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ON DIVISIBILITY OF BINOMIAL COEFFICIENTS

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In memory of Professor Alf van der Poorten

Abstract

In this paper, motivated by Catalan numbers and higher-order Catalan numbers, we study factors of products of at most two binomial coefficients.

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1. Introduction

There are many papers on the divisibility of sums of binomial coefficients. See, for example, [2–4, 7, 8, 10].

Bober (see [1]) made sophisticated use of the theory of hypergeometric series to determine all positive integers $a_1, \ldots, a_r, b_1, \ldots, b_{r+1}$ such that

$$a_1 + \cdots + a_r = b_1 + \cdots + b_{r+1}$$

and

$$\frac{(a_1 n)! \cdots (a_r n)!}{(b_1 n)! \cdots (b_{r+1} n)!}$$

is an integer for any $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. In particular, if $k, l \in \mathbb{Z}^+$, then

$$\frac{\binom{ln}{n}\binom{kln}{ln}}{\binom{kn}{n}} = \frac{(kln)!((k-1)n)!}{(kn)!((l-1)n)!((k-1)ln)!} \in \mathbb{Z} \quad \forall n \in \mathbb{Z}^+$$
$$\iff k = l \quad \text{or } \{k, l\} \cap \{1, 2\} \neq \emptyset \quad \text{or } \{k, l\} = \{3, 5\}.$$

In this paper we study factors of products of at most two binomial coefficients. Our methods are elementary and combinatorial and the proofs may be easily understood.

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Recall that for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ the nth (usual) Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}.$$

The Catalan numbers arise naturally in many enumeration problems in discrete mathematics (see, for example, [6, pp. 219–229]). For $h, n \in \mathbb{N}$ the nth (generalized) Catalan number of order h is defined to be

$$C_n^{(h)} = \frac{1}{hn+1}\binom{(h+1)n}{n} = \binom{(h+1)n}{n} - h\binom{(h+1)n}{n-1}.$$

We extend the basic fact that $(hn+1) \mid \binom{(h+1)n}{n}$ in the following theorem.

Theorem 1.1. *Let $k, l, n \in \mathbb{Z}^+$. Then*

$$\frac{ln+1}{\gcd(k, ln+1)} \,\Big|\, \binom{kn+ln}{kn}, \tag{1.1}$$

where $\gcd(k, ln+1)$ denotes the greatest common divisor of k and $ln+1$. In particular, $(ln+1) \mid \binom{kn+ln}{kn}$ if l is divisible by all the prime factors of k.

The following conjecture seems difficult to prove.

Conjecture 1.2. Let k and l be positive integers. If $(ln+1) \mid \binom{kn+ln}{kn}$ for all sufficiently large positive integers n, then each prime factor of k divides l. In other words, if k has a prime factor not dividing l, then there are infinitely many positive integers n such that $(ln+1) \nmid \binom{kn+ln}{kn}$.

In order to study Conjecture 1.2 we introduce a new function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{N}$ as follows. For positive integers k and l, if $(ln+1) \mid \binom{kn+ln}{kn}$ for all $n \in \mathbb{Z}^+$ (which happens if all prime factors of k divide l), then we set $f(k, l) = 0$. Otherwise we define $f(k, l)$ to be the smallest positive integer n such that $(ln+1) \nmid \binom{kn+ln}{kn}$. We have computed the following values of f using `Mathematica`.

$$f(7, 36) = 279,\ f(10, 192) = 362,\ f(11, 100) = 1187,\ f(22, 200) = 6462,$$
$$f(74, 62) = 885,\ f(213, 3) = 3384,\ f(223, 93) = 13\,368,\ f(307, 189) = 31\,915.$$

Wee turn to our results on the factors of products of two binomial coefficients. They are related to either Catalan numbers or higher-order Catalan numbers. Note that $nC_n^{(h)} = \binom{(h+1)n}{n-1}$ for all $h, n \in \mathbb{Z}^+$. Recall that the odd part of an integer k is the largest odd divisor of k.

Theorem 1.3. *Let $k, n \in \mathbb{Z}^+$.*

(i) *Then*

$$\binom{kn}{n} \,\Big|\, (2k-1)C_n\binom{2kn}{2n}.$$

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Moreover,

$$(2k-1)C_n\binom{2kn}{2n}\Big/\binom{kn}{n}$$

is odd if and only if $n+1$ *is a power of two.*

(ii) *Let* $(k+1)'$ *be the odd part of* $k+1$. *Then*

$$\binom{2n}{n}\,\Big|\,(k+1)'C_n^{(k-1)}\binom{2kn}{kn}.$$

Moreover,

$$(k+1)'C_n^{(k-1)}\binom{2kn}{kn}\Big/\binom{2n}{n}$$

is odd if and only if $(k-1)n+1$ *is a power of two.*

By Theorem 1.3(ii), if $n\in\mathbb{Z}^+$ and $k=2^l-1$ for some $l\in\mathbb{N}$, then

$$\binom{2n}{n}\,\Big|\,\binom{2kn}{kn}C_n^{(k-1)} \iff n\binom{2n}{n}\,\Big|\,\binom{kn}{n-1}\binom{2kn}{kn}.$$

Using `Mathematica` we find that this result can be further strengthened.

Theorem 1.4. *For every* $k,n\in\mathbb{Z}^+$,

$$2^{k-1}\binom{2n}{n}\,\Big|\,\binom{2(2^k-1)n}{(2^k-1)n}C_n^{(2^k-2)}. \tag{1.2}$$

A key step in our proof of Theorem 1.4 is to prove the first assertion in the following conjecture for prime values of m.

Conjecture 1.5. Let m be an integer greater than 1 and let k and n be positive integers. Then the sum of all digits in the expansion of $(m^k-1)n$ in base m is at least $k(m-1)$. Also, the expansion of $n(m^k-1)/(m-1)$ in base m has at least k nonzero digits.

The following result relies on certain particular properties of the integers 3 and 5.

Theorem 1.6. *For every* $n\in\mathbb{Z}^+$,

$$(6n+1)\binom{5n}{n}\,\Big|\,\binom{3n-1}{n-1}C_{3n}^{(4)}$$

and

$$\binom{3n}{n}\,\Big|\,\binom{5n-1}{n-1}C_{5n}^{(2)}.$$

We define two new sequences $\{s_n\}_{n\geq1}$ and $\{t_n\}_{n\geq1}$ of integers by

$$s_n=\frac{\binom{3n-1}{n-1}C_{3n}^{(4)}}{(6n+1)\binom{5n}{n}}=\frac{\binom{3n-1}{n-1}\binom{15n}{3n}}{(6n+1)(12n+1)\binom{5n}{n}}=\frac{\binom{3n}{n}\binom{15n}{3n-1}}{9n(6n+1)\binom{5n}{n}}$$

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and

$$t_n = \frac{\binom{5n-1}{n-1}C_{5n}^{(2)}}{\binom{3n}{n}} = \frac{\binom{5n-1}{n-1}\binom{15n}{5n}}{(10n+1)\binom{3n}{n}} = \frac{\binom{5n}{n}\binom{15n}{5n-1}}{25n\binom{3n}{n}}.$$

It would be interesting to find recursion formulae or combinatorial interpretations for s_n and t_n.

Based on our computations using `Mathematica`, we formulate the following conjecture about the sequence $\{t_n\}_{n\geq 1}$.

Conjecture 1.7. Let $n \in \mathbb{Z}^+$. Then $(10n+3) \mid 21t_n$.

If p is a prime, then the p-adic valuation of an integer m is given by

$$\nu_p(m) = \sup\{a \in \mathbb{N} : p^a \mid m\}.$$

For a rational number $x = m/n$ where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we set $\nu_p(x) = \nu_p(m) - \nu_p(n)$ for any prime p.

The following lemma is fundamental for our approach in this paper.

Lemma 1.8.

(i) *A rational number x is an integer if and only if $\nu_p(x) \geq 0$ for all primes p.*

(ii) (Legendre's theorem) *If p is prime and $n \in \mathbb{N}$, then*

$$\nu_p(n!) = \sum_{i=1}^{\infty}\left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - \rho_p(n)}{p-1}$$

where $\rho_p(n)$ is the sum of the digits in the expansion of n in base p.

(iii) *Let n be a positive integer. Then $\nu_2(n!) \leq n-1$. Also $\nu_2(n!) = n-1$ if and only if n is a power of two.*

Proof. Part (i) is obvious. Part (ii) is well known and may be found in [5, pp. 22–24]. Part (iii) follows immediately from part (ii); see also [9, Lemma 4.1]. □

Example 1.9. Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$ and set

$$Q(m,n) := \frac{\binom{2n}{n}\binom{2m+2n}{2n}}{2\binom{m+n}{n}}.$$

Then

$$Q(m,n) = \frac{2^{n-1}}{n!}\prod_{j=1}^{n}(2m+2j-1) = (-1)^n 2^{2n-1}\binom{-m-1/2}{n}.$$

Applying Lemma 1.8, we see that $Q(m,n) \in \mathbb{Z}$ and that $2 \nmid Q(m,n)$ if and only if n is a power of two. When $n > 1$ we see that

$$\frac{\binom{2n}{n}\binom{2m+2n}{2n-1}}{8\binom{m+n}{n}} = Q(m+1, n-1) \in \mathbb{Z}.$$

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Also,

$$\binom{2n}{n}\binom{2m+2n}{2n-1}\Big/\left(8\binom{m+n}{n}\right)$$

is odd if and only if $n-1$ is a power of two.

By Example 1.9 we see that $\binom{kn}{n} \mid \binom{2n}{n}\binom{2kn}{2n-1}$ for any $k, n \in \mathbb{Z}^+$. In view of this and Theorems 1.3, 1.4 and 1.6, we raise the following conjecture.

Conjecture 1.10. Let k and l be integers greater than one. If $\binom{kn}{n} \mid \binom{ln}{n}\binom{kln}{ln-1}$ for all $n \in \mathbb{Z}^+$, then $k = l$ or $l = 2$ or $\{k, l\} = \{3, 5\}$. If $\binom{kn}{n} \mid \binom{ln}{n-1}\binom{kln}{ln}$ for all $n \in \mathbb{Z}^+$, then $k = 2$ and $l + 1$ is a power of two.

We will prove Theorems 1.1 and 1.3 in the next section. Section 3 is devoted to the sophisticated proofs of Theorems 1.4 and 1.6. Throughout this paper, for a real number x we let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.

2. Proofs of Theorems 1.1 and 1.3

Proof of Theorem 1.1. Clearly (1.1) holds if and only if $(ln+1) \mid k\binom{kn+ln}{kn}$. For any prime p, we calculate

$$\begin{aligned}\nu_p\left(\frac{k\binom{kn+ln}{kn}}{ln+1}\right) &= \nu_p\left(\frac{(kn+ln)!k!}{(kn)!(ln+1)!(k-1)!}\right)\\ &= \sum_{j=1}^{\infty}\left(\left\lfloor\frac{kn+ln}{p^j}\right\rfloor - \left\lfloor\frac{kn}{p^j}\right\rfloor - \left\lfloor\frac{ln+1}{p^j}\right\rfloor + \left\lfloor\frac{k}{p^j}\right\rfloor - \left\lfloor\frac{k-1}{p^j}\right\rfloor\right).\end{aligned}$$

So it suffices to show that for any $m \in \mathbb{Z}^+$ the inequality

$$\left\lfloor\frac{kn+ln}{m}\right\rfloor - \left\lfloor\frac{kn}{m}\right\rfloor - \left\lfloor\frac{ln+1}{m}\right\rfloor + \left\lfloor\frac{k}{m}\right\rfloor - \left\lfloor\frac{k-1}{m}\right\rfloor \geq 0 \tag{2.1}$$

is satisfied. If $m \nmid kn$, then

$$\left\lfloor\frac{kn}{m}\right\rfloor + \left\lfloor\frac{ln+1}{m}\right\rfloor = \left\lfloor\frac{kn-1}{m}\right\rfloor + \left\lfloor\frac{ln+1}{m}\right\rfloor \leq \left\lfloor\frac{(kn-1)+(ln+1)}{m}\right\rfloor.$$

If $m \nmid (ln+1)$, then

$$\left\lfloor\frac{kn}{m}\right\rfloor + \left\lfloor\frac{ln+1}{m}\right\rfloor = \left\lfloor\frac{kn}{m}\right\rfloor + \left\lfloor\frac{ln}{m}\right\rfloor \leq \left\lfloor\frac{kn+ln}{m}\right\rfloor.$$

When $m \mid kn$ and $m \mid (ln+1)$, clearly $\gcd(m, n) = 1$, $m \mid k$ and hence

$$\left\lfloor\frac{kn+ln}{m}\right\rfloor - \left\lfloor\frac{kn}{m}\right\rfloor - \left\lfloor\frac{ln+1}{m}\right\rfloor + \left\lfloor\frac{k}{m}\right\rfloor - \left\lfloor\frac{k-1}{m}\right\rfloor = 0.$$

Therefore inequality (2.1) holds and this concludes the proof. □

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LEMMA 2.1. *Let $m \in \mathbb{Z}^+$ and $k, n \in \mathbb{Z}$. Then*

$$\left\lfloor \frac{2kn}{m} \right\rfloor - \left\lfloor \frac{kn}{m} \right\rfloor + \left\lfloor \frac{(k-1)n}{m} \right\rfloor - \left\lfloor \frac{2(k-1)n}{m} \right\rfloor \ge \left\lfloor \frac{n+1}{m} \right\rfloor - \left\lfloor \frac{2k-1}{m} \right\rfloor + \left\lfloor \frac{2k-2}{m} \right\rfloor, \tag{2.2}$$

unless $2 \mid m$, $k \equiv m/2 + 1 \bmod m$ and $n \equiv -1 \bmod m$, in which case the left-hand side of inequality (2.2) minus the right-hand side of inequality (2.2) is equal to -1.

PROOF. As $\lfloor x \rfloor = x - \{x\}$ for any rational number x and

$$2kn - kn + (k-1)n - 2(k-1)n + (2k-1) - (2k-2) = n + 1,$$

inequality (2.2) holds if and only if

$$\left\{ \frac{2kn}{m} \right\} - \left\{ \frac{kn}{m} \right\} + \left\{ \frac{(k-1)n}{m} \right\} - \left\{ \frac{2(k-1)n}{m} \right\} + \left\{ \frac{2k-1}{m} \right\} - \left\{ \frac{2k-2}{m} \right\} < 1. \tag{2.3}$$

Clearly inequality (2.3) holds when $m = 1$. Below we assume that $m \ge 2$. There are three cases to consider.

Case 1. Either both $\{kn/m\} < 1/2$ and $\{(k-1)n/m\} < 1/2$, or both $\{kn/m\} \ge 1/2$ and $\{(k-1)n/m\} \ge 1/2$.

In this case, the left-hand side of inequality (2.3) is equal to

$$C := \left\{ \frac{kn}{m} \right\} - \left\{ \frac{(k-1)n}{m} \right\} + \left\{ \frac{2k-1}{m} \right\} - \left\{ \frac{2k-2}{m} \right\}.$$

If $m \nmid (k-1)n$, then

$$C < \{kn/m\} + 1/m \le 1.$$

If $m \mid (k-1)n$ and $n \not\equiv -1 \bmod m$, then

$$C \le \{n/m\} + 1/m < 1.$$

If $m \mid (k-1)n$ and $n \equiv -1 \bmod m$, then

$$\{kn/m\} = (m-1)/m \ge 1/2 > \{(k-1)n/m\} = 0,$$

which leads to a contradiction.

Case 2. In this case

$$\{kn/m\} < 1/2 \le \{(k-1)n/m\}$$

and thus the left-hand side of inequality (2.3) is equal to

$$D := \left\{ \frac{kn}{m} \right\} - \left\{ \frac{(k-1)n}{m} \right\} + 1 + \left\{ \frac{2k-1}{m} \right\} - \left\{ \frac{2k-2}{m} \right\}.$$

If $n \not\equiv -1 \bmod m$, then

$$\{(k-1)n/m\} - \{kn/m\} \ne 1/m$$

and so

$$D < -1/m + 1 + 1/m = 1.$$

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If $n \equiv -1 \bmod m$ and $2k \equiv 1 \bmod m$, then

$$D = -1/m + 1 - (m-1)/m < 1.$$

If $n \equiv -1 \bmod m$ and $2k \not\equiv 1 \bmod m$, then we must have $2 \mid m$ and

$$k \equiv m/2 + 1 \mod m$$

since

$$\{-k/m\} < 1/2 \le \{(1-k)/m\}.$$

If $2 \mid m$, $k \equiv m/2 + 1 \bmod m$ and $n \equiv -1 \bmod m$, then it is easy to verify that the right-hand side of inequality (2.2) minus the left-hand side of inequality (2.2) is equal to 1.

Case 3. In this case

$$\{kn/m\} \ge 1/2 > \{(k-1)n/m\}$$

and thus the left-hand side of (2.3) is

$$\left\{\frac{kn}{m}\right\} - 1 - \left\{\frac{(k-1)n}{m}\right\} + \left\{\frac{2k-1}{m}\right\} - \left\{\frac{2k-2}{m}\right\} \le \left\{\frac{kn}{m}\right\} - 1 + \frac{1}{m} \le 0.$$

Thus Lemma 2.1 is satisfied in all cases. □

Lemma 2.2. *Let $m > 2$ be an integer. For any $k, n \in \mathbb{Z}$,*

$$\left\lfloor\frac{2kn}{m}\right\rfloor + \left\lfloor\frac{n}{m}\right\rfloor + \left\lfloor\frac{k+1}{m}\right\rfloor \ge \left\lfloor\frac{k}{m}\right\rfloor + \left\lfloor\frac{2n}{m}\right\rfloor + \left\lfloor\frac{kn}{m}\right\rfloor + \left\lfloor\frac{(k-1)n+1}{m}\right\rfloor. \tag{2.4}$$

Proof. As

$$k + ((k-1)n + 1) + kn - 2kn + 2n - n = k + 1,$$

inequality (2.4) is equivalent to the inequality $M \ge 0$ where

$$M := \left\{\frac{k}{m}\right\} + \left\{\frac{(k-1)n+1}{m}\right\} + \left\{\frac{kn}{m}\right\} - \left\{\frac{2kn}{m}\right\} + \left\{\frac{2n}{m}\right\} - \left\{\frac{n}{m}\right\}.$$

If $\{n/m\} < 1/2 \le \{kn/m\}$ or both $\{n/m\} < 1/2$ and $\{kn/m\} < 1/2$ or both $\{n/m\} \ge 1/2$ and $\{kn/m\} \ge 1/2$, then one can easily show that $M \ge 0$.

Below we suppose that $\{kn/m\} < 1/2 \le \{n/m\}$. Clearly $m \nmid n$ and

$$M = \left\{\frac{k}{m}\right\} + \left\{\frac{(k-1)n+1}{m}\right\} - \left\{\frac{kn}{m}\right\} + \left\{\frac{n}{m}\right\} - 1.$$

If

$$(k-1)n + 1 \equiv 0 \mod m,$$

then

$$\{(n-1)/m\} = \{kn/m\} < 1/2 \le \{n/m\}$$

and hence m is odd (otherwise $n \equiv m/2 \bmod m$ and thus $1 \equiv 0 \bmod m/2$, which is impossible). Moreover,

$$n \equiv (m+1)/2 \mod m,$$

from which it follows that

$$k-1 \equiv (k-1)2n \equiv -2 \pmod m$$

and

$$M = \left\{\frac{k}{m}\right\} - \left\{\frac{n-1}{m}\right\} + \left\{\frac{n}{m}\right\} - 1 = \left\{\frac{k}{m}\right\} - \frac{m-1}{m} = 0.$$

If

$$(k-1)n + 1 \not\equiv 0 \pmod m,$$

then $\{kn/m\} < \{(n-1)/m\}$ and hence

$$M = \left\{\frac{k}{m}\right\} + \left(\left\{\frac{kn}{m}\right\} - \left\{\frac{n-1}{m}\right\} + 1\right) - \left\{\frac{kn}{m}\right\} + \left\{\frac{n}{m}\right\} - 1 \geq \frac{1}{m}.$$

This concludes the proof. □

Proof of Theorem 1.3. To prove part (i) we observe that

$$Q_1 := \frac{(2k-1)C_n\binom{2kn}{kn}}{\binom{kn}{n}} = \frac{(2kn)!((k-1)n)!(2k-1)!}{(n+1)!(kn)!(2(k-1)n)!(2k-2)!}.$$

So, for any prime p,

$$\nu_p(Q_1) = \sum_{i=1}^{\infty} A_{p^i}(k,n)$$

where $A_m(k,n)$ denotes the left-hand side of inequality (2.2) minus the right-hand side of inequality (2.2). By Lemma 2.1, $A_{p^i}(k,n) \geq 0$ unless $p = 2$, $k \equiv 2^{i-1} + 1 \bmod 2^i$ and $n \equiv -1 \bmod 2^i$ in which case $A_{p^i}(k,n) = -1$. Therefore $2Q_1 \in \mathbb{Z}$.

Note that

$$Q_1 = \frac{2^n(2k-1)}{(n+1)!}\prod_{j=1}^{n}((2k-2)n + 2j - 1)$$

and thus

$$\nu_2(Q_1) = n - \nu_2((n+1)!).$$

By Lemma 1.8(iii), $Q_1 \in \mathbb{Z}$, and Q_1 is odd if and only if $n+1$ is a power of two.

We now prove part (ii). Obviously

$$Q_2 := \frac{(k+1)C_n^{(k-1)}\binom{2kn}{kn}}{\binom{2n}{n}} = \frac{(k+1)!(2kn)!n!}{k!(kn)!((k-1)n+1)!(2n)!}.$$

As in the proof of part (i), by Lemma 2.2, we have $\nu_p(Q_2) \geq 0$ for any odd prime p.

We now consider $\nu_2(Q_2)$. Set $m = (k-1)n$. Then

$$Q_2 = \frac{2^m(k+1)}{(m+1)!}\prod_{j=1}^{m}(2j + 2n - 1)$$

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and therefore

$$\nu_2(Q_2) = \nu_2(k+1) + m - \nu_2((m+1)!).$$

Applying Lemma 1.8(iii), we see that $\nu_2(Q_2) \geq \nu_2(k+1)$. So $Q_2/2^{\nu_2(k+1)}$ is an integer. With the help of Lemma 1.8(iii), we also see that

$$\begin{aligned} &\frac{Q_2}{2^{\nu_2(k+1)}} = \frac{(k+1)'C_n^{(k-1)}\binom{2kn}{kn}}{\binom{2n}{n}} \text{ is odd} \\ \iff\ & \nu_2((m+1)!) = m \\ \iff\ & m+1 = (k-1)n+1 \text{ is a power of two.} \end{aligned}$$

This concludes the proof of Theorem 1.3(ii). □

3. Proofs of Theorems 1.4 and 1.6

Lemma 3.1. *Let p be a prime and let $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. Then*

$$\frac{\rho_p((p^k-1)n)}{p-1} = \sum_{j=1}^{\infty}\left\{\frac{(p^k-1)n}{p^j}\right\} \geq k \tag{3.1}$$

and hence the expansion of $(p^k-1)n$ in base p has at least k nonzero digits.

Proof. For any $m \in \mathbb{Z}^+$, by Lemma 1.8 (ii),

$$\frac{\rho_p(m)}{p-1} = \frac{m}{p-1} - \nu_p(m!) = \sum_{j=1}^{\infty}\frac{m}{p^j} - \sum_{j=1}^{\infty}\left\lfloor\frac{m}{p^j}\right\rfloor = \sum_{j=1}^{\infty}\left\{\frac{m}{p^j}\right\}.$$

If the expansion of m in base p has less than k nonzero digits, then $\rho_p(m) < k(p-1)$. So it remains to show that the inequality in formula (3.1) holds.

Observe that

$$p^k\binom{p^kn-1}{n-1} = \binom{p^kn}{n} = \frac{(p^kn)!}{n!((p^k-1)n)!}$$

and

$$\begin{aligned} &\nu_p((p^kn)!) - \nu_p(n!) - \nu_p(((p^k-1)n)!) \\ &\quad = \sum_{j=1}^{\infty}\left\lfloor\frac{p^kn}{p^j}\right\rfloor - \sum_{j=1}^{\infty}\left\lfloor\frac{n}{p^j}\right\rfloor - \sum_{j=1}^{\infty}\left\lfloor\frac{(p^k-1)n}{p^j}\right\rfloor \\ &\quad = \sum_{j=1}^{k} p^{k-j}n - \sum_{j=1}^{\infty}\left\lfloor\frac{(p^k-1)n}{p^j}\right\rfloor = \sum_{j=1}^{\infty}\left\{\frac{(p^k-1)n}{p^j}\right\}. \end{aligned}$$

So the inequality in formula (3.1) holds and we are done. □

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Proof of Theorem 1.4. Since the odd part of $(2^k-1)+1$ is 1 by Theorem 1.3(ii) and its proof, we see that

$$Q_3 := \frac{\binom{2(2^k-1)n}{(2^k-1)n} C_n^{(2^k-2)}}{\binom{2n}{n}} \in \mathbb{Z}$$

and also that

$$\nu_2(Q_3) = m - \nu_2((m+1)!)$$

where $m = ((2^k-1)-1)n$ is even. We now apply Lemma 1.8(ii) and Lemma 3.1 with $p=2$ to deduce that

$$\nu_2(Q_3) = m! - \nu_2(m!) = \rho_2(m) = \rho_2((2^{k-1}-1)n) \ge k-1.$$

Therefore $2^{k-1} \mid Q_3$ and hence formula (1.2) holds. □

Lemma 3.2. *Let x be a real number.*

(i) *Then*

$$\{12x\} + \{5x\} + \{2x\} \ge \{4x\} + \{15x\}. \tag{3.2}$$

(ii) *Suppose also that $\{5x\} \ge \{2x\} \ge 1/2$. Then $\{5x\} \ge 2/3$.*

Proof. Since

$$12x + 5x + 2x - 4x = 15x,$$

inequality (3.2) reduces to

$$\{12x\} + \{5x\} + \{2x\} - \{4x\} \ge 0$$

which can be easily checked and part (i) is proved.

As $\{5x\} \ge \{2x\} \ge 1/2$ we can easily see that

$$\{x\} \in [1/3, 2/5) \cup [3/4, 4/5).$$

It follows that $\{5x\} \ge 2/3$ and (ii) is proved. □

Lemma 3.3. *Let $m > 1$ and n be integers.*

(i) *If $3 \nmid m$, then*

$$\left\lfloor \frac{15n-1}{m} \right\rfloor + \left\lfloor \frac{2}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor \ge \left\lfloor \frac{12n+2}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor + \left\lfloor \frac{5n-1}{m} \right\rfloor. \tag{3.3}$$

(ii) *If $5 \nmid m$, then*

$$\left\lfloor \frac{15n-1}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor \ge \left\lfloor \frac{10n+1}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n-1}{m} \right\rfloor. \tag{3.4}$$

Proof. First we prove (i). Clearly (3.3) holds when $m=2$. Below we assume that $m>2$ and $3 \nmid m$.

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Since $m \mid 15n$ if and only if $m \mid 5n$,

$$\left\{\frac{5n-1}{m}\right\} - \left\{\frac{15n-1}{m}\right\} = \left\{\frac{5n}{m}\right\} - \left\{\frac{15n}{m}\right\}$$

and thus inequality (3.3) has the following equivalent form:

$$\left\{\frac{12n+2}{m}\right\} + \left\{\frac{5n}{m}\right\} + \left\{\frac{2n}{m}\right\} - \left\{\frac{4n}{m}\right\} \geq \left\{\frac{15n}{m}\right\} + \frac{2}{m}. \tag{3.5}$$

If

$$12n+1,\ 12n+2 \not\equiv 0 \pmod{m},$$

then inequality (3.5) is equivalent to the inequality

$$\{12x\} + \{5x\} + \{2x\} - \{4x\} \geq \{15x\}$$

where $x = n/m$, which holds by Lemma 3.2(i).

Below we assume that

$$12n + \delta \equiv 0 \pmod{m}$$

for some $\delta \in \{1, 2\}$. Clearly m does not divide $3n$ and inequality (3.5) can be rewritten as

$$\left\{\frac{5n}{m}\right\} + \left\{\frac{2n}{m}\right\} - \left\{\frac{4n}{m}\right\} \geq \left\{\frac{3n-\delta}{m}\right\} + \frac{\delta}{m} = \left\{\frac{3n}{m}\right\}.$$

(Note that if $m \mid (12n+2)$ and $m \mid (3n-1)$, then m divides

$$12n + 2 - 4(3n-1) = 6$$

which contradicts the conditions that $m > 2$ and $3 \nmid m$.)

Now it suffices to show that

$$f(x) := \{5x\} + \{2x\} - \{4x\} - \{3x\} \geq 0$$

where $x = \{n/m\}$. Clearly

$$\begin{aligned} f(x) &= \lfloor 3x \rfloor + \lfloor 4x \rfloor - \lfloor 2x \rfloor - \lfloor 5x \rfloor \\ &= \left|\left\{\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}\right\} \cap (0, x]\right| - \left|\left\{\tfrac{1}{2}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}\right\} \cap (0, x]\right|. \end{aligned}$$

It follows that $f(x) < 0$ if and only if

$$x \in [1/5, 1/4) \cup [3/5, 2/3).$$

Clearly

$$a := 12x + \delta/m \in \{1, \ldots, 11\}$$

and

$$x = \frac{a}{12} - \frac{\delta/m}{12} \in \left(\frac{a-1}{12}, \frac{a}{12}\right).$$

Note that

$$\left[\tfrac{1}{5}, \tfrac{1}{4}\right) \subseteq \left(\tfrac{2}{12}, \tfrac{3}{12}\right) \quad \text{and} \quad \left[\tfrac{3}{5}, \tfrac{2}{3}\right) \subseteq \left(\tfrac{7}{12}, \tfrac{8}{12}\right).$$

Also $a \neq 3, 8$ since 12 divides neither $3m - \delta$ nor $8m - \delta$. We have thus proved part (i).

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To prove part (ii), suppose that $5 \nmid m$. Then $m \mid 15n$ if and only if $m \mid 3n$. Note also that
$$(10n+1) - 1 + 3n + 4n - 2n = 15n.$$
Thus inequality (3.4) has the following equivalent form:
$$W := \left\{\frac{10n+1}{m}\right\} - \frac{1}{m} + \left\{\frac{3n}{m}\right\} + \left\{\frac{4n}{m}\right\} - \left\{\frac{2n}{m}\right\} \geq 0. \tag{3.6}$$
In the case where $m \mid 3n$, inequality (3.6) reduces to
$$\{(n+1)/m\} + \{n/m\} \geq \{2n/m\} + 1/m,$$
which holds whether m divides $2n+1$ or not.

Below we assume that $m \nmid 3n$. Then
$$W := \left\{\frac{10n+1}{m}\right\} + \left\{\frac{3n-1}{m}\right\} + \left\{\frac{4n}{m}\right\} - \left\{\frac{2n}{m}\right\}.$$
If $\{2n/m\} < 1/2$, then
$$\{4n/m\} - \{2n/m\} = \{2n/m\} \geq 0.$$
Moreover, if $\{2n/m\} \geq 1/2$ and $\{(5n-1)/m\} < \{2n/m\}$, then
$$W = \left\{\frac{10n+1}{m}\right\} + \left\{\frac{3n-1}{m}\right\} + \left\{\frac{2n}{m}\right\} - 1 \geq \left\{\frac{5n-1}{m}\right\} \geq 0.$$
We now consider the remaining case, that is, when
$$\{(5n-1)/m\} \geq \{2n/m\} \geq 1/2.$$
Note that
$$W = \left\{\frac{10n+1}{m}\right\} + \left\{\frac{3n-1}{m}\right\} + \left\{\frac{2n}{m}\right\} - 1 = \left\{\frac{10n+1}{m}\right\} + \left\{\frac{5n-1}{m}\right\} - 1.$$
Clearly $W = 0$ if $m \mid 5n$. If $m \mid (10n+1)$, then $2 \nmid m$,
$$5n \equiv (m-1)/2 \pmod m$$
and hence $\{(5n-1)/m\} < 1/2$.

Now we simply assume that $m \nmid 5n$ and $m \nmid (10n+1)$. Then $\{5x\} \geq \{2x\} \geq 1/2$, where $x = n/m$. Thus
$$W = 2\{5x\} - 1 + \{5x\} - 1 \geq 0$$
by Lemma 3.2(ii). This concludes the proof. □

Proof of Theorem 1.6. Observe that
$$A := \frac{\binom{3n-1}{n-1} C_{3n}^{(4)}}{(6n+1)\binom{5n}{n}} = \frac{(15n-1)!2!(4n)!}{(12n+2)!(2n)!(5n-1)!}$$

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and

$$B := \frac{\binom{5n-1}{n-1} C_{5n}^{(2)}}{\binom{3n}{n}} = \frac{(15n-1)!(2n)!}{(10n+1)!(4n)!(3n-1)!}.$$

By Lemma 3.3, $\nu_p(A) \geq 0$ for any prime $p \neq 3$, and $\nu_p(B) \geq 0$ for any prime $p \neq 5$. Thus it suffices to show that $\nu_3(A) \geq 0$ and $\nu_5(B) \geq 0$. In fact

$$\frac{C_{3n}^{(4)}}{(6n+1)\binom{5n}{n}} = \frac{1}{(6n+1)(12n+1)} \prod_{\substack{j=1 \\ 3 \nmid j}}^{3n} \frac{12n+j}{j}$$

is a 3-adic integer and

$$\frac{C_{5n}^{(2)}}{\binom{3n}{n}} = \frac{1}{10n+1} \prod_{\substack{j=1 \\ 5 \nmid j}}^{5n} \frac{10n+j}{j}$$

is a 5-adic integer. We are done. □

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