On Professor Whittaker's solution of differential equations by definite integrals: Part II

Applications of the methods of non-commutative Algebra

By W. O. KERMACK (Royal College of Physicians, Edinburgh), and W. H. McCREA, (Edinburgh University).

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§ 1. *Introduction.* In Part I it has been shown that, given a contact transformation, two equations

$$Q(q, p) = Q \tag{1.1}$$

$$P(q, p) = P \tag{1.2}$$

can be derived which lead to the compatible differential equations

$$Q(q, -\partial/\partial q)\, \chi = Q\chi, \tag{1.3}$$

$$P(q, -\partial/\partial q)\, \chi = \partial\chi/\partial Q. \tag{1.4}$$

It will be shown in the present communication that the necessary and sufficient condition that (1.3), (1.4) should be compatible is that

$$Q(q, p)\, P(q, p) - P(q, p)\, Q(q, p) = 1, \tag{1.5}$$

regarded as an equation in the non-commutative variables q, p which themselves satisfy the condition

$$qp - pq = 1. \tag{1.6}$$

We shall call functions satisfying this condition *conjugate functions.* From this point of view the method employed by Professor Whittaker in his original paper, involving the use of a contact transformation, was really a particular method of generating conjugate functions. This powerful method may be supplemented and extended by the other methods developed in the following pages.

In working out this theory it has been found necessary to develop somewhat the algebra of non-commutative variables obeying the law (1.6). Section A contains certain results, including an extension of Taylor's Theorem to this algebra, which appear to possess an interest of their own.

The next Section deals with the general theory of conjugate functions and canonical transformations and the relation of this theory to the solution of differential equations by the methods of Professor Whittaker.

In the last Section some results are given relating to infinitesimal canonical transformations, and it is shown how this theory may be used to obtain further identities in non-commutative algebra.

It may be added that although for the sake of simplicity we have considered only the case of one pair of conjugate variables, practically all the results may without difficulty be generalised to the case of n such pairs, provided that the variables belonging to different pairs commute.

A. *Non-commutative algebra.*

§ 2. We shall give a few theorems in the algebra of a pair of variables q, p which obey all the laws of ordinary algebra except the commutative law, in place of which we have

$$qp - pq = 1. \tag{2.1}$$

Clearly, if we have any identity holding between functions of q, p, it remains an identity when both sides are pre-multiplied or both post-multiplied by the same function of q, p.

For partial differentiation with respect to q, p we shall use the symbols $\delta/\delta q$, $\delta/\delta p$. This will avoid confusion in subsequent applications to contact transformations. These operations are always to be carried out without transposing any non-numerical factors, *e.g.*,

$$\delta pq^2/\delta q = p\delta q^2/\delta q = 2pq,$$

$$\delta pqp/\delta p = qp + pq.$$

When this is done it is clear that the ordinary rule for differentiating a product remains valid.

All mixed derivatives are independent of the order of differentiation, provided the necessary differentiability conditions are satisfied. It would, however, be difficult to formulate conditions of convergence or continuity or differentiability in the present variables. In this paper we leave aside such considerations and assume that the functions with which we deal are such that we may legitimately perform the required operations upon them.

§ 3. *Theorem I. If $K(q, p)$ be any function of q, p then*

$$K(q, p)\,p - pK(q, p) = \delta K(q, p)/\delta q, \tag{3.1}$$

$$qK(q, p) - K(q, p)\,q = \delta K(q, p)/\delta p. \tag{3.2}$$

This result has been given by Dirac.[1]

§ 4. *Theorem II. Any function $K(q, p)$ possesses a unique derivative with respect to q or p.*

For let $L(q, p)$ be some other way of writing $K(q, p)$, so that

$$K(q, p) = L(q, p),$$

giving

$$K(q, p)\,p - p\,K(q, p) = L(q, p)\,p - p\,L(q, p).$$

Hence from (3.1)

$$\delta K(q, p)/\delta q = \delta L(q, p)/\delta q,$$

and similarly

$$\delta K(q, p)/\delta p = \delta L(q, p)/\delta p.$$

This shows that the permutation of the variables according to the law (2.1) and differentiation are processes whose order may be interchanged. It is further to be noted that the fundamental rule (2.1) is consistent with the definition of differentiation.

§ 5. We define *integration* as the process inverse to differentiation.

Theorem III. There exists a unique integral of any function $K(q, p)$ with respect to q, apart from an arbitrary function of p, and a unique integral with respect to p, apart from an arbitrary function of q.

For if there exist two functions $L(q, p)$, $M(q, p)$ such that

$$\delta L(q, p)/\delta q = K(q, p), \quad \delta M(q, p)/\delta q = K(q, p)$$

and

$$L(q, p) - M(q, p) = N(q, p) \qquad \text{(say)},$$

then

$$\delta N(q, p)/\delta q = 0;$$

or, from (3.1)

$$N(q, p)\;p - p\;N(q, p) = 0.$$

Therefore $N(q, p)$ commutes with p, and so is a function[2] of p. Similarly for the second part of the Theorem.

[1] Dirac, *Principles of Quantum Mechanics* (1930), 34. The present algebra differs from Dirac's only in taking $qp - pq = 1$ instead of $qp - pq = i$. In some previous work Dirac used the first relation, *cf. Proc. Camb. Phil. Soc.*, 23 (1936), 412.

[2] Dirac, *Op. cit.*, 41.

Theorem IV. If a function $L(q, p)$ *is such that*

$$\delta L/\delta q = \delta L/\delta p = M(q, p) \tag{4.1}$$

then $L(q, p)$ *is a function of* $(q + p)$ since it follows from (4.1) that $L(q, p)$ commutes with $(q + p)$. Consequently $M(q, p)$ is a function of $(q + p)$, *and hence we have*

$$\int M(q + p)\,\delta q = \int M(q + p)\,\delta p = \int M(a)\,da \tag{4.2}$$

the last integral being the ordinary integral with respect to $a\ (\equiv q + p)$.

§ 6. *Theorem V. The derivatives of* $(p + q)^n$, *where* n *is a positive integer, are given by*

$$\delta(p + q)^n/\delta q = n(p + q)^{n-1}, \tag{6.1}$$

$$\delta(p + q)^n/\delta p = n(p + q)^{n-1}. \tag{6.2}$$

If the theorem is true for n we have

$$\begin{aligned}\delta(p + q)^{n+1}/\delta q &= \delta(p + q)(p + q)^n/\delta q\\ &= (p + q)^n + (p + q)\,n\,(p + q)^{n-1}\\ &= (n + 1)(p + q)^n,\end{aligned}$$

from which the Theorem follows by induction, since it is certainly true for $n = 1$.

It is evident that $(p + q)^n$ cannot be expanded by the ordinary binomial theorem, since the variables do not commute. But let us denote the formal binomial expansion

$$p^n + \binom{n}{1} p^{n-1} q + \binom{n}{2} p^{n-2} q^2 + \dots + \binom{n}{n-1} p q^{n-1} + q^n \tag{6.3}$$

by

$$(p_1 + q)^n.$$

Thus $(p_1 + q)^n$ is to be expanded formally by the ordinary binomial theorem, keeping p_1 factors to the left in each term, and p_1 finally replaced by p. We may now prove the following theorem, which may be looked upon as the first step in generalising the binomial theorem to the present algebra.

Theorem VI. The expansion of $(p + q)^n$ *is given by*

$$\begin{aligned}\frac{(p + q)^n}{n!} = \frac{(p_1 + q)^n}{n!} + \frac{1}{2}\frac{(p_1 + q)^{n-2}}{(n-2)!} + \left(\frac{1}{2}\right)^2 \frac{1}{2!}\frac{(p_1 + q)^{n-4}}{(n-4)!}\\ + \left(\frac{1}{2}\right)^3 \frac{1}{3!}\frac{(p_1 + q)^{n-6}}{(n-6)!} + \dots\end{aligned} \tag{6.4}$$

where the series stops with the last exponent which is positive or zero.

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From the definition (6.3) it follows that

$$\int (p_1 + q)^n \, \delta q = \frac{(p_1 + q)^{n+1}}{n+1} + f(p), \quad \int (p_1 + q)^n \, \delta p = \frac{(p_1 + q)^{n+1}}{n+1} + g(q) \quad (6.5)$$

where $f(p)$, $g(q)$ are arbitrary functions.

Assume the Theorem holds for n, and integrate both sides of (6.4) with respect to $(q + p)$. Using Theorem IV and (6.5) we find for $n = 2m$,

$$\frac{(p+q)^{2m+1}}{(2m+1)!} = \frac{(p_1+q)^{2m+1}}{(2m+1)!} + \frac{1}{2}\,\frac{(p_1+q)^{2m-1}}{(2m-1)!} + \ldots + \left(\frac{1}{2}\right)^m \frac{1}{m!}(p_1 + q) + \text{const.} \quad (6.6)$$

But we may also get $(p + q)^{n+1}/(n + 1)!$ by multiplying both sides of (6.4) by $(p + q)/(n + 1)$, and this shows that the constant in (6.6) is zero.

Similarly if $n = 2m + 1$, we find

$$\frac{(p+q)^{2m+2}}{(2m+2)!} = \frac{(p_1+q)^{2m+2}}{(2m+2)!} + \frac{1}{2}\,\frac{(p_1+q)^{2m}}{(2m)!} + \ldots + \left(\frac{1}{2}\right)^m \frac{1}{m!}\,\frac{(p_1+q)^2}{2!} + \text{const.} \quad (6.7)$$

Again multiplying both sides of (6.4) by $(p + q)/(n + 1)$, it is clear that the constant in (6.7) comes from the difference between $(p + q)(p_1 + q)$ and $(p_1 + q)^2$, and so it is just $(\frac{1}{2})^{m+1}/(m + 1)!$.

Hence in either case the Theorem is true for $(n + 1)$ if it is true for n. It is obviously true for $n = 1, 2$. Hence it is true in general.

§ 7. *The exponential function.* We define this function by the exponential series. This presents no novelty for commuting variables but demands special treatment for non-commuting variables. The first case we consider is e^{p+q} and we prove :

Theorem VII. The function e^{p+q} is given by

$$e^{p+q} = e^p e^q e^{\frac{1}{2}}. \quad (7.1)$$

For by definition,

$$\begin{aligned} e^{p+q} &= 1 + (p+q) + \frac{(p+q)^2}{2!} + \frac{(p+q)^3}{3!} + \ldots \\ &= 1 + (p_1+q) + \frac{(p_1+q)^2}{2!} + \frac{(p_1+q)^3}{3!} + \frac{(p_1+q)^4}{4!} + \ldots \\ &\quad + \frac{1}{2} \cdot 1 + \frac{1}{2}(p_1+q) + \frac{1}{2}\,\frac{(p_1+q)^2}{2!} + \ldots \\ &\quad + \frac{1}{2^2}\,\frac{1}{2!} + \ldots \\ &\quad + \ldots \end{aligned} \quad (7.2)$$

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using Theorem VI. and putting the successive terms of $(p+q)^n/n!$ in (7.1) into successive lines on the right hand side of (7.2). From this last result

$$e^{p+q} = e^{1/2}\, e^{p_1+q}$$
$$= e^{1/2}\, e^p\, e^q$$

since in every term of e^{p_1+q} the p's are written first. This is the Theorem.

Suppose now we write $p \equiv -d/dq$, which is consistent with (2.1), and operate on any function $f(q)$. We find

$$e^{-\frac{d}{dq}+q} f(q) = e^{1/2}\, e^p\, e^q f(q)$$
$$= e^{\frac{1}{2}}\, e^{q-1} f(q-1)$$
$$= e^{q-\frac{1}{2}} f(q-1). \tag{7.3}$$

We have similarly:

Theorem VIII. $$e^{p+q} = e^{-\frac{1}{2}}\, e^q\, e^p. \tag{7.4}$$

Comparing (7.1) and (7.4), we have

$$e^{q-1} = e^p\, e^q\, e^{-p},$$

which is a special case of the general theorem

$$f(q-1) = e^p f(q)\, e^{-p}, \tag{7.5}$$

where $f(q)$ is any function of q, and is merely the symbolic expression of Taylor's Theorem for that function.

A simple but important rule that may be inserted here is: *In any identity in q, p we may interchange the variables if we simultaneously change the sign of one of them.*

So, for example, (7.5) gives

$$f(p-1) = e^{-q} f(p)\, e^q. \tag{7.6}$$

§ 8. We may now write Theorem VI. in a different form. For we have

$$(p_1+q)^n = q^n + npq^{n-1} + \frac{n(n-1)}{2!} p^2 q^{n-2} + \ldots .$$
$$= q^n + p\frac{\delta}{\delta q} q^n + \frac{1}{2!} p^2 \left(\frac{\delta}{\delta q}\right)^2 q^n + \ldots .$$
$$= e^{p_1 \delta/\delta q}\, q^n, \tag{8.01}$$

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where $e^{p_1\delta/\delta q}$ stands for the formal exponential series with p kept to the left in each term. But we also have

$$\left(\frac{1}{2}\right) n\,(n-1)\,(p_1+q)^{n-2} = \left(\frac{1}{2}\frac{\delta^2}{\delta q^2}\right)(p_1+q)^n$$

$$\left(\frac{1}{2}\right)^2 \frac{1}{2\,!}\, n\,(n-1)\,(n-2)\,(n-3)\,(p_1+q)^{n-4} = \frac{1}{2\,!}\left(\frac{1}{2}\frac{\delta^2}{\delta q^2}\right)^2 (p_1+q)^n,$$

and so on.

Hence, together with (8.01), these results allow us to write Theorem VI. as

$$(q+p)^n = \left\{1 + \tfrac{1}{2}\frac{\delta^2}{\delta q^2} + \frac{1}{2\,!}\left(\frac{1}{2}\ \frac{\delta^2}{\delta q^2}\right)^2 + \ldots.\right\} e^{p_1\delta/\delta q}\, q^n$$

$$= e^{\frac{1}{2}\delta^2/\delta q^2 + p_1\delta/\delta q}\, q^n. \tag{8.02}$$

It can now be verified that the results holds also for negative integral values of n. For it would give

$$(q+p)^{-m} = (p_1+q)^{-m} + \tfrac{1}{2}\frac{(m)_2}{1\,!}(p_1+q)^{-m-2} + \left(\frac{1}{2}\right)^2 \frac{(m)_4}{2\,!}(p_1+q)^{-m-4} + \ldots \tag{8.03}$$

where m is a positive integer and $(m)_r = m\,(m+1)\ldots.(m+r-1)$.

Here $(p_1+q)^{-m}$ stands formally for the expansion

$$\left(1 - mpq^{-1} + \frac{m\,(m+1)}{1.2}\,p^2 q^{-2} - \ldots.\right) q^{-m}. \tag{8.04}$$

It is easy to prove that, for any positive integer l,

$$(q+p)\,(p_1+q)^{-l} = (p_1+q)^{-l+1} - l\,(p_1+q)^{-l-1}. \tag{8.05}$$

Multiplying (8.03) by $(q+p)$ and making use of (8.05), we should have

$$(q+p)\,(q+p)^{-m} =$$

$$(p_1+q)^{-(m-1)} + \frac{1}{2}\,\frac{(m-1)_2}{1\,!}(p_1+q)^{-(m+1)} + \left(\frac{1}{2}\right)^2 \frac{(m-1)^4}{2\,!}(p_1+q)^{-(m+3)} + \ldots$$

Repeating this process m times, we then get

$$(q+p)^m\,(q+p)^{-m} = 1. \tag{8.06}$$

Hence $(q+p)^{-m}$ as defined by (8.03) is that function, assumed unique, which when multiplied by $(q+p)^m$ gives unity. This is what is required.

Hence (8.02) holds for all integral n, positive or negative. *Therefore if $f(q)$ is any function which may be expanded in positive or negative integral powers of q, we have*

$$f(q+p) = e^{\frac{1}{2}\delta^2/\delta q^2 + p_1 \delta/\delta q} f(q). \tag{8.07}$$

Theorem IX. If $\phi(p)$ be any function of p, the function $\{q+\phi(p)\}^n$ is given by

$$\{q+\phi(p)\}^n = \left\{ \exp\left[\phi_1\,\delta/\delta q + \frac{1}{2!}\phi_1'\,(\delta/\delta q)^2 + \frac{1}{3!}\,\phi_1''\left(\frac{\delta}{\delta q}\right)^3 + \ldots\right]\right\} q^n, \tag{8.08}$$

where $\phi_1^{(r)} = \delta^r\phi_1(p)/\delta p^r$, and $\phi_1 = \phi$, and the suffix indicates that all ϕ-factors must be kept on the left of each term.

Consider the right hand side of (8.08), with q^{n+1} substituted for q. This we shall write as

$$\left\{ \exp\left[-\phi_1\,P + \frac{1}{2!}\phi_1'\,P^2 - \frac{1}{3!}\phi_1''\,P^3 + \ldots\right]\right\} q.q^n, \tag{8.09}$$

where $P \equiv -\delta/\delta q$, and so P satisfies the relation (2.1), *viz.*,

$$qP - Pq = 1. \tag{8.10}$$

Hence, by Theorem I, if $F(P)$ be any function of P, we have

$$F(P)q = q\,F(P) - F'(P). \tag{8.11}$$

Applying this to the expression (8.09), we may write it

$$\begin{aligned}&\left\{(q) \exp\left[-\phi_1\,P + \frac{1}{2!}\phi_1'\,P^2 - \frac{1}{3!}\phi_1''\,P^3 + \ldots\right]\right\} q^n \\ &- \left\{\frac{\delta}{\delta P} \exp\left[-\phi_1\,P + \frac{1}{2!}\phi_1'\,P^2 - \frac{1}{3!}\phi_1''\,P^3 + \ldots\right]\right\} q^n,\end{aligned} \tag{8.12}$$

when the first bracket means that the q has been brought to the left of the P-factors in each term, but remains to the right of the p-factors. But we may repeat the process for the latter factors, since

$$G(p)q = qG(p) - G'(p), \tag{8.13}$$

where $G(p)$ is any function of p.

The expression (8.12) then becomes

$$\left\{\left[q - \frac{\delta}{\delta p} - \frac{\delta}{\delta P}\right] \exp\left[-\phi_1\,P + \frac{1}{2!}\phi_1'\,P^2 - \frac{1}{3!}\phi_1''\,P^3 + \ldots\right]\right\} q^n. \tag{8.14}$$

But

$$\frac{\delta}{\delta p}\exp\left[-\phi_1 P+\frac{1}{2!}\phi_1' P^2-\ldots\right]$$
$$=\left[-\phi_1' P+\frac{1}{2!}\phi_1'' P^2-\ldots\right]\exp\left[-\phi_1 P+\frac{1}{2!}\phi_1' P^2-\ldots\right],$$
$$\frac{\delta}{\delta P}\exp\left[-\phi_1 P+\frac{1}{2!}\phi_1' P^2-\ldots\right]$$
$$=\left[-\phi_1+\phi_1' P-\frac{1}{2!}\phi_1'' P^2+\ldots\right]\exp\left[-\phi_1 P+\frac{1}{2!}\phi_1' P^2-\ldots\right].$$

Hence (8.14) reduces to

$$(q+\phi)\left\{\exp\left[-\phi_1 P+\frac{1}{2!}\phi_1' P^2-\ldots\right]\right\}q^n \tag{8.15}$$
$$=(q+\phi)\,(q+\phi)^n$$
$$=(q+\phi)^{n+1},$$

if Theorem IX holds for n. Therefore the theorem is true for $(n+1)$ if it is true for n. The result for positive n follows by induction, and the proof may be completed for negative n as in the previous case, *and so for any function of q expansible in powers of q.*

The result may be further generalised in either of two ways. First, it may be noticed that the proof is not affected if ϕ is a function also of q. Thus we may expand any function of the form

$$f\{q+\phi\,(q,\,p)\}. \tag{8.16}$$

Second, if ϕ remains a function of p only, but $f(q)$ is replaced by $F(q,\,p)$ a function also of p, the proof is again unaffected. Thus we may expand any function of the form

$$F\{q+\phi(p),\,p\}. \tag{8.17}$$

It is not, however, possible to introduce both extensions simultaneously since the additional parts are then not permutable as required. The final generalisation can only be obtained by transforming the variables, as shown in §16.

B. *The solution of differential equations.*

§9. The main object of this section is to give certain general theorems connected with our preceding paper. A few preliminary results must first be stated.

Theorem X. Given any function $\rho(q,\,p)$, there exists a function $\varpi\,(q,\,p)$, such that

$$\rho(q,\,p)=\varpi(q,\,p)\,q\,\varpi^{-1}(q,\,p). \tag{9.1}$$

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Here q, p still satisfy the condition (2.1) and ϖ^{-1} is defined by $\varpi\varpi^{-1} = 1$, which requires $\varpi^{-1}\varpi = 1$.

We may write (9.1) as

$$\rho(q, p)\varpi(q, p) = \varpi(q, p)q$$
$$= q\varpi(q, p) - \delta\varpi(q, p)/\delta p$$

or

$$\delta\varpi/\delta p - (q - \rho)\varpi = 0, \tag{9.2}$$

which is a linear differential equation for ϖ.

It must be noted that if ϖ can be factorised in the form $\varpi = \varpi_1\varpi_2$, then its inverse is given by $\varpi^{-1} = \varpi_2^{-1}\varpi_1^{-1}$. In particular we notice that ϖ in (9.1) may be post-multiplied by any arbitrary function of q.

Hence if we assume ϖ expansible in the form

$$\varpi(q, p) = a_0(q) + a_1(q)p + a_2(q)p^2 + \ldots., \tag{9.3}$$

then $a_0(q)$ is arbitrary.

Let us assume that $(q - \rho)$ may be expanded as

$$q - \rho = A_0(q) + A_1(q)p + A_2(q)p^2 + \ldots. \tag{9.4}$$

Now we have

$$\begin{aligned} p^r a_s(q) &= p^{r-1}\{a_s(q)p - a_s'(q)\} \\ &= p^{r-2}\{a_s(q)p^2 - 2a_s'(q)\,p + a_s''(q)\} \\ &= \ldots. \\ &= a_s p^r - \binom{r}{1} a_s' p^{r-1} + \ldots + (-)^r a_s^{(r)}. \end{aligned} \tag{9.5}$$

Hence, substituting (9.3), (9.4) in (9.2) we obtain

$$\begin{aligned} & a_1 + 2a_2 p + 3a_3 p^2 + \ldots. \\ &= (A_0 + A_1 p + A_2 p^2 + \ldots.)(a_0 + a_1 p + a_2 p^2 + \ldots.) \\ &= A_0 a_0 + A_0 a_1 p + A_0 a_2 p^2 + \ldots. \\ &\quad - A_1 a_0' + A_1 a_0 p \\ &\qquad - A_1 a_1' p + A_1 a_1 p^2 \\ &\qquad\qquad - \ldots. \\ &\quad + A_2 a_0'' - 2A_2 a_0' p + A_2 a_0' p^2 \\ &\qquad + A_2 a_1'' p - 2A_2 a_1' p^2 + A_2 a_1 p^3 \\ &\qquad\qquad + \ldots. \end{aligned}$$

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where we have employed (9.5) to bring all the p's to the right in each term. Since this has been done we may equate coefficients of powers of p_1 obtaining

$$\begin{aligned} a_1 &= A_0 a_0 - A_1 a_0' + A_2 a_0'' - \ldots. \\ 2a_2 &= (A_1 a_0 - 2A_2 a_0' + \ldots.) + (A_0 a_1 - A_1 a_1' + A_2 a_1'' - \ldots.) \\ 3a_3 &= (A_2 a_0 - \ldots.) + (A_1 a_1 - \ldots.) + (A_0 a_2 - \ldots.) \end{aligned}$$

and so on.

Since a_0 is an arbitrary function of q, and $A_0, A_1, \ldots.$ are assumed known, this process determines *formally* the successive coefficients $a_1, a_2, \ldots.$ Thus wherever $\rho(q, p)$ is formally expansible in powers of p there exists a *formal* solution in series of equation (9.2).

In what follows we shall take it for granted that the functions involved are such that solutions of (9.2) do exist.

Theorem XI. Conjugate functions. Given any function $\rho(q, p)$ there exists another function $\sigma(q, p)$ such that

$$\rho\sigma - \sigma\rho = 1. \tag{9.6}$$

Take $\varpi(q, p)$ as defined in Theorem X and define $\sigma(q, p)$ by the relation

$$\sigma(q, p) = \varpi(q, p) p \varpi^{-1}(q, p). \tag{9.7}$$

Then we have

$$\begin{aligned} \rho\sigma - \sigma\rho &= \varpi q \varpi^{-1} \varpi p \varpi^{-1} - \varpi p \varpi^{-1} \varpi q \varpi^{-1} \\ &= \varpi (qp - pq) \varpi^{-1} \\ &= \varpi\varpi^{-1} \\ &= 1. \end{aligned}$$

Hence $\sigma(q, p)$ given by (9.7) is the function required.

We call ρ and σ *conjugate functions*, and the transformation from q, p to ρ, σ a *canonical transformation.* It is clear that we may add to σ any function of ρ and still preserve the condition (9.6). It will be seen that any identity holding for q, p must hold for any pair of conjugate functions. *Such identities are therefore invariant under canonical transformations.*

Theorem XII. If ρ, σ are conjugate functions of q, p, then there exists a function $\varpi(q, p)$ such that

$$\rho(q, p) = \varpi(q, p)\, q\, \varpi^{-1}(q, p), \tag{9.8}$$

$$\sigma(q, p) = \varpi(q, p)\, p\, \varpi^{-1}(q, p). \tag{9.9}$$

For by Theorem X a function $\varpi^*(q, p)$ satisfying (9.8) does exist. But we have

$$\rho\sigma - \sigma\rho = 1,$$

from which we obtain, using this function ϖ^*,

$$\varpi^{*-1}\rho\varpi^*\varpi^{*-1}\sigma\varpi^* - \varpi^{*-1}\sigma\varpi^*\varpi^{*-1}\rho\varpi^* = 1,$$

or

$$q\varpi^{*-1}\sigma\varpi^* - \varpi^{*-1}\sigma\varpi^* q = 1,$$

from which it follows that

$$\varpi^{*-1}\sigma\varpi^* = p + \phi(q),$$

where $\phi(q)$ is some function of q only.

Now let $\psi(q)$ be such that

$$\psi(q) = d\phi/dq.$$

Then we have

$$\begin{aligned} e^{-\psi}\varpi^{*-1}\sigma\varpi^* e^{\psi} &= e^{-\psi} p e^{\psi} + \phi \\ &= e^{-\psi}(e^{\psi} p - \delta e^{\psi}/\delta q) + \phi \\ &= p. \end{aligned}$$

Therefore

$$\sigma = \varpi^* e^{\psi} p e^{-\psi}\varpi^{*-1}.$$

But we have

$$\rho = \varpi^* q \varpi^{*-1} = \varpi^* e^{\psi} q e^{-\psi}\varpi^{*-1}$$

since e^{ψ}, being a function of q, only commutes with q,

Hence the required function $\varpi(q,p)$ exists and is given by

$$\varpi(q, p) = \varpi^*(q, p) e^{\psi(q)}.$$

§ 10. *Theorem XIII. A necessary and sufficient condition that the relations*

$$\mathrm{Q}(q, p)\chi = Q\chi \tag{10.01}$$

$$\mathrm{P}(q, p)\chi = P\chi \tag{10.02}$$

should yield a pair of compatible differential equations when p and P are interpreted respectively as $-\partial/\partial q$ and $\partial/.\partial Q$ is that $\mathrm{Q}(q, p)$ and $\mathrm{P}(q,p)$ should be conjugate functions.

For if (10.01), (10.02) are compatible with the solution $\chi(q, Q)$, we have

$$\begin{aligned} Q(q, p)\,\mathrm{P}(q, p)\chi &= \mathrm{Q}(q, p)\,P\chi = P.\mathrm{Q}(q, p)\chi = PQ\chi, \\ \mathrm{P}(q, p)\,\mathrm{Q}(q, p)\chi &= \mathrm{P}(q, p)\,Q\chi = Q.\mathrm{P}(q, p)\chi = QP\chi. \end{aligned}$$

Therefore

$$\{\mathrm{Q}(q, p)\,\mathrm{P}(q, p) - \mathrm{P}(q, p)\,\mathrm{Q}(q, p)\}\chi = (PQ - QP)\chi = \chi, \tag{10.03}$$

since

$$(PQ - QP)\chi = \left(\frac{\partial}{\partial Q}Q - Q\frac{\partial}{\partial Q}\right)\chi = \chi. \tag{10.04}$$

It follows from (10.03) that, either

$$\mathrm{Q}(q, p)\,\mathrm{P}(q, p) - \mathrm{P}(q, p)\,\mathrm{Q}(q, p) = 1 \tag{10.05}$$

identically, *i.e.* $\mathrm{Q}(q, p)$, $\mathrm{P}(q, p)$ *are conjugate functions*, or else that

$$\chi(q, Q) = \lambda(q)\,\mu(Q), \tag{10.06}$$

where $\lambda(q)$ is a solution of the equation

$$\left\{\mathrm{Q}\left(q,-\frac{d}{dq}\right)\mathrm{P}\left(q,-\frac{d}{dq}\right)-\mathrm{P}\left(q,-\frac{d}{dq}\right)\mathrm{Q}\left(q,-\frac{d}{dq}\right)-1\right\}\lambda(q)=0. \tag{10.07}$$

But it is clear that a function of the form (10.06) will *not* satisfy equations of the form (10.01), (10.02).

Hence a necessary condition is that $\mathrm{Q}(q, p)$, $\mathrm{P}(q, p)$ should be conjugate.

Conversely, if these functions are conjugate, then by a variant of Theorem XII there exists a function $\varpi(q, p)$ such that

$$\mathrm{Q}(q, p) = \varpi p \varpi^{-1}, \tag{10.08}$$

$$\mathrm{P}(q, p) = -\varpi q \varpi^{-1}. \tag{10.09}$$

Now considered as differential equations for χ^*, the equations

$$p\chi^* = Q\chi^* \tag{10.10}$$

$$-q\chi^* = P\chi^* \tag{10.11}$$

are compatible and have the solution $\chi^* = e^{-qQ}$.

Operating on both sides of (2.10), (2.15) with $\varpi(q, p)$ we obtain

$$\varpi p \varpi^{-1}\varpi\chi^* = \varpi p\chi^* = \varpi Q\chi^* = Q\varpi\chi^*$$

$$-\varpi q \varpi^{-1}\varpi\chi^* = -\varpi q\chi^* = \varpi P\chi^* = P\varpi\chi^*,$$

or, by (10.08), (10.09)

$$\mathrm{Q}(q, p)\,\chi = Q\chi \tag{10.01}$$

$$\mathrm{P}(q, p)\,\chi = P\chi, \tag{10.02}$$

where

$$\chi = \varpi\chi^*. \tag{10.12}$$

Thus the equations (10.01), (10.02) are compatible, having the common solution $\chi = \varpi(q, p)\, e^{-qQ}$.

§ 11. *Relation to solution by definite integrals.* We are now in a position to give the discussion, relegated from Paper I, of the degrees of generality of the methods there given.

The first method employs the contact transformation derived from any function $W(q, Q)$. The nucleus of the integral is then $\chi = e^{W}$.

The second method employs the canonical transformation specified by $\varpi(q, p)$, say, giving the transformation $Q = \varpi p \varpi^{-1}$,

$P = -\varpi q \varpi^{-1}$. The nucleus of the integral is then ϖe^{-qQ}. Any arbitrary function ϖ may be used.

It is evident that these two methods must be equivalent. In practice a combination of them may be most useful, as is illustrated in Paper I. The equivalence is made more explicit by the following theorem.

Theorem XIV. To any given ϖ-function there corresponds a χ-function given by $\chi(q, Q) = \varpi(q, p)\, e^{-qQ} = e^{-qQ}\varpi(q, Q)$, and to any χ-function there corresponds a ϖ-function given by $\varpi(q, Q) = e^{qQ}\chi(q, Q)$.

Throughout we suppose that in each term of $\varpi(q, p)$ the part that involves p has been brought to the extreme right. This Theorem is an immediate consequence of equation (10.12).

Finally we may notice that there are certain general groups of canonical transformation exemplified by the following result:

Theorem XV. If $k(q, p)$, $h(q, p)$ are conjugate functions of q, p and if $\sigma(hk)$ is any function of the product hk, then the transformation given by

$$\left.\begin{aligned} Q &= k\sigma \\ P &= \sigma^{-1} h \end{aligned}\right\} \tag{11.01}$$

is canonical.

For $QP = \sigma^{-1} hk\sigma = hk$, since σ is a function of hk and must consequently commute with hk.

Also

$$PQ = k\sigma\sigma^{-1} h = kh.$$

Therefore

$$PQ - QP = kh - hk = 1,$$

since k, h are conjugate, and so the transformation is canonical.[1]

It is to be noted that when σ is fixed the arbitrariness of k implies that the function giving Q is still completely arbitrary, so that actually it must be possible to put *any* canonical transformation into this form in any number of ways.

Theorem XVI. The solution of a differential equation

$$g(q, p)\,\psi = 0, \quad (p \equiv -\partial/\partial q) \tag{11.02}$$

is given by

$$\psi(q) = \int \varpi(q, p)\, e^{-qt}\, dt, \tag{11.03}$$

where the function $\varpi(q, p)$ is such that

$$g(q, p) = -\varpi q \varpi^{-1}. \tag{11.04}$$

[1] A particular example of this is $k = q$, $h = p$, which gives the type of transformation used by Professor Whittaker in § 7 of his paper.

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By Theorem X a function ϖ satisfying (11.04) does exist, and by Theorem XIV the corresponding χ-function is

$$\chi = \varpi(q, p)\, e^{-qQ}. \tag{11.05}$$

Now define $\psi(q)$ by the integral

$$\psi(q) = \int \chi(q, t)\, \phi(t)\, dt \tag{11.06}$$

along a suitable contour. Further, let $\phi(t)$ be a solution of the differential equation

$$d\phi/dt = 0 \quad \text{or} \quad P\phi = 0. \tag{11.07}$$

Then by Paper I, § 3.1, and equation (10.09) of this paper, the function $\psi(q)$ satisfies the equation

$$g(q, p)\, \psi = 0.$$

Since from (11.07) ϕ is merely a constant, we have the required result (11.03).

Thus the solution of any linear equation has been reduced to the determination of a function ϖ. Theorem X shows that this is the solution of a linear partial differential equation of the *first* order. Since however the variables involved in this equation are non-commutative, it is in general difficult to solve except by the formal series we have discussed.

We can now prove that there exists a χ-function connecting the solutions of *any* two linear differential equations.

Theorem XVII. If $\psi(q)$, $\phi(Q)$ are respectively solutions of the linear differential equations

$$g(q, p)\, \psi = 0, \qquad G(Q, P)\, \phi = 0, \tag{11.08}$$

where $p \equiv -\partial/\partial q$, $P \equiv \partial/\partial Q$, then there exists a function $\chi(q, Q)$ such that

$$\psi(q) = \int \chi(q, \dot{Q})\, \phi(Q)\, dQ \tag{11.09}$$

for a suitable path of integration.

For by Theorem X there exist functions $\varpi(q, p)$, $\Pi(Q, P)$ such that

$$g(q, p) = -\varpi q \varpi^{-1}, \qquad \widetilde{G}(Q, P) = \Pi P \Pi^{-1},$$

where $\widetilde{G}(Q, P)$ is the "adjoint" of $G(Q, P)$ obtained from it by reversing the order in each term and changing the sign of P. Then if the equations for $\chi(q, Q)$ are taken to be

$$-\varpi q \varpi^{-1} \chi = \Pi P \Pi^{-1} \chi \tag{11.10}$$

$$\varpi p \varpi^{-1} \chi = \Pi Q \Pi^{-1} \chi, \tag{11.11}$$

they have the solution

$$\chi = \varpi \Pi e^{-qQ}. \tag{11.12}$$

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Since then (11.10), (11.11) are compatible equations they may be taken to define the transformation from q, p to Q, P. But (11.10) then gives

$$g(q, p) = \widetilde{G}(Q, P). \tag{11.13}$$

By Paper I, § 3.1, however, this is just the condition that, if in (11.09) $\phi(Q)$ satisfies the equation

$$G(Q, P)\,\phi = 0,$$

then $\psi(q)$ must satisfy

$$g(q, p)\,\psi = 0.$$

Hence the χ-function required is given by (11.12), viz.,

$$\chi(q, Q) = \varpi(q, p)\,\Pi(Q, P)\,e^{-qQ} = e^{-qQ}\varpi(q, Q)\,\Pi(Q, -q), \tag{11.14}$$

where as usual the p and P parts of each term in ϖ and Π respectively have first been brought to the right hand side.

§ 12. We revert for a moment to the consideration of the function ϖ of Theorem XIV. Suppose we are given some function $Q(q, p)$ and we wish to find the function ϖ such that $Q(q, p) = \varpi p \varpi^{-1}$. If we may suppose the contact transformation corresponding to (10.01), (10.02) known, then from Paper I we know χ, and from Theorem XIV we obtain ϖ immediately. The question arises, can we find ϖ^{-1} by ordinary algebra?

Let us define the function $\theta(Q, t)$ by the equation

$$e^{-qQ} = \int \theta(Q, t)\,\chi(q, t)\,dt \tag{12.1}$$

that is

$$e^{-qQ} = \int \theta(Q, t)\,\varpi(q, p)\,e^{-qt}\,dt,$$

giving

$$\varpi^{-1}(q, p)\,e^{-qQ} = \int \theta(Q, t)\,\varpi^{-1}\varpi e^{-qt}\,dt$$

$$= \int \theta(Q, t)\,e^{-qt}\,dt. \tag{12.2}$$

Hence just as Theorem XIV gives ϖ derived from the function χ, so from (12.2) we may give ϖ^{-1} derived analogously from a function χ^*, viz.,

$$\varpi^{-1}(q, p)\,e^{-qQ} = \chi^*(q, Q) \tag{12.3}$$

where

$$\chi^* = \int \theta(Q, t)\,e^{-qt}\,dt,$$

and $\theta(Q, t)$ is defined by (12.1) with a suitable path of integration. This provides formally a method of obtaining ϖ^{-1} without recourse to non-commutative algebra.

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C. *The identical transformation and infinitesimal transformations.*

§ 13. *The identical transformation.* This is given by

$$\left.\begin{aligned} Q &= q \\ P &= p \end{aligned}\right\} \tag{13.1}$$

so that the corresponding equations for the function $\chi(q, Q)$ are

$$(Q - q)\,\chi = 0$$
$$(\partial/\partial Q + \partial/\partial q)\,\chi = 0.$$

These have the solution

$$\chi = \delta\,(Q - q) \tag{13.2}$$

where $\delta\,(Q - q)$ is Dirac's δ-function.[1]

In this case Professor Whittaker's general theorem reduces to the property of the δ-function expressed by

$$\psi\,(q) = \int \psi\,(t)\;\delta\,(t - q)\,dt \tag{13.3}$$

where ψ is any function.

In the preceding Section we studied the general transformation by relating it to the transformation $Q = p$, $P = -q$, that is, in effect, just the Laplace Transformation. The theory can be worked out by relating it instead to the identical transformation (13.1). But this involves the use of the δ-function in place of the function e^{-qQ}; and so on account of difficulties in the theory of the δ-function we thought it better to sacrifice some of the symmetry introduced by it and to use the simpler function.

§ 14. *Theorem XVIII. Infinitesimal Contact Transformation. Let $\theta\,(q, p)$ be any function of q, p. Then the transformation given by*

$$Q = q - \epsilon\delta\theta/\delta p \tag{14.1}$$
$$P = p + \epsilon\delta\theta/\delta q \tag{14.2}$$

is canonical, when powers of ϵ greater than the first are neglected.

We have from (14.1), (14.2)

$$\begin{aligned}\mathrm{Q}(q,p)\mathrm{P}(q,p)-\mathrm{P}(q,p)\mathrm{Q}(q,p) &= \left(q-\epsilon\frac{\delta\theta}{\delta p}\right)\left(p+\epsilon\frac{\delta\theta}{\delta q}\right)-\left(p+\epsilon\frac{\delta\theta}{\delta q}\right)\left(q-\epsilon\frac{\delta\theta}{\delta p}\right)\\ &= qp-pq+\epsilon\left(q\frac{\delta\theta}{\delta q}-\frac{\delta\theta}{\delta q}q\right)-\epsilon\left(\frac{\delta\theta}{\delta p}p-p\frac{\delta\theta}{\delta p}\right),\end{aligned}$$

neglecting terms in ϵ^2,

$$\begin{aligned} &= 1+\epsilon\left(\frac{\delta^2\theta}{\delta p\delta q}-\frac{\delta^2\theta}{\delta q\delta p}\right)\\ &= 1\end{aligned}$$

using[2] Theorem I. Hence (14.1) (14.2) define a canonical trans-

[1] Dirac, *op. cit.*, 63.

[2] Recalling the remarks in § 2 on the type of function used.

formation. We may therefore write it in the form, using Theorem XII,

$$Q = \varpi q \varpi^{-1} \tag{14.3}$$

$$P = \varpi p \varpi^{-1}. \tag{14.4}$$

Since it reduces to the identical transformation when $\epsilon = 0$, we shall write $\quad \varpi(q, p) = 1 + \epsilon\lambda(q, p)$.

Then $\quad \varpi^{-1}(q, p) = 1 - \epsilon\lambda(q, p)$,

since this gives $\quad \varpi\varpi^{-1} = 1 = \varpi^{-1}\varpi$

neglecting terms in ϵ^2. We have then to this order

$$\begin{aligned}\varpi q \varpi^{-1} &= (1 + \epsilon\lambda)\, q\, (1 - \epsilon\lambda) \\ &= q - \epsilon(q\lambda - \lambda q) \\ &= q - \epsilon\frac{\delta\lambda}{\delta p} \\ &= q - \epsilon\frac{\delta\theta}{\delta p}\end{aligned}$$

by the definition (14.1), and so we may take

$$\lambda(q, p) = \theta(q, p), \tag{14.5}$$

that is

$$\varpi(q, p) = (1 + \epsilon\theta). \tag{14.6}$$

It can be verified immediately that this satisfies also equation (14.2).

Comparing the equations (14.1), (14.2) with (13.1) it is evident that the corresponding χ-function is given by

$$\begin{aligned}\chi(q, Q) &= \varpi\delta(Q - q) \\ &= (1 + \epsilon\theta)\,\delta(Q - q)\end{aligned} \tag{14.7}$$

This result can also be obtained directly from the differential equations for χ.

§ 15. *Algebraic Identities.* Consider now any function $p(Q, P)$. We can change to the variables q, p in two ways.

$$\begin{aligned}\text{(i)}\quad \rho(Q, P) &= \rho(q - \epsilon\delta\theta/\delta p,\ p + \epsilon\delta\theta/\delta q) \\ &= \rho(q, p) + \epsilon\rho'(q, p),\end{aligned} \tag{15.1}$$

where ρ' is derived from ρ by replacing, for instance, a term $ppppqqq$ by the expression

$$\frac{\delta\theta}{dq}pppqqq + p\frac{\delta\theta}{\delta q}ppqqq + \ldots - pppp\frac{\delta\theta}{\delta p}qq - \ldots - ppppqq\frac{\delta\theta}{\delta p}. \tag{15.2}$$

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(ii) By the general properties of canonical transformations

$$\begin{aligned}\rho(Q,P) &= \varpi\rho(q,p)\varpi^{-1}\\ &= (1+\epsilon\theta)\rho(1-\epsilon\theta)\\ &= \rho(q,p)+\epsilon(\theta\rho-\rho\theta)\end{aligned} \tag{15.3}$$

Identifying (15.1) and (15.3) we obtain the general identity

$$\theta\rho - \rho\theta = \rho'(q,p). \tag{15.4}$$

This is a general theorem on the permutation of two arbitrary functions. It is seen from (15.2) that in (15.4) the degree of the right hand side is two less than the degree of either term on the left hand side. When θ, ρ are conjugate, the right hand side reduces, of course, to unity.

Example. Take $\rho = p^n$, $\theta = q^m$; then, from (15.4), we have

$$q^m p^n - p^n q^m = m(q^{m-1}p^{n-1} + pq^{m-1}p^{n-2} + \ldots + p^{n-1}q^{m-1}).$$

Interchanging ρ, θ we find

$$p^n q^m - q^m p^n = -n(p^{n-1}q^{m-1} + qp^{n-1}q^{m-2} + \ldots + q^{m-1}p^{n-1}).$$

These identities are, of course, derivable by other means, but are scarcely self-evident.

§ 16. *Taylor's Theorem.* We are now in a position to extend Theorem IX. By applying a canonical transformation to the function (8.16) we get a new one given by

$$\varpi f\{q+\phi(q,p)\}\varpi^{-1} = f\{\varpi q\varpi^{-1} + \varpi\phi(q,p)\varpi^{-1}\},$$

which may be written

$$f\{\lambda(q,p)+\mu(q,p)\} \tag{16.1}$$

where λ, μ are *arbitrary* functions, since ϖ, ϕ are arbitrary. Applying then the same transformation to the expansion of (8.16) given by Theorem IX we get a method of expanding any function of the form (16.1). This may be regarded as the complete generalisation of *Taylor's Theorem* to non-commuting variables.

§ 17. Now let the function (8.16) be

$$f\{q-\epsilon\theta'(q,p)\}.$$

Then Theorem IX gives

$$\begin{aligned}f\{q-\epsilon\theta'(q,p)\} &= \exp\left\{-\epsilon\left[\theta'\frac{\delta}{\delta q}+\frac{\theta''}{2!}\left(\frac{\delta}{\delta q}\right)^2+\frac{\theta'''}{3!}\left(\frac{\delta}{\delta q}\right)^3+\ldots\right]\right\}f(q)\\ &= f(q)-\epsilon\left[\theta'f'+\frac{1}{2!}\theta''f''+\frac{1}{3!}\theta'''f'''+\ldots\right],\end{aligned} \tag{17.1}$$

neglecting terms in ϵ of degree higher than the first. The accents on

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θ denote differentiation with respect to p, while those on f denote differentiation with respect to q.

But applying to $f(q)$ the canonical transformation given by (14.6), we have

$$\begin{aligned} f(q-\epsilon\theta') &= (1+\epsilon\theta)\,f(q)\,(1-\epsilon\theta) \\ &= f(q)+\epsilon\,(\theta f - f\theta). \end{aligned} \tag{17.2}$$

Hence comparing (17.1), (17.2) we have

$$f\theta = \theta f + \theta' f' + \frac{1}{2!}\,\theta'' f'' + \frac{1}{3!}\,\theta''' f''' + \ldots, \tag{17.3}$$

another general theorem on the permutation of two arbitrary functions, subject only to the restriction that one of them is a function of only one of the variables q, p.

By applying a canonical transformation to the result (17.3) we get an expression for $(f\theta - \theta f)$ for completely arbitrary f, θ.

Example. An elementary example of (17.3) is got by taking $f(q) = q^m$, $\theta(q, p) = p^n$; we find that

$$q^m p^n = p^n q^m + nmp^{n-1} q^{m-1} + \frac{n(n-1)\,m(m-1)}{2!}\,p^{n-2} q^{m-2} + \ldots.$$

§ 18. RELATION TO OTHER ALGEBRAS.

Every non-commutative algebra in a pair of variables q, r in which the difference $qr - rq$ is an explicit function of the variables, say

$$qr - rq = F(r, q), \tag{18.1}$$

can be reduced to the algebra for which

$$qp - pq = 1. \tag{18.2}$$

For we merely have to change from the variables q, r to new variables q, p according to

$$\begin{aligned} q &= q \\ r &= r(q, p), \end{aligned}$$

where from (18.1) $r(q, p)$ satisfies the equation

$$\delta\, r(q, p)/\delta p = F\{r(q, p),\ q\}. \tag{18.3}$$

For example, in the algebra specified by $qr + rq = 0$, we take $r = q^{-1} e^{2qp}$.

When differentiation is involved it is of course necessary to change to q, p before any differentiations are actually performed.

The variables used may throughout be regarded as infinite matrices. When however the difference $qr - rq$ is not any given function of the variables we have the case of finite matrices, and the methods used above are not applicable. Taylor's Theorem for finite matrices has been given by Turnbull.[1]

[1] H. W. Turnbull, *Proc. Edin. Math. Soc.* (2) **2** (1930), 33.