Proceedings of the Edinburgh Mathematical Society (1990) 33, 241–250 ©

AN UPPER BOUND FOR λ_1 FOR $\Gamma(q)$ AND $\Gamma_0(q)$

by C. J. MOZZOCHI

(Received 21st December 1988)

Under the assumption of the Selberg conjecture I establish by means of the Selberg trace formula the following:

Theorem. *Let Γ denote $\Gamma(q)$ or $\Gamma_0(q)$, q square-free. Let Δ_q denote the Laplace operator on $L^2(\Gamma\backslash H)$, and let Σ_q denote its discrete spectrum. Then there exists an absolute positive constant A such that for $q \geqq A$*

$$\Sigma_q \cap \left[\frac{1}{4}, \frac{1}{4} + \frac{475}{(\ln q)^2}\right] \neq \emptyset.$$

1980 *Mathematics subject classification* (1985 Revision): 11F72.

1. Introduction

In this paper we concentrate on obtaining an upper bound for λ_1, the smallest positive eigenvalue of the noneuclidean Laplacian for $\Gamma(q)$ or $\Gamma_0(q)$ for q square-free.

The Riemannian geometry of the Laplace operator is concerned with (among other problems) the closed eigenvalue problem, the Neumann eigenvalue problem, the Dirichlet eigenvalue problem, and the mixed eigenvalue problem. These problems are discussed in depth in [3]. For a report on Cheeger's lower bound for the first eigenvalue λ_1 of the Laplacian on a compact Riemannian manifold, see [1]. For a discussion of lowest-eigenvalue inequalities see [2], and for a general discussion of the Laplace operator on compact Riemann surfaces, see [6]. Interesting insights on estimates for λ_1 of the Laplace operator of a compact, oriented, connected n-manifold isometrically immersed in R^{n+p} are discussed in [15], and for geometric bounds on the low eigenvalues of a compact surface, see [17].

The central consideration for the groups that we explore is the Selberg conjecture [18] that states that for any congruence group we have $\lambda_1 \geqq \frac{1}{4}$ i.e. λ_1 lies on the continuous spectrum. The conjecture has been proved for groups of small level by H. Maass, W. Roelcke, and M.-F. Vignéras, with the record for $\Gamma_0(q)$ being $q \leqq 17$ due to M. N. Huxley (see [8] and [9]). The first step toward a proof of this conjecture is due to Selberg himself [18], who was able to show unconditionally that $\lambda_1 \geqq \frac{3}{16}$.

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

Recently, H. Iwaniec [10] has made remarkable progress by showing that for almost all p there is no spectrum below 44/225 of the Laplace–Beltrami operator acting on automorphic functions with respect to the group $\Gamma_0(p)$ and with a multiplier system given by the quadratic character (mod p).

B. Randol [14] constructed groups for which the first positive eigenvalue is as small as one desires (this was known to Selberg), and J.-M. Deshouillers and H. Iwaniec [unpublished] have established that for $\Gamma(q)$ and $\Gamma_0(q)$, q square-free, as q tends to infinity, one may find eigenvalues arbitrarily close to 1/4.

Specifically, in this paper we consider $\Gamma=\Gamma(q)$ and $\Gamma_0(q)$ where q is square-free.

Let

$$\lambda_0=0<\lambda_1\leqq\lambda_2\leqq\cdots$$

be the eigenvalues of the hyperbolic Laplacian associated with Γ. We set

$$\lambda_j=s_j(1-s_j) \quad \text{and} \quad s_j=\tfrac{1}{2}+it_j; \quad \text{so} \quad \lambda_j=\tfrac{1}{4}+t_j^2.$$

Therefore, the t_j's must be purely real or purely imaginary in order for the λ_j's to be real and positive. Clearly, if the Selberg conjecture is false, $\lambda_1<\frac{1}{4}$. Hence it is assumed that the Selberg conjecture is true; so the t_j's must be real.

We establish the following:

Theorem. *Let Γ denote $\Gamma(q)$ or $\Gamma_0(q)$, q square-free. Let Δ_q denote the Laplace operator on $L^2(\Gamma\backslash H)$, and let Σ_q denote its discrete spectrum. Then there exists an absolute positive constant A such that for $q\geqq A$*

$$\Sigma_q\cap\left[\frac{1}{4},\frac{1}{4}+\frac{475}{(\ln q)^2}\right]\neq\emptyset.$$

Clearly, since an eigenvalue for $\Gamma_0(q)$ is also one for $\Gamma(q)$, it is sufficient to establish the theorem for $\Gamma=\Gamma_0(q)$.

2. An application of the trace formula

The proof is based on the Selberg trace formula for $\Gamma=\Gamma_0(q)$.

$$h\left(\frac{i}{2}\right)+\sum_{j\geqq 1}h(t_j)=C+E+H+P=F(q),$$

say where C is the contribution of the identity and E, H, and P respectively stand for the contribution of the conjugacy classes of the elliptic, hyperbolic, and parabolic elements of Γ.

We apply the trace formula with the test function defined

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

$$h(t)=\frac{\cos^2(kLt)\sin^6(cLt)}{(1-(2kL/\pi)^2t^2)L^4t^4},$$

where $L=L(q)$ is real-valued and goes to infinity with q, and the parameters c,k will be chosen later in Section 3 subject to $0<c\leqq k$.

Observe that $h(t)$ satisfies the conditions for the trace formula, in particular h is even. Further, we have

$$h(t)>0 \quad \text{for} \quad 0<t<\left(\frac{\pi}{2kL}\right)$$

and

$$h(t)\leqq 0 \quad \text{if} \quad t\geqq\frac{\pi}{2kL} \quad \text{or } t \text{ is purely imaginary.}$$

Since $h(i/2)<0$, if one could show $F(q)>0$ for $q\geqq A$, then $t_1^2\leqq(\pi/2kL)^2$.

3. A lower bound for C

$$C=\frac{\operatorname{vol}(\Gamma/H)}{4\pi}\int_{-\infty}^{\infty} t\tanh(\pi t)h(t)\,dt.$$

Choose $q\geqq q_1$ so that $(1/L(q))\leqq\frac{1}{3}$. Then, using the inequality

$$x-\tfrac{1}{3}x^3\leqq\tanh x\leqq x$$

it is easy to see that

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} t\tanh(\pi t)h(t)\,dt\geqq\tfrac{1}{2}\int_0^{\infty} t^2h(t)\,dt-\frac{\pi^2}{6}\int_0^{\pi/2kL} t^4h(t)\,dt.$$

By a simple change of variable we have

$$\int_0^{\infty} t^2h(t)\,dt=L^{-3}I(c,k)$$

where

$$I(c,k)=\int_0^{\infty}\frac{\cos^2(kx)(\sin cx)^6}{(1-(2k/\pi)^2x^2)x^2}\,dx.$$

It is easy to see that

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

$$I(c,k)=cI_1(\theta)+\frac{2k}{\pi}I_2(\theta)$$

where

$$I_1(\theta)=\int_0^\infty \frac{\cos^2(t/\theta)(\sin t)^6}{t^2}\,dt,$$

$$I_2(\theta)=\int_0^\infty \frac{\cos^2((\pi/2)t)(\sin(\pi/2)t\theta)^6}{1-t^2}\,dt,$$

and

$$\theta=\frac{c}{k}.$$

Clearly, $I_1(\theta)>0$ for all $\theta>0$, and it is very easy to see that $I_2(\theta)=0$ if $\theta=1$; so that if we choose $c=k$, then $I(c,k)>0$. However, for reasons which will become clear later we also want to choose c and k so that θ is as small as possible.

Consider

$$F(\theta)=\frac{(2/\pi)|I_2(\theta)|}{I_1(\theta)}.$$

We want to choose c and k so that θ is as small as possible and

$$\theta>F(\theta).$$

The following two lemmas, together with the proofs presented here, were kindly communicated to me by Richard Bumby.

Lemma 3.1.

(1) *For* $0<\theta\leqq\frac{1}{3}$ *we have* $I_1(\theta)=\dfrac{3\pi}{32}$.

(2) *For* $\frac{1}{3}\leqq\theta\leqq\frac{1}{2}$ *we have* $I_1(\theta)=\dfrac{\pi}{64}\left(9-\dfrac{1}{\theta}\right)$.

(3) *For* $\frac{1}{2}\leqq\theta\leqq 1$ *we have* $I_1(\theta)=\dfrac{\pi}{128}\left(\dfrac{10}{\theta}-6\right)$.

(4) *For* $1\leqq\theta$ *we have* $I_1(\theta)=\dfrac{\pi}{32}\left(6-\dfrac{5}{\theta}\right)$.

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

Proof. This follows immediately from the identities $\cos^2 x=\frac{1}{2}(1+\cos 2x)$; $\sin^6 x=\frac{1}{32}(10-15\cos 2x+6\cos 4x-\cos 6x)$; $\cos x\cos y=\frac{1}{2}(\cos(x+y)+\cos(x-y))$; and $\cos 2x=(1-2\sin^2 x)$, and (9, page 446 in [4]).

Lemma 3.2.

(1) *For* $0<\theta\leqq\frac{1}{3}$ *we have*

$$I_2(\theta)=-\frac{\pi}{128}(15\sin(\pi\theta)-6\sin(2\pi\theta)+\sin(3\pi\theta)).$$

(2) *For* $\frac{1}{3}\leqq\theta\leqq\frac{1}{2}$ *we have*

$$I_2(\theta)=-\frac{\pi}{128}(15\sin(\pi\theta)-6\sin(2\pi\theta)).$$

(3) *For* $\frac{1}{2}\leqq\theta\leqq 1$ *we have*

$$I_2(\theta)=-\frac{\pi}{128}(15\sin(\pi\theta)).$$

(4) *For* $1\leqq\theta$ *we have*

$$I_2(\theta)=0.$$

The result follows by straightforward calculation from the identities:

$$\sin^6 x-\sin^6 y=\frac{1}{32}(-15(\cos 2x-\cos 2y)+6(\cos 4x-\cos 4y)-(\cos 6x-\cos 6y)),$$

$$\cos x-\cos y=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right);\quad\text{and}$$

$$\sin^2 x\sin y=\frac{1}{4}(2\sin y-\sin(2x+y)-\sin(y-2x)),$$

and (3.721(1), page 405 in [4]).

For

$$\theta\geqq 0.821897500$$

we have

$$\theta>F(\theta).$$

In particular

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

$$0.822 > F(0.822) = 0.821677012.$$

Choose $c_1 > 0$, $k_1 > 0$ so that $(c_1/k_1) = \theta = 0.822$. Hence

$$\tfrac{1}{2}\int_0^\infty t^2 h(t)\,dt = \tfrac{1}{2}L^{-3} I(c_1, k_1) = C_1 L^{-3} > 0.$$

Clearly,

$$|h(t)| \leqq C_2;$$

so that

$$\frac{\pi^2}{6}\int_0^{\pi/2k_1L} t^4 h(t)\,dt \leqq C_3 L^{-5};$$

so that

$$C \geqq C_4 \operatorname{vol}(\Gamma/H) L^{-3} \geqq C_5 q L^{-3}.$$

4. The parabolic contribution

Lemma 4.1. *For $\Gamma = \Gamma_0(q)$, q square-free we have*

$$P \ll q^{1/2} L.$$

Proof. By (4.6), page 538 in [5], we have

$$P = \left(P_{SL(2,z)} - \sum_{p\mid q}\sum_{k=0}^{\infty} \frac{\log p}{p^k} g(2k\log p)\right) d(q)$$

where

$$g(\theta) = 2\pi\ [\text{Fourier transform of } h \text{ evaluated at } (-\theta)];$$

so that

$$P \ll L\left(1 + \sum_{p\mid q} \frac{\log p}{p}\right) d(q) \ll q^{1/2} L.$$

5. The elliptic contribution

Lemma 5.1. *For $\Gamma = \Gamma_0(q)$ we have $E \ll q^{1/2} L$.*

Proof. We have

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

$$E=\sum_{\{R\}}\sum_{k=1}^{m-1}\frac{1}{2m\sin(\pi k/m)}\int_{-\infty}^{\infty}\frac{e^{-2\pi k/m}}{1+e^{-2\pi r}}h(r)\,dr,$$

where $m=\operatorname{ord}\{R\}=2$ or 3. The number of elliptic classes is bounded by $2\prod_{p|q}(1+1/p)\ll q^{1/2}$ (cf. [**19**]), and the result follows.

6. The hyperbolic contribution

Let $\mathcal{M}(q,n)$ denote the multiplicity of distinct primitive hyperbolic classes in $\Gamma(q)$ or $\Gamma_0(q)$ with trace $n\geqq 3$, and set

$$A(n)=\left(\frac{n+(n^2-4)^{1/2}}{2}\right).$$

Denote by $\mathcal{D}$ the set of positive ring discriminants, that is, $\{d>0\,|\,d\equiv 0,1\,(\mathrm{mod}\,4),\ d$ not a square$\}$. To each such d let $h(d)$ denote the number of inequivalent primitive binary quadratic forms of discriminant d, and let (x_d,y_d) be the fundamental solution of the Pellian equation

$$x^2-dy^2=4.$$

Let

$$\varepsilon_d=\frac{x_d+\sqrt{d}\,y_d}{2}.$$

Lemma 6.1. *The norms, $A(n)$, of the conjugacy classes with trace n of primitive hyperbolic transformations of $\Gamma_0(1)$ are ε_d^2 where $d\in\mathcal{D}$, with multiplicity $h(d)$. Or put another way, the lengths of the closed geodesics on H/Γ are the numbers $2\log\varepsilon_d$ with multiplicity $h(d)$.*

Proof. See [**16**] for a recent proof.

Therefore, $\mathcal{M}(1,n)$ can be viewed either as a class number or as the number of closed geodesics with a given length. See Section 2 in [**11**] for a discussion of this fact, giving more insight.

The following lemma is established in [7].

Lemma 6.2. *For $\Gamma=\Gamma_0(q)$ we have $\mathcal{M}(q,n)\ll q^{\varepsilon}n^2(\ln n)^8$.*

We now compute the contribution of the hyperbolic classes for $\Gamma_0(q)$.

Clearly, we have $n\leqq A(n)\leqq n^2$, and by Lemma 6.2 we obtain

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

$$H \ll q^{\varepsilon} \lim_{x \to \infty} \sum_{n=3}^{x} n^2 (\ln n)^8 \sum_{k=1}^{\infty} \frac{|g(k \ln A(n))| \ln A(n)}{A(n)^{k/2} - A(n)^{-k/2}}.$$

We will show

$$|g(k \ln A(n))| \leqq B(n, q)$$

with $B(n, q)$ to be specified later. From this we deduce that

$$H \ll q^{\varepsilon} \lim_{x \to \infty} \sum_{n=3}^{x} n^2 (\ln n)^9 B(n, q) \sum_{k=1}^{\infty} \frac{1}{A(n)^{k/2} - A(n)^{-k/2}}.$$

Since $3 \leqq A(n)$, it follows that

$$H \ll q^{\varepsilon} \lim_{x \to \infty} \sum_{n=3}^{x} n^2 (\ln n)^9 B(n, p) \sum_{k=1}^{\infty} \frac{1}{A(n)^{k/2}}.$$

But

$$\sum_{k=1}^{\infty} \left(\frac{1}{A(n)^{1/2}} \right)^k = \sum_{k=1}^{\infty} \left\{ \frac{2}{(n + (n^2 - 4)^{1/2}} \right\}^k \leqq \sum_{k=1}^{\infty} \left(\frac{2}{n} \right)^k = \frac{2}{n-2}.$$

Hence

$$H \ll q^{\varepsilon} \lim_{x \to \infty} \sum_{n=3}^{x} n (\ln n)^9 B(n, q).$$

It is immediate by Section 28 in [**20**] that $g(\alpha) = 0$ for each α such that

$$|zh(z)\, e^{i\alpha z}| \to 0 \quad \text{as} \quad z \to \infty \quad \text{and} \quad \operatorname{Im} z \geqq 0.$$

Using the estimates

$$|\cos z| \leqq \cosh y \leqq e^y \quad \text{for} \quad y \geqq 0$$

$$|\sin z| \leqq \cosh y \leqq e^y \quad \text{for} \quad y \geqq 0,$$

we obtain

$$|zh(z)\, e^{i\alpha z}| \leqq \frac{e^{[2k_1 L + 6c_1 L - \alpha] y}}{L^4 |1 - (2k_1 L/\pi) z^2| |z|^3},$$

so that

$$g(\alpha) = 0 \quad \text{if} \quad (2k_1 L + 6c_1 L) < \alpha.$$

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

Since $n \geqq 3$, $\theta = k \ln A(n) \geqq k \ln n \geqq 1$; so $g(\theta) = 0$, if $\ln n > (2k_1 + 6c_1)L$. Hence $g(\theta) = 0$, if $n > e^{(2k_1+6c_1)L}$. Consequently, $|g(\theta)| \leqq C_6 L^{-1}$, if $1 \leqq n \leqq e^{(2k_1+6c_1)L}$. Hence

$$B(n, q) = \begin{cases} 0 & \text{if} \quad n > e^{(2k_1+6c_1)L} \\ C_6 L^{-1} & \text{if} \quad 1 \leqq n \leqq e^{(2k_1+6c_1)L}. \end{cases}$$

Hence, for $\Gamma = \Gamma_0(q)$ it is easy to see

$$H \ll q^{\varepsilon} L^8 e^{2(2k_1+6c_1)L}.$$

Let

$$L = L(q) = \frac{\ln q}{(2+\varepsilon)(2k_1 + 6c_1)B} \quad \text{for} \quad B > 1.$$

Then we have

$$\lambda_1 \leqq \tfrac{1}{4} + \left[\frac{B\pi(2 + 6(c_1/k_1)(1 + \varepsilon/2))}{\ln q}\right]^2$$

for $q \geqq A$, and the proof is completed.

Acknowledgements. I would like to thank Professor Henryk Iwaniec for several helpful conversations concerning this problem and for communicating to me the proof of Lemma 5.1. Further, I would like to express my appreciation to Professor Martin Huxley for providing me with a copy of [7]. The two integral evaluations by Professor Richard Bumby in Section 3 are most appreciated, as are the helpful suggestions by the referee concerning exposition.

This paper is based in part on work done during the summer of 1988 at the Institute for Advanced Study, and I would like to thank the Institute for providing me with excellent working conditions.

REFERENCES

1. M. Buser, On Cheeger's inequality $\lambda_1 \geqq h^2/4$, *Proc. Sympos. Pure Math.* **36** (1980), 29–77.

2. I. Chavel, Lowest-eigenvalue inequalities, *Proc. Sympos. Pure Math.* **36,** (1980), 79–89.

3. I. Chavel, *Eigenvalues in Riemannian Geometry* (Academic Press, Inc., New York, 1984).

4. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals Series and Products* (Academic Press, New York, 1980).

5. D. Hejhal, *The Selberg Trace Formula for PSL(2, R)* II (Lecture Notes in Math. **1001,** Springer-Verlag, Berlin, New York, 1983).

6. H. Huber, On the spectrum of the Laplace operator on compact Riemann surfaces, *Proc. Sympos. Pure Math.* **36** (1980), 181–184.

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press

7. M. N. Huxley, Conjugacy classes in congruence subgroups, preprint.

8. M. N. Huxley, Introduction to Kloostermania, in *Elementary and Analytical Theory of Numbers* (Banach Center Publ. **17**, PWN, Warsaw, 1985), 217–306.

9. M. N. Huxley, Exceptional eigenvalues and congruence subgroups, *Contemp. Math.* **53** (1986), 341–349.

10. H. Iwaniec, Small eigenvalues for congruence groups, preprint.

11. H. Iwaniec, Prime geodesic theorem, *J. Reine Angew. Math.* **349** (1984), 136–159.

12. H. Iwaniec, Character sums and small eigenvalues for $\Gamma_0(p)$, *Glasgow Math. J.* **27** (1985), 99–116.

13. H. Iwaniec and J. Szmidt, Density theorems for exceptional eigenvalues of Laplacian for congruence groups, *Banach Center Publ.* **17** (1984), 317–331.

14. B. Randol, Small eigenvalues of the Laplace operator on compact Riemann surfaces, *Bull. Amer. Math. Soc.* **80** (1974), 996–1000.

15. R. C. Reilly, Extrinsic estimates for λ_1, *Proc. Sympos. Pure Math* **36** (1980), 275–278.

16. P. Sarnak, Class numbers of indefinite binary quadratic forms, *J. Number Theory* **15** (1982), 229–247.

17. R. Schoen, S. Wolpert and S. T. Yau, Geometric bounds on the low eigenvalues of a compact surface, *Proc. Sympos. Pure Math.* **36** (1980), 279–285.

18. A. Selberg, On the estimation of Fourier coefficients of modular forms, *Proc. Sympos. Pure Math.* **8** (1965), 1–15.

19. G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions* (Princeton University Press, 1971).

20. I. N. Sneddon, *The Use of Integral Transforms* (McGraw-Hill, New York, 1972).

P.O. Box 1424
Princeton, NJ 08542
USA

https://doi.org/10.1017/S0013091500018162 Published online by Cambridge University Press