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Extremal Sequences for the Bellman Function of the Dyadic Maximal Operator and Applications to the Hardy Operator

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Abstract. We prove that the extremal sequences for the Bellman function of the dyadic maximal operator behave approximately as eigenfunctions of this operator for a specific eigenvalue. We use this result to prove the analogous one with respect to the Hardy operator.

1 Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is a useful tool in analysis and is defined by

$$\mathcal{M}_d\phi(x) = \sup\Big\{ \frac{1}{|Q|}\int_Q |\phi(y)|dy \ : x \in Q,\ Q \subseteq \mathbb{R}^n \text{ in a dyadic cube}\Big\},$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$, where the dyadic cubes are those formed by the grids

$$2^{-N}\mathbb{Z}^n, \quad \text{for } N = 0, 1, 2, \dots.$$

As is well known, it satisfies the weak type (1,1) inequality

$$|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) > \lambda\}| \le \frac{1}{\lambda}\int_{\{\mathcal{M}_d\phi>\lambda\}} |\phi(u)|du, \tag{1.1}$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$.

It is easily seen that (1.1) implies the following L^p-inequality:

$$\|\mathcal{M}_d\phi\|_p \le \frac{p}{p-1}\|\phi\|_p. \tag{1.2}$$

It is also easy to see that the weak type inequality (1.1) is best possible, while (1.2) is also sharp. (See [1] for general martingales and [15] for dyadic ones).

For further study of the dyadic maximal operator the function

$$B_p(f,F) = \sup\left\{ \frac{1}{|Q|}\int_Q (\mathcal{M}_d\phi)^p : \phi \ge 0, \frac{1}{|Q|}\int_Q \phi = f, \frac{1}{|Q|}\int_Q \phi^p = F\right\}, \tag{1.3}$$

has been introduced, where Q is a fixed dyadic cube and $0 < f^p \le F$.

The function (1.3), which is called the Bellman function of two variables of the dyadic maximal operator, is in fact independent of the cube Q, and its value was given

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in [2]. More precisely, it was proved there that

$$B_p(f,F) = F\omega_p(f^p/F)^p,$$

where $\omega_p\colon[0,1]\to[1,\frac{p}{p-1}]$ denotes the inverse function H_p^{-1} of H_p, which is defined by

$$H_p(z) = -(p-1)z^p + pz^{p-1}, \quad \text{for } z\in\Big[1,\frac{p}{p-1}\Big].$$

In fact, this evaluation has been done in a much more general setting, where the dyadic sets are now given as elements of a tree $\mathcal{T}$ on a non-atomic probability space (X,μ). Then the associated dyadic maximal operator is defined by

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\Big\{\frac{1}{\mu(I)}\int_I|\phi|d\mu : x\in I\in\mathcal{T}\Big\}.$$

Additionally, inequalities (1.1) and (1.2) remain true and sharp in this setting. Moreover, if we define

$$B'_{p,\mathcal{T}}(f,F) = \sup\Big\{\int_X(\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \phi\ge 0, \int_X\phi d\mu = f, \int_X\phi^p d\mu = F\Big\} \tag{1.4}$$

for $0<f^p\le F$, then $B'_{p,\mathcal{T}}(f,F) = B_p(f,F)$. In particular, the Bellman function of the dyadic maximal operator is independent of the structure of the tree $\mathcal{T}$.

Another approach for finding the value of $B_p(f,F)$ is given in [3], where the following function of two variables was introduced:

$$S_p(f,F) = \sup\Big\{\int_0^1\Big(\frac{1}{t}\int_0^t g\Big)^p dt : g\colon(0,1]\to\mathbb{R}^+ : \text{ non-increasing, continuous and } \int_0^1 g = f, \int_0^1 g^p = F\Big\}. \tag{1.5}$$

The first step, as seen in [3], is to prove that $S_p(f,F) = B_p(f,F)$. This can be viewed as a symmetrization principle of the dyadic maximal operator with respect to the Hardy operator. The second step is to prove that $S_p(f,F)$ has the expected value mentioned above.

Now the proof that $S_p = B_p$ can be given in an alternative way, as can be seen in [8]. More precisely, the following result is proved there.

Theorem 1.1 ([8]) *Given $g,h\colon(0,1]\to\mathbb{R}^+$ non-increasing integrable functions and a non-decreasing function $G\colon[0,+\infty)\to[0,+\infty)$, the following equality holds:*

$$\sup\Big\{\int_K G[(\mathcal{M}_{\mathcal{T}}\phi)^*]h(t)dt : \phi\ge 0, \phi^* = g,\ K \textit{ measurable subset of } [0,1] \textit{ with } |K| = k\Big\} = \int_0^k G\Big(\frac{1}{t}\int_0^t g\Big)h(t)dt,$$

for any $k\in(0,1]$, where ϕ^* denotes the equimeasurable decreasing rearrangement of ϕ.

It is obvious that Theorem 1.1 implies the equation $S_p = B_p$, and gives an immediate connection of the dyadic maximal operator with the Hardy operator.

An interesting question that arises now is the behaviour of the extremal sequences of functions for the quantities (1.4) and (1.5). The problem concerning (1.4) was solved in [6], where the following theorem is proved.

Theorem 1.2 ([6]) *If $\phi_n\colon (X,\mu) \to \mathbb{R}^+$ satisfies $\int_X \phi_n d\mu = f$, $\int_X \phi_n^p d\mu = F$, for every $n \in N$, then the following are equivalent:*

(i) $\lim_n \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p$,

(ii) $\lim_n \int_X |\mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n|^p d\mu = 0$, *where* $c = \omega_p(f^p/F)$.

Now it is interesting to consider the opposite problem concerning (1.5). In fact, we will prove the following theorem.

Theorem 1.3 *Let $g_n\colon (0,1] \to \mathbb{R}^+$ be a sequence of non-increasing functions continuous such that $\int_0^1 g_n(u)du = f$ and $\int_0^1 g_n^p(u)du = F$ for every $n \in \mathbb{N}$. Then the following are equivalent:*

(i) $$\lim_n \int_0^1 \Big(\frac{1}{t} \int_0^t g_n \Big)^p dt = F\omega_p(f^p/F)^p,$$

(ii) $$\lim_n \int_0^1 \Big| \frac{1}{t} \int_0^t g_n - c g_n(t) \Big|^p dt = 0, \text{ where } c = \omega_p(f^p/F).$$

The proof is based on the proof of Theorem 1.1 and on the statement of Theorem 1.2.

Concerning problem (1.4), it can be easily seen that extremal functions do not exist (when the tree $\mathcal{T}$ differentiates $L^1(X,\mu)$). That is, for every $\phi \in L^p(X,\mu)$ with $\phi \geq 0$ and $\int_X \phi d\mu = f$, $\int_X \phi^p d\mu = F$ we have the strict inequality $\int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu < F\omega_p(f^p/F)^p$.

This is because of a self-similar property that is mentioned in [7], which states that for every extremal sequence (ϕ_n) for (1.4) the following is true:

$$\lim_n \frac{1}{\mu(I)} \int_I \phi_n d\mu = f, \text{ while } \lim_n \frac{1}{\mu(I)} \int_I \phi_n^p d\mu = F.$$

So if ϕ is an extremal function for (1.4), then we must have that $\frac{1}{\mu(I)} \int_I \phi d\mu = f$ and $\frac{1}{\mu(I)} \int_I \phi^p d\mu = F$, and since the tree $\mathcal{T}$ differentiates $L^1(X,\mu)$ (because of (1.1)), we must have that $\phi(x) = f$ and $\phi^p(x) = F$ hold μ-a.e; that is, $f^p = F$, which is the trivial case.

It turns out that the above does not hold for the extremal problem (1.5). That is there exist extremal functions for (1.5). We state it as the following theorem.

Theorem 1.4 *There exists unique $g\colon (0,1] \to \mathbb{R}^+$ non-increasing and continuous with $\int_0^1 g(u)du = f$ and $\int_0^1 g^p(u)du = F$ such that*

$$\int_0^1 \Big(\frac{1}{t} \int_0^t g \Big)^p dt = F\omega_p(f^p/F)^p. \tag{1.6}$$

As expected, due to Theorem 1.3, g satisfies the equality

$$\frac{1}{t} \int_0^t g(u)du = \omega_p(f^p/F)g(t)$$

for every $t \in (0,1]$, which immediately gives (1.6).

After proving Theorem 1.4 we will be able to prove the following theorem.

Theorem 1.5 *Let g_n be as in Theorem 1.3. Then the following are equivalent*

(i) $\lim_n \int_0^1 \left(\frac{1}{t}\int_0^t g_n\right)^p dt = F\omega_p(f^p/F)^p$;

(ii) $\lim_n \int_0^1 |g_n - g|^p dt = 0$, *where g is the function constructed in Theorem 1.4.*

In this way we complete the discussion about the characterization of the extremal functions of the corresponding problem related to the Hardy operator. We also remark that for the proof of Theorem 1.3 we need to fix a non-atomic probability space (X,μ) equipped with a tree structure $\mathcal{T}$ that differentiates $L^1(X,\mu)$. We use this measure space as a base in order to work there with measurable non-negative rearrangements of certain non-increasing functions on $(0,1]$.

We should also mention that the exact evaluation of (1.3) for $p > 1$ was also given in [10] by L. Slavin, A. Stokolos and V. Vasyunin who linked the computation of it to solving certain PDE's of the Monge-Ampère type, and in this way they obtained an alternative proof of the results in [2]. This method is different from that used in [2] or [9]. However, the techniques that appear in the last two articles and this one, give us the possibility to provide effective and powerful stability results (see for example [6]).

We also remark that there are several problems in harmonic analysis where Bellman functions arise. Such problems (including the dyadic Carleson imbedding theorem and weighted inequalities) are described in [10] (one can also see [4] and [5]) and also connections to stochastic optimal control are provided, from which it follows that the corresponding Bellman functions satisfy certain nonlinear second-order PDE's. Finally, we remark that the exact evaluation of a Bellman function is a difficult task and is connected with the deeper structure of the corresponding harmonic analysis problem. We mention also that until now several Bellman functions have been computed (see [2–5, 9, 11–14]).

The paper is organized as follows. In Section 2 we give some preliminary definitions and results. In Section 4 we give an alternative proof of Theorem 1.2, which is based on the proof of the evaluation of the Bellman function of two variables for the dyadic maximal operator and is presented in Section 3. At last we prove Theorems 1.3, 1.4, and 1.5 in Sections 5, 6, and 7, respectively.

2 Preliminaries

Let (X,μ) be a non-atomic probability measure space.

Definition 2.1 A set $\mathcal{T}$ of measurable subsets of X will be called a *tree* if it satisfies the following conditions:

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have that $\mu(I) > 0$.

(ii) For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I) \subseteq \mathcal{T}$ containing at least two elements such that

  (a) the elements of $C(I)$ are pairwise disjoint subsets of I,

  (b) $I = \cup C(I)$.

(iii) $\mathcal{T} = \bigcup_{m\geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I\in\mathcal{T}_{(m)}} C(I)$.
(iv) We have that $\lim_{m\to\infty} \sup_{I\in\mathcal{T}_{(m)}} \mu(I) = 0$.

Examples of trees are given in [2]. The most known is the one given by the family of all dyadic subcubes of $[0,1]^n$. The following was proved in [3].

Lemma 2.2 *For every $I \in \mathcal{T}$ and every a such that $0 < a < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of pairwise disjoint subsets of I such that*

$$\mu\Big(\bigcup_{J\in\mathcal{F}(I)} J\Big) = \sum_{J\in\mathcal{F}(I)} \mu(J) = (1-a)\mu(I).$$

We will also need the following fact, obtained in [8].

Lemma 2.3 *Let $\phi: (X,\mu) \to \mathbb{R}^+$ and let $(A_j)_j$ be a measurable partition of X such that $\mu(A_j) > 0$, for all j. If $\int_X \phi\, d\mu = f$, then there exists a rearrangement h of ϕ, ($h^* = \phi^*$), such that $\frac{1}{\mu(A_j)} \int_{A_j} h\, d\mu = f$, for every j.*

Now given a tree on (X,μ) we define the associated dyadic maximal operator as

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\Big\{ \frac{1}{\mu(I)} \int_I |\varphi|\, d\mu : \ x \in I \in \mathcal{T} \Big\},$$

where $\phi \in L^1(X,\mu)$. We also recall the following from [8].

Lemma 2.4 *Let $k \in (0,1]$ and K measurable subset of X with $\mu(K) = k$. Then the following inequality holds*

$$\int_K G[\mathcal{M}_{\mathcal{T}}\phi]\, d\mu \leq \int_0^k G\Big(\frac{1}{t}\int_0^t g(u)\, du\Big)\, dt,$$

where $g = \phi^*$, $\phi \in L^1(X,\mu)$, and $G: [0,+\infty) \to [0,+\infty)$ is a non-decreasing function.

3 The Bellman Function of the Dyadic Maximal Operator

In this section we provide a proof of the evaluation of the Bellman function of the dyadic maximal operators with respect to two variables f, F. The result appears in [9] in a more general form, but we give a proof for completeness. For this purpose we will need the following lemma.

Lemma 3.1 *Let $\phi: (X,\mu) \to \mathbb{R}^+$ be such that*

$$\int_X \phi\, d\mu = f \quad \textit{and} \quad \int_X \phi^p\, d\mu = F,$$

where $0 < f^p \leq F$. Then

$$\int_X (\mathcal{M}_{\mathcal{T}}\phi)^p\, d\mu \leq F \cdot \omega_p(f^p/F)^p.$$

Proof We consider the integral

$$I = \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p\, d\mu.$$

By using Fubini's theorem we can write

$$I = \int_{\lambda=0}^{+\infty} p\lambda^{p-1}\mu\big(\{\mathcal{M}_{\mathcal{T}}\phi > \lambda\}\big)\,d\lambda = \int_{\lambda=0}^{f} + \int_{\lambda=f}^{+\infty} p\lambda^{p-1}\mu\big(\{\mathcal{M}_{\mathcal{T}}\phi > \lambda\}\big)\,d\lambda = I_1 + I_2, \tag{3.1}$$

where

$$I_1 = \int_{\lambda=0}^{f} p\lambda^{p-1}\mu\big(\{\mathcal{M}_{\mathcal{T}}\phi > \lambda\}\big)\,d\lambda = \int_{\lambda=0}^{f} p\lambda^{p-1}\mu(X)d\lambda = \int_{\lambda=0}^{f} p\lambda^{p-1}d\lambda = f^p, \tag{3.2}$$

since $\mathcal{M}_{\mathcal{T}}\phi(x) \geq f$, for every $x \in X$. Then I_2 is defined by

$$I_2 = \int_{\lambda=f}^{+\infty} p\lambda^{p-1}\mu(\{\mathcal{M}_{\mathcal{T}}\phi > \lambda\})d\lambda.$$

Using inequality (1.1), we conclude that

$$\begin{aligned} I_2 &\leq \int_{\lambda=f}^{+\infty} p\lambda^{p-1}\frac{1}{\lambda}\Big(\int_{\{\mathcal{M}_{\mathcal{T}}\phi>\lambda\}} \phi\,d\mu\Big)\,d\lambda \\ &= \int_{\lambda=f}^{+\infty} p\lambda^{p-2}\Big(\int_{\{\mathcal{M}_{\mathcal{T}}\phi>\lambda\}} \phi\,d\mu\Big)\,d\lambda = \frac{p}{p-1}\int_X \phi(x)\big[\lambda^{p-1}\big]_{\lambda=f}^{\mathcal{M}_{\mathcal{T}}\phi(x)}\,d\mu(x), \end{aligned}$$

where in the last step we have used Fubini's theorem and the fact that $\mathcal{M}_{\mathcal{T}}\phi(x) \geq f$, for all $x \in X$. Therefore,

$$I_2 \leq \frac{p}{p-1}\int_X \phi\cdot(\mathcal{M}_{\mathcal{T}}\phi)^{p-1}d\mu - \frac{p}{p-1}f^p. \tag{3.3}$$

Thus, as a consequence of (3.1), (3.2), and (3.3) we have that

$$I = \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu \leq -\frac{1}{p-1}f^p + \frac{p}{p-1}\int_X \phi\cdot(\mathcal{M}_{\mathcal{T}}\phi)^{p-1}d\mu. \tag{3.4}$$

Using Hölder's inequality now, it is easy to see that for every ϕ as above, the following inequality is true:

$$\int_X \phi(\mathcal{M}_{\mathcal{T}}\phi)^{p-1}d\mu \leq \Big(\int_X \phi^p d\mu\Big)^{1/p}\cdot\Big(\int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu\Big)^{(p-1)/p}. \tag{3.5}$$

By (3.4) and (3.5), we thus have

$$I = \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu \leq -\frac{1}{p-1}f^p + \frac{p}{p-1}\cdot F^{1/p}\cdot I^{(p-1)/p} \tag{3.6}$$

therefore,

$$\frac{I}{F} \leq -\frac{1}{p-1}\cdot\frac{f^p}{F} + \Big(\frac{p}{p-1}\Big)\Big(\frac{I}{F}\Big)^{(p-1)/p}.$$

Because of (3.6), if we set $J = (\frac{I}{F})^{1/p}$, we have that

$$J^p \leq -\frac{1}{p-1}\cdot\frac{f^p}{F} + \frac{p}{p-1}J^{p-1}. \tag{3.7}$$

We distinguish the following two cases:

(a) $J \le 1$: Then $J \le \omega_p(f^p/F)$, since ω_p takes values on $[1, p/(p-1)]$. Thus,

$$\left(\frac{I}{F}\right)^{1/p} \le \omega_p(f^p/F), \quad \text{hence} \quad I \le F\omega_p(f^p/F)^p,$$

and our result is trivial in this case.

(b) $J > 1$: Then because of (3.7) we conclude that

$$pJ^{p-1} - (p-1)J^p \ge \frac{f^p}{F} \quad \text{or} \quad H_p(J) \ge \frac{f^p}{F}$$

hence, $J \le \omega_p(\frac{f^p}{F})$, since $\omega_p = H_p^{-1}$. As a consequence, we have that

$$\int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu \le F \cdot \omega_p\left(\frac{f^p}{F}\right)^p,$$

which concludes the proof of our lemma. ∎

As we shall see in Section 6, for every f, F fixed such that $0 < f^p \le F$ and $p > 1$ there exists $g: (0,1] \to \mathbb{R}^+$ non-increasing, continuous that satisfies $\int_0^1 g(u)du = f$, $\int_0 g^p(u)du = F$, and $\frac{1}{t}\int_0^t g(u)du = cg(t)$ for every $t \in (0,1]$ where

$$c = \omega_p\left(\frac{f^p}{F}\right).$$

Thus, the next theorem is a consequence of Theorem 1.1 and the results of this section.

Theorem 3.2 *Let f, F be fixed such that $0 < f^p \le F$ where $p > 1$. Then the following equality is true*

$$\sup\left\{ \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \phi \ge 0, \int_X \phi d\mu = f, \int_X \phi^p d\mu = F\right\} = F\omega_p\left(\frac{f^p}{F}\right)^p.$$

4 Characterization of the Extremal Sequences for the Bellman Function

In this section we will provide an alternative proof of Theorem 1.2, different from that in [6], based on the proof of the evaluation of the Bellman function of the dyadic maximal operator, which is given in Section 3.

Proof of Theorem 1.2 (i) ⇒ (ii) Let $(\phi_n)_n$ be a sequence of functions $\phi_n: (X, \mu) \to \mathbb{R}^+$ such that $\int_X \phi_n d\mu = f$, $\int_X \phi_n^p d\mu = F$ for which

$$\lim_n \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p.$$

We will prove that

$$\lim_n \int_X |\, \mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n \,|^p \, d\mu = 0,$$

where $c = \omega_p(\frac{f^p}{F})$.

By setting $\Delta_n = \{\mathcal{M}_\mathcal{T}\phi_n > c\phi_n\}$ and $\Delta'_n = X \setminus \Delta_n = \{\mathcal{M}_\mathcal{T}\phi_n \le c\phi_n\}$, it is immediate to see that it is enough to define

$$I_n = \int_{\Delta_n} (\mathcal{M}_\mathcal{T}\phi_n - c\phi_n)^p d\mu \quad \text{and} \quad J_n = \int_{\Delta'_n} (c\phi_n - \mathcal{M}_\mathcal{T}\phi_n)^p d\mu,$$

and then prove that $I_n, J_n \to 0$, as $n \to \infty$.

For the evaluation of the Bellman function, as described in the previous section we used the following inequality:

$$\int_X \phi \cdot (\mathcal{M}_\mathcal{T}\phi)^{p-1} d\mu \le \Big(\int_X \phi^p d\mu\Big)^{1/p} \cdot \Big(\int_X (\mathcal{M}_\mathcal{T}\phi)^p d\mu\Big)^{(p-1)/p},$$

which, for our sequence $(\phi_n)_n$, must hold as an equality in the limit (we pass to a subsequence if necessary). We write this fact as

$$\int_X \phi_n \cdot (\mathcal{M}_\mathcal{T}\phi_n)^{p-1} d\mu \approx \Big(\int_X \phi_n^p d\mu\Big)^{1/p} \cdot \Big(\int_X (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu\Big)^{(p-1)/p}.$$

Now, we are going to state and prove the following lemma.

Lemma 4.1 *Under the above notation and hypotheses we have that*

$$\int_{X_n} \phi_n (\mathcal{M}_\mathcal{T}\phi_n)^{p-1} d\mu \approx \Big(\int_{X_n} \phi_n^p d\mu\Big)^{1/p} \cdot \Big(\int_{X_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu\Big)^{(p-1)/p},$$

where X_n may be replaced either by Δ_n or Δ'_n.

Proof Certainly the following inequalities hold true in view of Hölder's inequality. These are

$$\int_{\Delta_n} \phi_n \cdot (\mathcal{M}_\mathcal{T}\phi_n)^{p-1} d\mu \le \Big(\int_{\Delta_n} \phi_n^p d\mu\Big)^{1/p} \cdot \Big(\int_{\Delta_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu\Big)^{(p-1)/p}, \tag{4.1}$$

$$\int_{\Delta'_n} \phi_n (\mathcal{M}_\mathcal{T}\phi_n)^{p-1} d\mu \le \Big(\int_{\Delta'_n} \phi_n^p d\mu\Big)^{1/p} \cdot \Big(\int_{\Delta'_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu\Big)^{(p-1)/p}, \tag{4.2}$$

for any $n \in \mathbb{N}$. Adding them we obtain

$$\begin{aligned} \int_X \phi_n \cdot (\mathcal{M}_\mathcal{T}\phi_n)^{p-1} d\mu \le &\Big(\int_{\Delta_n} \phi_n^p d\mu\Big)^{1/p} \cdot \Big(\int_{\Delta_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu\Big)^{(p-1)/p} \\ &+ \Big(\int_{\Delta'_n} \phi_n^p d\mu\Big)^{1/p} \cdot \Big(\int_{\Delta'_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu\Big)^{(p-1)/p}. \end{aligned} \tag{4.3}$$

We now use the following elementary inequality for which the proof is given below.

For every $t, t' > 0$, $s, s' > 0$ such that $t + t' = a > 0$ and $s + s' = b > 0$ and any $q \in (0,1)$, we have that

$$t^q \cdot s^{1-q} + (t')^q \cdot (s')^{1-q} \le a^q \cdot b^{1-q}, \tag{4.4}$$

Applying it for $q = 1/p$ we obtain the following inequality from (4.3):

$$\int_X \phi_n \cdot (\mathcal{M}_\mathcal{T}\phi_n)^{p-1} d\mu \le \Big(\int_X \phi_n^p d\mu\Big)^{1/p} \cdot \Big(\int_X (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu\Big)^{(p-1)/p},$$

which in fact is an equality in the limit, because of our hypothesis. Thus, we must have equality in both (4.1) and (4.2) in the limit, and our lemma is proved as soon as we prove (4.4).

Fix $t \in (0, a]$ and consider the function F_t of the variable $s \in (0, b)$ defined by

$$F_t(s) = t^q \cdot s^{1-q} + (a-t)^q \cdot (b-s)^{1-q}.$$

Then

$$F_t'(s) = (1-q)\Big[\Big(\frac{t}{s}\Big)^q - \Big(\frac{a-t}{b-s}\Big)^q\Big], \quad s \in (0,b)$$

so that $F_t'(s) > 0$ for every $s \in (0, \frac{tb}{a})$, and $F_t'(s) < 0$ for $s \in (\frac{tb}{a}, b)$. Thus, F attains its maximum on the interval $[0, b]$ at the point $\frac{tb}{a}$. The result is now easily derived. ■

We continue now with the proof of Theorem 1.2. We write

$$\int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu = \int_{\Delta_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu + \int_{\Delta_n'} (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu.$$

We first assume that

$$\int_{\Delta_n} \phi_n^p \, d\mu > 0 \quad \text{and} \quad \int_{\Delta_n'} \phi_n^p \, d\mu > 0 \quad \text{for any } n \in \mathbb{N}.$$

Thus, in view of Hölder's inequality, (4.1), (4.2), and (4.5) we must have that

$$\int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu \geq \frac{\Big(\int_{\Delta_n} \phi_n \cdot (\mathcal{M}_{\mathcal{T}}\phi_n)^{p-1} d\mu\Big)^{p/(p-1)}}{\Big(\int_{\Delta_n} \phi_n^p d\mu\Big)^{1/(p-1)}} + \frac{\Big(\int_{\Delta_n'} \phi_n \cdot (\mathcal{M}_{\mathcal{T}}\phi_n)^{p-1} d\mu\Big)^{p/(p-1)}}{\Big(\int_{\Delta_n'} \phi_n^p d\mu\Big)^{1/(p-1)}}. \tag{4.5}$$

We now use Hölder's inequality in the following form:

$$\frac{a^k}{b^{k-1}} + \frac{c^k}{d^{k-1}} \geq \frac{(a+c)^k}{(b+d)^{k-1}}, \text{ for any } a, c \geq 0, b, d > 0, \text{ where } k > 1. \tag{4.6}$$

The above inequality is true as an equality if and only if

$$\frac{a}{b} = \frac{c}{d} = \lambda, \text{ for some } \lambda \in \mathbb{R}, \ \ \lambda \geq 0.$$

Thus, in view of (4.6), (4.5) becomes

$$\int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu \geq \frac{\Big(\int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^{p-1}\phi_n d\mu\Big)^{(p-1)/p}}{\Big(\int_X \phi_n^p d\mu\Big)^{1/(p-1)}}, \tag{4.7}$$

which is an equality in the limit, in view of the fact that ϕ_n is extremal for the Bellman function; that is, $\lim_n \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p\big(\frac{f^p}{F}\big)^p$. From all the above we conclude, by passing to a subsequence if necessary, that

$$\lim_n \frac{\int_{\Delta_n} \phi_n \cdot (\mathcal{M}_{\mathcal{T}}\phi_n)^{p-1} d\mu}{\int_{\Delta_n} \phi_n^p d\mu} = \lim_n \frac{\int_{\Delta_n'} \phi_n \cdot (\mathcal{M}_{\mathcal{T}}\phi_n)^{p-1} d\mu}{\int_{\Delta_n'} \phi_n^p d\mu} = \lambda \in \mathbb{R}^+. \tag{4.8}$$

Thus, by the equality that holds in the limit in (4.5), which is true because of the equality in (4.7), we conclude that

$$\lambda^{p/(p-1)} \lim_n \Big[\int_{\Delta_n} \phi_n^p d\mu + \int_{\Delta_n'} \phi_n^p d\mu \Big] = \lim_n \int_X (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu$$

or that

$$\lambda^{p/(p-1)} \cdot F = F \omega_p \Big(\frac{f^p}{F} \Big)^p,$$

hence

$$\lambda = \omega_p \Big(\frac{f^p}{F} \Big)^{p-1}.$$

Thus, by (4.8) we conclude

$$\int_{\Delta_n} \phi_n \cdot (\mathcal{M}_{\mathcal{T}} \phi_n)^{p-1} d\mu \approx \omega_p \Big(\frac{f^p}{F} \Big)^{p-1} \cdot \Big(\int_{\Delta_n} \phi_n^p d\mu \Big),$$
$$\int_{\Delta_n'} \phi_n (\mathcal{M}_{\mathcal{T}} \phi_n)^{p-1} d\mu \approx \omega_p \Big(\frac{f^p}{F} \Big)^{p-1} \cdot \Big(\int_{\Delta_n'} \phi_n^p d\mu \Big).$$

Then, because of Lemma 4.1, we obtain that

$$\int_{\Delta_n} (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu \approx \omega_p \Big(\frac{f^p}{F} \Big)^p \cdot \int_{\Delta_n} \phi_n^p d\mu,$$
$$\int_{\Delta_n'} (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu \approx \omega_p \Big(\frac{f^p}{F} \Big)^p \cdot \int_{\Delta_n'} \phi_n^p d\mu.$$

We will now need the following lemma.

Lemma 4.2 *Suppose we are given $\omega_n: X_n \to \mathbb{R}^+$, where $X_n \subseteq X$ for $n \in \mathbb{N}$, and $w: X \to \mathbb{R}^+$ satisfying $w_n \geq w$ on X_n. Suppose also that*

$$\lim_n \int_{X_n} w_n^p d\mu = \lim_n \int_{X_n} w^p d\mu, \text{ where } p > 1.$$

Then

$$\lim_n \int_{X_n} (w_n - w)^p d\mu = 0.$$

Proof It is a simple matter to prove this lemma because of the following inequality. For any $x > y > 0$, $p > 1$, the following holds: $(x - y)^p \leq x^p - y^p$. Thus,

$$\int_{X_n} (w_n - w)^p d\mu \leq \int_{X_n} w_n^p d\mu - \int_{X_n} w^p d\mu \to 0, \text{ as } n \to \infty$$

and the proof is complete. ∎

In view of Lemma 4.2 and the definitions of Δ_n, Δ_n', we see immediately that

$$\int_{\Delta_n} (\mathcal{M}_{\mathcal{T}} \phi_n - c\phi_n)^p d\mu \to 0 \quad \text{and} \quad \int_{\Delta_n'} (c\phi_n - \mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu \to 0, \text{ as } n \to \infty.$$

As a consequence, $\int_X | \mathcal{M}_{\mathcal{T}} \phi_n - c\phi_n |^p \, d\mu \to 0$, as $n \to \infty$, and our result is proved in the case where

$$\int_{\Delta_n} \phi_n^p > 0 \quad \text{and} \quad \int_{\Delta_n'} \phi_n^p d\mu > 0, \text{ for any } n \in \mathbb{N}. \tag{4.9}$$

The same proof holds even if we have that (4.9) is true for every $n \geq n_0$ for some $n_0 \in \mathbb{N}$.

Assume now that $\int_{\Delta'_n} \phi_n^p d\mu = 0$ for a fixed $n \in \mathbb{N}$. Since $\Delta'_n = \{\mathcal{M}_{\mathcal{T}}\phi_n \leq c\phi_n\}$ and $\mathcal{M}_{\mathcal{T}}\phi_n(x) \geq f$ for every $x \in X$, we conclude that

$$f^p \mu(\Delta'_n) \leq \int_{\Delta'_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \leq c^p \int_{\Delta'_n} \phi_n^p = 0,$$

therefore $\mu(\Delta'_n) = 0$, hence $\mathcal{M}_{\mathcal{T}}\phi_n > c\phi_n$ $\mu - a.c.$ on X. As a consequence, for our fixed $n \in \mathbb{N}$ we must have that

$$\int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu > c^p \cdot \int_X \phi_n^p d\mu = F \cdot \omega_p(f^p/F)^p,$$

which cannot hold in view of Lemma 3.1.

Now suppose that for some subsequence of $(\phi_n)_n$, which we suppose without loss of generality is the same as (ϕ_n), we have that

$$\int_{\Delta_n} \phi_n^p d\mu = 0 \tag{4.10}$$

Remember that $\Delta_n = \{\mathcal{M}_{\mathcal{T}}\phi_n > c\phi_n\}$.

Let $x \in \{\phi_n = 0\}$. Then if $x \in \Delta'_n$, we would have that $\mathcal{M}_{\mathcal{T}}\phi_n(x) \leq c\phi_n(x)$ or that $\mathcal{M}_{\mathcal{T}}\phi_n(x) = 0$, which is impossible, since $\mathcal{M}_{\mathcal{T}}\phi_n(y) \geq f$, for every $y \in X$. Thus, $\{\phi_n = 0\} \subseteq \Delta_n$, hence $\Delta'_n \subseteq \{\phi_n > 0\}$. But from (4.10) we have that $\int_{\Delta'_n} \phi_n^p d\mu = F$, so if $\mu(\{\phi_n > 0\} \setminus \Delta'_n)$ were positive, we would obtain $\int_{\{\phi_n > 0\}} \phi_n^p d\mu > F$, which is impossible. Thus, we have that

$$\Delta'_n \subseteq \{\phi_n > 0\} \quad \text{and} \quad \mu(\Delta'_n) = \mu(\{\phi_n > 0\})$$

for every $n \in \mathbb{N}$. Since integrals are not affected by adding or deleting a set of measure zero, we suppose that

$$\Delta'_n = \{\phi_n > 0\}. \tag{4.11}$$

Because of Lemma 4.1, we have that

$$\int_{\Delta'_n} \phi_n (\mathcal{M}_{\mathcal{T}}\phi_n)^{p-1} \approx \Big(\int_{\Delta'_n} \phi_n^p d\mu \Big)^{1/p} \cdot \Big(\int_{\Delta'_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \Big)^{(p-1)/p}. \tag{4.12}$$

Since (4.11) holds, we conclude by (4.12) that

$$\int_X \phi_n (\mathcal{M}_{\mathcal{T}}\phi_n)^{p-1} d\mu \approx F^{1/p} \Big(\int_{\Delta'_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \Big)^{(p-1)/p}.$$

But the next inequality is true in view of the extremality of the sequence of (ϕ_n) (see the beginning of this section):

$$\int_X \phi_n (\mathcal{M}_{\mathcal{T}}\phi_n)^{p-1} d\mu \approx F^{1/p} \cdot \Big(\int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \Big)^{(p-1)/p}.$$

Thus,

$$\int_{\Delta'_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \approx \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu, \quad \text{hence} \quad \int_{\Delta_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \approx 0,$$

and since $\mathcal{M}_{\mathcal{T}}\phi_n \geq f$ on X, we conclude that $\mu(\Delta_n) \to 0$. Then

$$\int_X |\, \mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n\,|^p \, d\mu = \int_{\Delta_n} + \int_{\Delta_n'} |\, \mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n\,|^p \, d\mu = I_n + J_n.$$

Then we proceed as follows:

$$I_n = \int_{\Delta_n} (\mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n)^p d\mu \leq \int_{\Delta_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu - c^p \int_{\Delta_n} (\phi_n)^p d\mu$$

in view of the elementary inequality used in the proof of Lemma 4.2. By all the above and by our hypothesis, we conclude that $I_n \approx 0$. As for J_n, we have

$$\begin{aligned} J_n &= \int_{\Delta_n'} (c\phi_n - \mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \leq c^p \int_{\Delta_n'} \phi_i^p d\mu - \int_{\Delta_n'} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \\ &\approx F\omega_p\Big(\frac{f^p}{F}\Big)^p - \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \approx 0, \end{aligned}$$

since (ϕ_n) is extremal.

Thus, in any case we conclude Theorem 1.2. ∎

5 Proof of Theorem 1.3

We will prove Theorem 1.3 by arguing as in the proof of Theorem 1.1 and by using Theorem 1.2.

Let $(g_n)_n$ be a sequence of non-increasing continuous functions $g_n\colon (0,1] \to \mathbb{R}^+$ such that $\int_0^1 g_n(u)du = f$ and $\int_0^1 g_n^p(u)du = F$, where $0 < f^p \leq F$. We set $c = \omega_p(f^p/F)$ and suppose that $(g_n)_n$ is extremal for (1.5); that is,

$$\lim_n \int_0^1 \Big(\frac{1}{t}\int_0^t g_n\Big)^p dt = F\omega_p(f^p/F)^p = F\cdot c^p.$$

Our aim is to prove that

$$\lim_n \int_0^1 \Big|\frac{1}{t}\int_0^t g_n - cg_n(t)\Big|^p dt = 0.$$

Set $A_n = \big\{ t \in (0,1] : \frac{1}{t}\int_0^t g_n > cg_n(t) \big\}$. Then for our purpose it is enough to prove that, as $n \to \infty$,

$$\int_{A_n} \Big[\frac{1}{t}\int_0^t g_n - cg_n(t)\Big]^p dt = I_{1,n} \to 0, \tag{5.1}$$

$$\int_{A_n} \Big[cg_n(t) - \frac{1}{t}\int_0^t g_n\Big]^p dt = I_{2,n} \to 0. \tag{5.2}$$

We consider (5.1), as (5.2) can be handled in a similar way. Since $(x-y)^p < x^p - y^p$, for $x > y > 0$ and $p > 1$, it is enough to prove that

$$II_n = \int_{A_n} \Big(\frac{1}{t}\int_0^t g_n\Big)^p dt - c^p \int_{A_n} g_n^p \to 0, \quad n \to \infty.$$

For each A_n that is an open set of $(0,1]$, we consider its connected components $I_{n,i}$, $i = 1, 2, \ldots$. So $A_n = \bigcup_{i=1}^{\infty} I_{n,i}$, where $I_{n,i}$ are open intervals in $(0,1]$ with $I_{n,i} \cap I_{n,j} = \varnothing$ for $i \neq j$.

Let $\varepsilon > 0$. For every $n \in \mathbb{N}$ choose $i_n \in \mathbb{N}$ such that

$$|III_n - III_{1,n}| < \varepsilon \quad \text{and} \quad |IV_n - IV_{1,n}| < \varepsilon,$$

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where

$$III_n = \int_{A_n} \Big(\frac{1}{t} \int_0^t g_n \Big)^p dt, \quad III_{1,n} = \int_{F_n} \Big(\frac{1}{t} \int_0^t g_n \Big)^p dt,$$
$$IV_n = c^p \int_{A_n} g_n^p, \qquad IV_{1,n} = c^p \int_{F_n} g_n^p,$$

and $F_n = \bigcup_{i=1}^{i_n} I_{n_1 i}$.

It is clear that such choice of i_n exists. Then $|II_n - II_{1,n}| < 2\varepsilon$, where

$$II_{1,n} = \int_{F_n} \Big(\frac{1}{t} \int_0^t g_n \Big)^p dt - c^p \int_{F_n} g_n^p.$$

We need to find a $n_0 \in \mathbb{N}$ such that $II_{1,n} < \varepsilon$, for all $n \geq n_0$. Fix a $g_n =: g$. We prove the following lemma.

Lemma 5.1 *There exists a family $\phi_a\colon (X,\mu) \to \mathbb{R}^+$ of rearrangements of g ($\phi_a^* = g$ for each $a \in (0,1)$) such that for each $\gamma \in (0,1]$ there exists a family of measurable subsets of X, $S_a^{(\gamma)}$ satisfying*

$$\lim_{a\to 0^+} \int_{S_a^{(\gamma)}} [\mathcal{M}_{\mathcal{T}}(\phi_a)]^p d\mu = \int_0^\gamma \Big(\frac{1}{t} \int_0^t g \Big)^p dt$$

and $\lim_{a\to 0^+} \mu(S_a^{(\gamma)}) = \gamma$. Moreover, we have that $S_a^{(\gamma)} \subseteq S_a^{(\gamma')}$ whenever $\gamma < \gamma' \leq 1$ and $a \in (0,1)$.

Proof We follow [8]. Let $a \in (0,1)$. Using Lemma 2.2 we choose for every $I \in \mathcal{T}$ a family $\mathcal{F}(I) \subseteq \mathcal{T}$ of disjoint subsets of I such that

$$\sum_{J\in\mathcal{F}(I)} \mu(J) = (1-a)\mu(I). \tag{5.3}$$

We define $S = S_a$ to be the smallest subset of $\mathcal{T}$ such that $X \in S$ and for every $I \in S$, $\mathcal{F}(I) \subseteq S$. We write $A_I = I \setminus \bigcup_{J\in\mathcal{F}(I)} J$ for $I \in S$. Then if $a_I = \mu(A_I)$, we have that $a_I = a\mu(I)$ because of (5.3). It is also clear that

$$S_a = \bigcup_{m\geq 0} S_{a,(m)}, \quad \text{where} \quad S_{a,(0)} = \{X\} \quad \text{and} \quad S_{a,(m+1)} = \bigcup_{I\in S_{a,(m)}} \mathcal{F}(I).$$

We also define $\operatorname{rank}(I) = r(I)$ to be the unique integer m for $I \in S_a$ such that $I \in S_{a,(m)}$.

Additionally, we define for every $I \in S_a$ with $r(I) = m$

$$\gamma(I) = \gamma_m = \frac{1}{a(1-a)^m} \int_{(1-a)^{m+1}}^{(1-a)^m} g(u)du.$$

We also set

$$b_m(I) = \sum_{\substack{S\ni J\subseteq I \\ r(J)=r(I)+m}} \mu(J)$$

for $I \in S_a$. We easily then see inductively that $b_m(I) = (1-a)^m \mu(I)$. It is also clear that for every $I \in S_a$, $I = \bigcup_{S_a \ni J \subseteq I} A_J$.

At last we define for every m the measurable subset of X, $S_m = \bigcup_{I\in S_{a,(m)}} I$.

Now, for each $m \geq 0$, we choose $\tau_a^{(m)} : S_m \smallsetminus S_{m+1} \to \mathbb{R}$ such that

$$\left[\tau_a^{(m)}\right]^* = \left(g / \left((1-a)^{m+1}, (1-a)^m\right]\right)^*.$$

This is possible, since $\mu(S_m \smallsetminus S_{m+1}) = \mu(S_m) - \mu(S_{m+1}) = b_m(X) - b_{m+1}(X) = (1-a)^m - (1-a)^{m+1} = a(1-a)^m$. It is obvious now that $S_m \smallsetminus S_{m+1} = \bigcup_{I \in S_{a,(m)}} A_I$ and that

$$\int_{S_m \smallsetminus S_{m+1}} \tau_a^{(m)} d\mu = \int_{(1-a)^{m+1}}^{(1-a)^m} g(u) du,$$

therefore

$$\frac{1}{\mu(S_m \smallsetminus S_{m+1})} \int_{S_m \smallsetminus S_{m+1}} \tau_a d\mu = \gamma_m.$$

Using Lemma 2.3 we see that there exists a rearrangement of $\tau_a / S_m \smallsetminus S_{m+1} = \tau_a^{(m)}$ called $\phi_a^{(m)}$ for which $\frac{1}{a_I} \int_{A_I} \phi_a^{(m)} = \gamma_m$ for every $I \in S_{a,(m)}$.

Now define $\phi_a : X \to \mathbb{R}^+$ by $\phi_a(x) = \phi_a^{(m)}(x)$, for $x \in S_m \smallsetminus S_{m+1}$. Of course, $\phi_a^* = g$. Let $I \in S_{a,(m)}$. Then

$$\begin{aligned}
Av_I(\phi_a) &= \frac{1}{\mu(I)} \int_I \phi_a d\mu = \frac{1}{\mu(I)} \sum_{S_a \ni J \subseteq I} \int_{A_J} \phi_a d\mu \\
&= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{\substack{S_a \ni J \subseteq I \\ r(J) = r(I) + \ell}} \int_{A_J} \phi_a d\mu = \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{S_a \ni J \subseteq I} \gamma_{m+\ell} a_J \\
&= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{S_a \ni J \subseteq I} a\mu(J) \frac{1}{a(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u) du \\
&= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \frac{1}{(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u) du \cdot \sum_{\substack{S_a \ni J \subseteq I \\ r(J) = m+\ell}} \mu(J) \\
&= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \frac{1}{(1-a)^{m+\ell}} \int_{A)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u) du \cdot b_\ell(I) \\
&= \frac{1}{(1-a)^m} \sum_{\ell \geq 0} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u) du = \frac{1}{(1-a)^m} \int_0^{(1-a)^m} g(u) du.
\end{aligned}$$

Now for $x \in S_m \smallsetminus S_{m+1}$, there exists $I \in S_{a,(m)}$ such that $x \in I$, so

$$\mathcal{M}_{\mathcal{T}}(\phi_a)(x) \geq Av_I(\phi_a) = \frac{1}{(1-a)^m} \int_0^{(1-a)^m} g(u) du =: \theta_m.$$

Since $\mu(S_m) = (1-a)^m$ for every $m \geq 0$ we easily see from the above that we have

$$[\mathcal{M}_{\mathcal{T}}(\phi_a)]^*(t) \geq \theta_m, \text{ for every } t \in \left((1-a)^{m+1}, (1-a)^m\right].$$

For any $a, \gamma \in (0,1]$, we now choose $m = m_a$ such that $(1-a)^{m+1} \leq \gamma < (1-a)^m$. So we have $\lim_{a \to 0^+} (1-a)^{m_a} = \gamma$.

Then using Lemma 2.4 we have that

$$\limsup_{a \to 0^+} \int_{\cup S_{a,(m_a)}} [\mathcal{M}_{\mathcal{T}}(\phi_a)]^p d\mu \leq \int_0^\gamma \left(\frac{1}{t} \int_0^t g\right)^p dt < +\infty, \tag{5.4}$$

where $\cup S_{a,(m_a)}$ denotes the union of the elements of $S_{a,(m_a)}$. This is $S_{m_a} = \bigcup_{I\in S_{a,(m_a)}} I$. This is true, since $\mu(S_{m_a}) \to \gamma$, as $a \to 0^+$.

Then

$$(5.5)\qquad \begin{aligned}\int_{S_{m_a}} (\mathcal{M}_{\mathcal{T}}\phi_a)^p d\mu &= \sum_{\ell\geq m_a} \int_{S_\ell\smallsetminus S_{\ell+1}} (\mathcal{M}_{\mathcal{T}}\phi_a)^p d\mu\\ &\geq \sum_{\ell\geq m_a}\Big(\frac{1}{(1-a)^\ell}\int_0^{(1-a)^\ell} g(u)du\Big)^p \mu(S_\ell\smallsetminus S_{\ell+1})\\ &= \sum_{\ell\geq m_a}\Big(\frac{1}{(1-a)^\ell}\int_0^{(1-a)^\ell} g(u)du\Big)^p \big|\big((1-a)^{\ell+1},(1-a)^\ell\big]\big|.\end{aligned}$$

Since $(1-a)^{m_a} \to \gamma$ and the right-hand side of (5.5) expresses a Riemann sum of the $\int_0^{(1-a)^{m_a}}\big(\frac{1}{t}\int_0^t g\big)^p dt$, we conclude that

$$(5.6)\qquad \limsup_{\ell\to 0^+}\int_{S_{m_a}} (\mathcal{M}_{\mathcal{T}}\phi_a)^p d\mu \geq \int_0^\gamma \Big(\frac{1}{t}\int_0^t g\Big)^p dt.$$

Then by (5.4) we have equality on (5.6).

We have thus constructed the family $(\phi_a)_{a\in(0,1)}$, for which we easily see that if $0<\gamma<\gamma'\leq 1$ then $S_a^{(\gamma)} \subseteq S_a^{(\gamma')}$ for each $a\in(0,1)$. ∎

Remark 5.2 It is not difficult to see by the proof of Lemma 5.1 that for every $\ell\in\mathbb{N}$ and $a\in(0,1)$ the following holds: $h = g/(0,(1-a)^\ell]$, where h is defined by $h := (\phi_a/S_{a,(\ell)})^*$ on $(0,(1-a)^\ell]$.

We now return to the proof of Theorem 1.3. Remember that

$$II_{1,n} = \int_{F_n}\Big(\frac{1}{t}\int_0^t g_n\Big)^p dt - c^p\int_{F_n} g_n^p = III_{1,n} - IV_{1,n}$$

with $F_n = \bigcup_{i=1}^{i_n} I_{n,i} = \bigcup_{i=n}^{i_n}(a_{n,i_1}b_{n,i})$, which is a disjoint union. Thus

$$III_{1,n} = \sum_n\Big[\int_0^{b_{n,i}}\Big(\frac{1}{t}\int_0^t g_n\Big)^p dt - \int_0^{a_{n,i}}\Big(\frac{1}{t}\int_0^t g_n\Big)^p dt\Big].$$

Now, for every $n\in\mathbb{N}$ we consider the corresponding to g_n, family $(\phi_{a,n})_{a\in(0,1)}$ and the respective subsets of X, $S_{a,n}^{(a_{n,i})}$, $S_{a,n}^{(b_{n,i})}$, $a\in(0,1)$, $i=1,2,\dots,n_i$ for which

$$\mu\big(S_{a,n}^{(a_{n,i})}\big) \longrightarrow a_{n,i} \quad\text{and}\quad \mu\big(S_{a,n}^{(b_{n,i})}\big) \longrightarrow b_{n,i},\ \text{as } a\to 0^+.$$

We can also suppose that

$$a_{n,i} < b_{n,i} \leq a_{n,i+1} < b_{n,i+1},\quad i=1,2,\dots,i_n-1.$$

Then we also have that $S_{a,n}^{(a_{n,i})} \subseteq S_{a,n}^{(b_{n,i})} \subseteq S_{a,n}^{(a_{n,i+1})}$, and of course

$$(5.7)\qquad \lim_{a\to 0^+}\int_{S_{a,n}^{(a_{n,i})}} [\mathcal{M}_{\mathcal{T}}(\phi_{a,n})]^p d\mu = \int_0^{a_{n,i}}\Big(\frac{1}{t}\int_0^t g_n\Big)^p dt,$$

and similarly for the other endpoint $b_{n,i}$ of $I_{n,i}$. Therefore, by (5.7) there exists for every $n \in \mathbb{N}$ an $a_{0,n} \in (0,1)$ such that $|III_{1,n} - V_n| < \frac{1}{n}$ for all $0 < a < a_{0,n}$, where

$$V_n = \sum_{i=1}^{i_n} \Big[\int_{S_{a,n}^{(b_{n,i})}} (\mathcal{M}_{\mathcal{T}}\phi_{a,n})^p d\mu - \int_{S_{a,n}^{(a_{n,i})}} (\mathcal{M}_{\mathcal{T}}\phi_{a,n})^p d\mu \Big]$$
$$= \int_{\Lambda_n^{(a)}} (\mathcal{M}_{\mathcal{T}}\phi_{a,n})^p d\mu, \quad \Lambda_n^{(a)} = \bigcup_{i=1}^{i_n} \big[S_{a,n}^{(b_{n,i})} \smallsetminus S_{a,n}^{(a_{n,i})} \big].$$

Additionally, we can suppose because of the relation

$$\lim_{a\to 0^+} \int_X (\mathcal{M}_{\mathcal{T}}\phi_{a,n})^p dt = \int_0^1 \Big(\frac{1}{t} \int_0^t g_n \Big)^p dt, \text{ for each } n \in \mathbb{N},$$

and since g_n is extremal for the problem (1.5), that $a_{0,n}$ can be chosen such that for every $a \in (0, a_{0,n})$

$$\Big| \int_X (\mathcal{M}_{\mathcal{T}}\phi_{a,n})^p d\mu - F\omega_p(f^p/F)^p \Big| < \frac{1}{n}, \text{ for every } n \in \mathbb{N}. \tag{5.8}$$

Choose $a'_n \in (0, a_n)$ and form the sequence $\phi_{a'_n,n} =: \phi_n$. Then, because of (5.8) and since $\phi_n^* = g_n$, we have that ϕ_n is extremal for (1.4).

Because of Remark 5.1, we now have for every $\ell \in \mathbb{N}$, each $n \in \mathbb{N}$ and $a \in (0,1)$, we have that

$$\Big(\phi_{a,n} / S_{a,(\ell)} \Big)^* : (0, \mu(S_\ell) = (1-a)^\ell] \longrightarrow \mathbb{R}^+$$

is equal to $g_n / (0, (1-a)^\ell]$. Since $\lim_{a\to 0^+} \mu(\Lambda_n^{(a)}) = |F_n|$, for every $n \in \mathbb{N}$ we can additionally suppose that $a_{0,n}$ satisfies the following

$$\big| \mu(\Lambda_n^{(a)}) - |F_n| \big| < \frac{1}{n}, \text{ for every } \quad a \in (0, a_{0,n}),$$

so if $\Lambda_n = \Lambda_n^{(a'_n)}$, since $\phi_{a'_{n,n}} = \phi_n$, we must have additionally that

$$\Big| \int_{F_n} \Big(\frac{1}{t} \int_0^t g_n \Big)^p dt - \int_{\Lambda_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu \Big| \le \frac{1}{n} \tag{5.9}$$

and $\big| \mu(\Lambda_n) - |F_n| \big| < \frac{1}{n}$, for every $n \in \mathbb{N}$.

It is also easy to see because of the above relations, Remark 5.1, and the form of Λ_n (by passing to a subsequence if necessary) that

$$\lim_n \int_{\Lambda_n} \phi_n^p = \lim_n \int_{F_n} g_n^p. \tag{5.10}$$

We now take advantage of Theorem 1.2.

Since ϕ_n is extremal for (1.4), we must have that $\int_X |\mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n|^p d\mu \to 0$, as $n \to \infty$, where $c = \omega_p(f^p/F)^p$. This implies that

$$\int_{\Lambda_n \cap \{\mathcal{M}_{\mathcal{T}}\phi_n \ge c\phi_n\}} (\mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n)^p d\mu \to 0,$$

as $n \to \infty$, or

$$\int_{\Lambda'_n} (\mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n)^p d\mu \to 0, \text{ as } n \to \infty,$$

where $\Lambda_n' = \Lambda_n \cap \{\mathcal{M}_{\mathcal{T}}\phi_n \geq c\phi_n\}$. Since

$$\Big[\int_{\Lambda_n'} (\mathcal{M}_{\mathcal{T}}\phi_n)^p\Big]^{1/p} \leq \Big[\int_{\Lambda_n'} (\mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n)^p\Big]^{1/p} + \Big[\int_{\Lambda_n'} (c\phi_n)^p\Big]^{1/p},$$

we must have, because of the definition of Λ_n' and the above inequality that

$$\lim_n \int_{\Lambda_n'} (\mathcal{M}_{\mathcal{T}}\phi_n)^p = c^p \lim_n \int_{\Lambda_n'} \phi_n^p.$$

In the same way we prove that

$$\lim_n \int_{\Lambda_n \smallsetminus \Lambda_n'} (\mathcal{M}_{\mathcal{T}}\phi_n)^p = c^p \lim_n \int_{\Lambda_n \smallsetminus \Lambda_n'} \phi_n^p,$$

so

$$\lim_n \int_{\Lambda_n} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = c^p \lim_n \int_{\Lambda_n} \phi_n^p d\mu.$$

Because of (5.9) and (5.10), we have that

$$\lim_n \int_{F_n} \Big(\frac{1}{t}\int_0^t g_n\Big)^p dt = \lim_n c^p \int_{F_n} g_n^p,$$

and from the choice of F_n we see that we must have that $II_n < 2\varepsilon$, for $n \geq n_0$, for a suitable $n_0 \in \mathbb{N}$, and this was our aim.

6 Uniqueness of Extremal Functions

In this section we will prove that there exists unique $g_0\colon (0,1] \to \mathbb{R}^+$ continuous, with

$$\int_0^1 g_0(u)du = f, \quad \int_0^1 g_0^p(u)du = F, \quad \text{and}$$
$$\int_0^1 \Big(\frac{1}{t}\int_0^t g_0(u)du\Big)^p dt = F\omega_p(f^p/F)^p.$$

This is the statement of Theorem 1.4.

Proof of Theorem 1.4 By Theorem 1.3 it is obvious that if such a function g_0 exists, it must satisfy

$$\frac{1}{t}\int_0^t g_0(u)du = cg_0(t), \text{ a.e. on } (0,1], \text{ where } c = \omega_p(f^p/F). \tag{6.1}$$

Because of the continuity of g_0 we must have equality on (6.1) in all $(0,1]$.

So in order for g_0 to satisfy (6.1), we need to set $g_0(t) = kt^{-1+\frac{1}{c}}$, $t \in (0,1]$ and search for a constant k (by solving the respective first order linear differential equation) such that

$$\int_0^1 g_0(u)du = f \quad \text{and} \quad \int_0^1 g_0^p(u)du = F.$$

The first equation becomes

$$\int_0^1 kt^{-1+\frac{1}{c}} dt = f \quad \Leftrightarrow \quad kc = f \quad \Leftrightarrow \quad k = f/c.$$

So we ask if g_0 for this k satisfies the second equation. This is equivalent to

$$\frac{k^p}{\left(-p+1+\frac{p}{c}\right)}=F,$$

or $f^p/F=\left[(-p+1)+\frac{p}{c}\right]c^p$; equivalently, $-(p-1)c^p+pc^{p-1}=f^p/F$. But this is true because of the choice of $c=\omega_p(f^p/F)$ and $\omega_p=H_p^{-1}$, where

$$H_p(z)=-(p-1)z^p+pz^{p-1}\quad\text{for } t\in\left[1,\frac{p}{p-1}\right].$$

Because of the form of $g_0\colon(0,1]\to\mathbb{R}^+$, we have that

$$\frac{1}{t}\int_0^t g_0(u)du=cg_0(t)\quad\text{for } t\in(0,1],$$

hence

$$\int_0^1\Big(\frac{1}{t}\int_0^t g_0(u)du\Big)^p du=F\omega_p(f^p/F)^p.$$

So g_0 is the only extremal function in $(0,1]$. ∎

7 Uniqueness of Extremal Sequences

We are now able to prove Theorem 1.5. The direction (ii) ⇒ (i) is obvious from the conditions that g satisfies. We now proceed to (ii) ⇒ (i).

We suppose that we are given $g_n\colon(0,1]\to\mathbb{R}^+$ non-increasing, continuous, such that $\int_0^1 g_n(u)du=f$, $\int_0^1 g_n^p(u)du=F$, and

$$\lim_n\int_0^1\Big(\frac{1}{t}\int_0^t g_n(u)du\Big)^p dt=F\omega_p(f^p/F)^p.$$

Using Theorem 1.4 we conclude that

$$\lim_n\int_0^1\Big|\frac{1}{t}\int_0^t g_n-cg_n(t)\Big|^p dt.$$

Thus, there exists a subsequence $(g_{k_n})_n$ such that if

$$F_n(t)=\frac{1}{t}\int_0^t g_n-cg_n(t),\quad t\in(0,1],\quad n\in\mathbb{N},$$

then $F_{k_n}\to 0$ almost everywhere (with respect to Lesbesgue measure). Because of the finiteness of the measure space $[0,1]$ and a well-known theorem in measure theory, we have that $F_{k_n}\to 0$ uniformly almost everywhere on $(0,1]$. This means that there exists a sequence of Lesbesgue measurable subsets of $(0,1]$, say $(H_n)_n$, such that $H_{n+1}\subseteq H_n$, $|H_n|\le\frac{1}{n}$ satisfying

$$\Big|\frac{1}{t}\int_0^t g_{k_n}-cg_{k_n}(t)\Big|=|F_{k_n}(t)|\le\frac{1}{n}\quad\text{for } t\in(0,1]\smallsetminus H_n.$$

Additionally, from the external regularity of the Lesbesgue measure, we can suppose that H_n is a disjoint union of closed intervals on $(0,1]$. Now let $t,t'\in[a,1]\smallsetminus H_{k_n}$, where a is a fixed element of $(0,1]$.

Then the following hold, where $c = \omega_p(f^p/F)$:

$$\begin{aligned}
&|cg_{k_n}(t) - cg_{k_n}(t')| \\
&\le \Big| cg_{k_n}(t) - \frac{1}{t}\int_0^t g_{k_n} \Big| + \Big| \frac{1}{t}\int_0^t g_{k_n} - \frac{1}{t'}\int_0^{t'} g_{k_n} \Big| + \Big| \frac{1}{t'}\int_0^{t'} g_{k_n} - cg_{k_n}(t') \Big| \\
&= I + II + III.
\end{aligned}$$

Then $I \le \frac{1}{k_n}$, since $t \notin H_{k_n}$. Similarly for III.

We look now at the second quantity II.

We may suppose that $t' > t$, so $t' = t + \delta$ for some $\delta > 0$. Then

$$\begin{aligned}
II &= \frac{1}{tt'}\Big| t' \int_0^t g_{k_n} - t\int_0^{t'} g_{k_n} \Big| \le \frac{1}{at}\Big| (t+\delta)\int_0^t g_{k_n} - t\int_0^t g_{k_n} - t\int_t^{t'} g_{k_n} \Big| \\
&= \frac{1}{at}\Big| \delta \int_0^t g_{k_n} - t\int_t^{t'} g_{k_n} \Big| \le \frac{\delta}{a^2} f + \frac{1}{a}\int_t^{t'} g_{k_n},
\end{aligned}$$

where $f = \int_0^1 g_{k_n}$. Now by Hölder's inequality, we have that

$$\int_t^{t'} g_{k_n} \le \Big(\int_t^{t'} g_{k_n}^p \Big)^{1/p} |t' - t|^{1-\frac{1}{p}} = F\delta^{1-\frac{1}{p}}.$$

Thus, $II \le \frac{\delta f}{a} + \frac{1}{a}\delta^{1-\frac{1}{p}} F$.

We consequently have that for a given $\varepsilon > 0$ and $a \in (0,1)$ there exists $\delta = \delta_{a,\varepsilon} > 0$ for which the following implication holds:

$$t, t' \in [a,1] \smallsetminus H_{k_n},\ \ |t - t'| < \delta \Rightarrow |g_{k_n}(t) - g_{k_n}(t')| < \varepsilon, \ \text{ for every } \ n \in \mathbb{N}. \tag{7.1}$$

Thus, $(g_{k_n})_n$ has a property of type of equicontinuity on a certain set that depends on a. We consider now an enumeration of the rationals in $(0,1]$; let $Q \cap (0,1] = \{q_1, q_2, \dots, q_{k_1}, \dots\}$.

For every $q \in Q \cap (0,1]$ we have that $(g_{k_n}(q))_n$ is a bounded sequence of real numbers, because g_{k_n} is a sequence of non-negative, non-increasing functions on $(0,1]$ satisfying $\int_0^1 g_{k_n} = f$.

By a diagonal argument we produce a subsequence that we denote again by g_{k_n} such that $g_{k_n}(q) \to \lambda_q$, $n \to \infty$ where $\lambda_q \in \mathbb{R}^+$, $q \in Q \cap (0,1]$.

Let $H = \bigcap_{n=1}^{\infty} H_{k_n}$, which is a set of Lebesgue measure zero, and suppose that $x \in (a,1) \smallsetminus H$. Then $x > a$, and there exist a $n_0 \in \mathbb{N}$ such that $x \notin H_{k_{n_0}}$, so that $x \notin H_{k_n}$, for all $n \ge n_0$. Additionally, choose a sequence $(p_k)_k$ of rationals on $(a,1) \smallsetminus H_{k_{n_0}}$ such that $p_k \to x$. This is possible, because the set $(a,1) \smallsetminus H_{k_{n_0}}$ is an open set. Thus, we have that $p_k > a$ and $p_k \notin H_{k_n}$, $n \ge n_0$, $k \in \mathbb{N}$.

Now let $k_0 \in \mathbb{N} : |p_k - x| < \delta$, for all $k \ge k_0$, where δ is the one given in (7.1).

We then have that $|g_{k_n}(x) - g_{k_m}(p_{k_0})| < \varepsilon$, for every $n \in \mathbb{N}$. Thus, for every such x and every $n, m \in \mathbb{N}$, we have that

$$\begin{aligned}
|g_{k_n}(x) - g_{k_m}(x)| &\le |g_{k_n}(x) - g_{k_n}(p_{k_0})| + |g_{k_n}(p_{k_0}) - g_{k_m}(p_{k_0})| \\
&\quad + |g_{k_m}(p_{k_0}) - g_{k_m}(x)| < 2\varepsilon + |g_{k_n}(p_{k_0}) - g_{k_m}(p_{k_0})|.
\end{aligned}$$

But $(g_{k_n}(p_{k_0}))_n$ is convergent sequence, thus Cauchy. Then $(g_{k_n}(x))_n$ is a Cauchy sequence for every $x \in (a,1) \smallsetminus H$ for every $a \in (0,1]$.

Thus, $(g_{k_n}(x))_n$ is a Cauchy sequence in all $(0,1] \smallsetminus H$.

As a consequence there exists $g_0'\colon (0,1] \to \mathbb{R}^+$ such that

$$g_{k_n} \to g_0' \text{ a.e. on } (0,1] \quad \Rightarrow \quad g_{k_n} \to g_0' \text{ uniformly a.e. on } (0,1]. \tag{7.2}$$

Since $F_{k_n}(t) \to 0$ a.e., this easily implies that

$$\frac{1}{t}\int_0^t g_0'(u)du = cg_0'(t), \text{ a.e. on } (0,1]. \tag{7.3}$$

Since (7.2) holds, we easily see that we have that $g_0' \in L^p((0,1])$ and that $\int_0^1 g_0' = f$ and $\int_0^1 (g_0')^p = F$.

Also, since the function $t \mapsto \int_0^t g_0'$ is continuous on $(0,1]$ we must have that g_0' can be considered continuous with equality on (7.3), everywhere on $(0,1]$.

This gives us that g_0' is the function constructed in Theorem 1.4.

Additionally, we obtain

$$\int_0^1 |g_{k_n}' - g_0'|^p dt \to 0, \quad \text{as } u \to \infty, \tag{7.4}$$

because of (7.2) and the fact that

$$\lim_{\delta \to 0^+} \Big(\sup \Big\{ \int_A g_{k_n}^p(u)du : \ n \in \mathbb{N},\ A \subseteq (0,1] \text{ with } |A| = \delta \Big\} \Big) = 0. \tag{7.5}$$

The validity of (7.5) can be concluded from the following remark.

Remark 7.1 In [2] it can be seen that the following is true:

$$\sup \Big\{ \int_K (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \phi \geq 0,\ \int_X \phi d\mu = f,\ \int_X \phi^p d\mu = F,\ \mu(k) = k \Big\} \to 0$$

as $k \to 0$, for $0 < f^p \leq F$. This is a result that can be seen in [4] and depends only on the statement of Theorem 3.2.

Because of the symmetrization principle (Theorem 1.1), this implies that if we define

$$B_p(f,F,k) = \sup \Big\{ \int_0^k \Big(\frac{1}{t}\int_0^t g \Big)^p dt : g\colon (0,1] \to \mathbb{R}^+ \\ \text{is non-increasing and continuous}, \int_0^1 g = f,\ \int_0^1 g = F \Big\},$$

then $\lim_{k \to 0} B_p(f,F,k) = 0$. Then the supremum in (7.5) is bounded above by $B_p(f,F,\delta)$ for every $\delta \in (0,1]$. Thus, if we work on every subsequence of $(g_n)_n$, which is again extremal, we produce a subsequence of it for which (7.4) is satisfied. Therefore, the proof of Theorem 1.5 is complete. ∎

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