is a polynomial of degree $|\nu|$, with real coefficients. This implies $(3N$-term) recurrencies of the following form: For $\nu \in \mathbb{Z}_+^N$, let $I_\nu = \{\mu \in \mathbb{Z}_+^N : \big||\mu|-|\nu|\big| \leqslant 1\}$. Then

$$(3.2) \qquad x_jH_\nu = \sum_{\mu\in I_\nu} c^j_{\nu,\mu}H_\mu \quad \text{and} \quad x_jh_\nu = \sum_{\mu\in I_\nu} c^j_{\nu,\mu}h_\mu \quad \text{for } j=1,\dots,N,$$

with coefficients $c^j_{\nu,\mu} \in \mathbb{R}$. In general, there are many possible choices of generalised Hermite systems. However, in the one-dimensional case $N = 1$ (with fixed parameter $k \geqslant 0$), the basis $\{\varphi_n, n \in \mathbb{Z}_+\}$ is uniquely determined. The associated generalised Hermite polynomials are orthogonal with respect to the weight function $|x|^{2k}e^{-|x|^2}$ on $\mathbb{R}$

and can be written explicitly in terms of Laguerre polynomials; for details, see [**12**] or [**3**, Chapter V].

We collect some further properties of the generalised Hermite functions $\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ which will be essential for the proof of Theorem 1.1.

LEMMA 3.1. [12]

(1) *$\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ is an orthonormal basis of $L^2(\mathbb{R}^N, w_k(x)dx)$.*

(2) *The h_ν are eigenfunctions of the Dunkl transform on $L^2(\mathbb{R}^N, w_k(x)dx)$, with $\widehat{h}_\nu^k = (-i)^{|\nu|} h_\nu$.*

(3) *The h_ν satisfy $\big(|x|^2 - \Delta_k\big)h_\nu = \big(2|\nu| + 2\gamma + N\big)\, h_\nu$.*

4. Proof of the uncertainty principle

From now on, $\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ is an arbitrary fixed system of generalised Hermite functions associated with G and $k \geqslant 0$. We shall need the dual counterparts of the recurrences (3.2):

$$T_j h_\nu = \sum_{\mu \in I_\nu} i^{1-|\nu|+|\mu|} c_{\nu,\mu}^j h_\mu \quad (j = 1, \ldots, N, \ \nu \in \mathbb{Z}_+^N.) \tag{4.1}$$

These are easily obtained from (3.2) by use of Proposition 2.1.(2) and Lemma 3.1.(2).

We write $\langle\, .\, , .\rangle_k$ for the scalar product in $L^2(\mathbb{R}^N, w_k(x)dx)$. The main part in the proof of Theorem 1.1 is the following Parseval-type identity.

LEMMA 4.1. *Let $f \in L^2(\mathbb{R}^N, w_k(x)dx)$. Then*

$$\int_{\mathbb{R}^N} |x|^2\Big(\big|f(x)\big|^2 + \big|\widehat{f}^k(x)\big|^2\Big) w_k(x)\, dx = \sum_{\nu \in \mathbb{Z}_+^N} \big(2|\nu| + 2\gamma + N\big) \cdot \big|\langle f, h_\nu\rangle_k\big|^2.$$

PROOF: Fix $j \in \{1, \ldots, N\}$. In view of Lemma 3.1.(1), we can write

$$\int_{\mathbb{R}^N} |x_j|^2 \big|f(x)\big|^2 w_k(x)dx = \sum_{\nu \in \mathbb{Z}_+^N} \big|\langle x_j f, h_\nu\rangle_k\big|^2 = \sum_{\nu \in \mathbb{Z}_+^N} \big|\langle f, x_j h_\nu\rangle_k\big|^2.$$

By use of (3.2), this becomes

$$\sum_{\nu \in \mathbb{Z}_+^N} \sum_{\mu, \rho \in I_\nu} c_{\nu,\mu}^j c_{\nu,\rho}^j \cdot \langle f, h_\mu\rangle_k \overline{\langle f, h_\rho\rangle_k} = \sum_{\mu, \rho \in \mathbb{Z}_+^N} \Big(\sum_{\nu \in I_\mu \cap I_\rho} c_{\nu,\mu}^j c_{\nu,\rho}^j\Big) \langle f, h_\mu\rangle_k \overline{\langle f, h_\rho\rangle_k}.$$

Here the last equality is justified by the facts that the involved index sets I_ν are finite, and that $\mu \in I_\nu \iff \nu \in I_\mu$ holds for all ν, $\mu \in \mathbb{Z}_+^N$. Exploiting Lemma 3.1.(2), Proposition 2.1.(2) and the Parseval identity for the Dunkl transform, one further obtains

$$\begin{aligned}\int_{\mathbb{R}^N} |x_j|^2 \big|\widehat{f}^k(x)\big|^2 w_k(x)dx &= \sum_{\nu \in \mathbb{Z}_+^N} \big|\langle x_j \widehat{f}^k, h_\nu\rangle_k\big|^2 = \sum_{\nu \in \mathbb{Z}_+^N} \big|\langle \widehat{f}^k, x_j \widehat{h}_\nu^k\rangle_k\big|^2 \\ &= \sum_{\nu \in \mathbb{Z}_+^N} \big|\langle f, T_j h_\nu\rangle_k\big|^2.\end{aligned}$$

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With the recurrence (4.1), this becomes

$$\sum_{\nu\in\mathbf{Z}_+^N}\sum_{\mu,\rho\in I_\nu} i^{|\nu|-|\mu|-1}c_{\nu,\mu}^j\cdot i^{1-|\nu|+|\rho|}c_{\nu,\rho}^j\cdot\langle f,h_\mu\rangle_k\overline{\langle f,h_\rho\rangle_k}$$
$$=\sum_{\mu,\rho\in\mathbf{Z}_+^N}\Big(\sum_{\nu\in I_\mu\cap I_\rho}c_{\nu,\mu}^jc_{\nu,\rho}^j\Big)i^{|\rho|-|\mu|}\cdot\langle f,h_\mu\rangle_k\overline{\langle f,h_\rho\rangle_k}.$$

Combining the previous results, we arrive at

$$(4.2)\qquad \int_{\mathbb{R}^N}|x|^2\big(|f(x)|^2+|\widehat{f}^k(x)|^2\big)w_k(x)\,dx=\sum_{\mu,\rho\in\mathbf{Z}_+^N}A_{\mu,\rho}\langle f,h_\mu\rangle\overline{\langle f,h_\rho\rangle},$$

where

$$A_{\mu,\rho}=\big(1+i^{|\rho|-|\mu|}\big)\cdot\sum_{j=1}^N\sum_{\nu\in I_\mu\cap I_\rho}c_{\nu,\mu}^jc_{\nu,\rho}^j.$$

On the other hand, a short calculation, using formulas (3.2) and (4.1), shows that

$$(4.3)\qquad \big(|x|^2-\Delta_k\big)h_\nu=\sum_{j=1}^N\sum_{\mu\in I_\nu}\sum_{\rho\in I_\mu}c_{\nu,\mu}^jc_{\mu,\rho}^j\big(1+i^{|\rho|-|\nu|}\big)h_\rho=\sum_{\rho\in\mathbf{Z}_+^N}A_{\nu,\rho}h_\rho,$$

where for the last identity, we used the fact that the coefficients $c_{\nu,\mu}^j$ are symmetric in their subscripts: $c_{\nu,\mu}^j=\int_{\mathbb{R}^N}x_jh_\nu(x)h_\mu(x)w_k(x)dx=c_{\mu,\nu}^j$. But by Lemma 3.1.(3), the left side of (4.3) is equal to $\big(2|\nu|+2\gamma+N\big)h_\nu$. The linear independence of the h_ν now implies that

$$A_{\nu,\rho}=\begin{cases}0 & \text{if } \rho\neq\nu,\\ 2|\nu|+2\gamma+N & \text{if } \rho=\nu.\end{cases}$$

Together with (4.2), this yields the assertion. ∎

In view of Lemma 3.1.(1), and as h_0 is a constant multiple of $e^{-|x|^2/2}$, we obtain as an immediate consequence the following:

Corollary 4.2. *For $f\in L^2(\mathbb{R}^N,w_k(x)dx)$,*

$$\int_{\mathbb{R}^N}|x|^2\Big(|f(x)|^2+|\widehat{f}^k(x)|^2\Big)w_k(x)\,dx\geqslant(2\gamma+N)\cdot\|f\|_{2,w_k}^2.$$

Moreover, equality holds if and only if $f(x)=c\,e^{-|x|^2/2}$ with some constant $c\in\mathbb{C}$.

Proof of Theorem 1.1 We may assume that $\|f\|_{2,w_k}=1$. For $s>0$ define $f_s(x):=s^{-\gamma-N/2}f(x/s)$. Since w_k is homogeneous we easily see that

$$\|f_s\|_{2,w_k}=1\quad\text{and}\quad\widehat{f_s}^k(\xi)=s^{\gamma+N/2}\cdot\widehat{f}^k(s\xi)\quad\text{for all } s>0 \text{ and } \xi\in\mathbb{R}^N.$$

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The above corollary implies that

$$\Phi_f(s) := \int_{\mathbb{R}^N} |x|^2\Big(\big|f_s(x)\big|^2 + \big|\widehat{f}_s^k(x)\big|^2\Big) w_k(x)\,dx \geqslant 2\gamma + N\,.$$

On the other hand, we can write

$$\Phi_f(s) = s^2 \cdot \big\| \, |x| f \, \big\|_{2,w_k}^2 + \frac{1}{s^2} \cdot \big\| \, |x| \widehat{f}^k \, \big\|_{2,w_k}^2\,.$$

It is easily checked that $s \mapsto \Phi_f(s)$ takes a minimum on $(0,\infty)$, namely

$$2 \cdot \big\| \, |x| f \, \big\|_{2,w_k} \cdot \big\| \, |x| \widehat{f}^k \, \big\|_{2,w_k}.$$

This implies (1.1). Further, equality in (1.1) holds exactly if $\min_{s\in(0,\infty)} \Phi_f(s) = 2\gamma + N$. By the second part of the corollary, this condition is satisfied if and only if $f(x) = c\, e^{-s^2|x|^2/2}$ with some constants $c \in \mathbb{C}$ and $s > 0$. This finishes the proof. ∎

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Zentrum Mathematik
Technische Universität München
Arcisstr. 21
D-80290 München
Germany
e-mail: roesler@mathematik.tu-muenchen.de