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A SINGULAR INVERSE OF A MATRIX BY RANK ANNIHILATION

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1. **Introduction.** Edelblute [1] has given a method of finding an inverse of a nonsingular matrix by rank annihilation. The purpose of this paper is to show that the method can be extended in the case of a singular matrix. This method will produce a singular inverse satisfying condition (3) of Penrose [3].

2. **Method.** We denote C^{-1} as the inverse of matrix C if C is nonsingular and C^{-} as the singular inverse of matrix C satisfying condition (3) of Penrose [3] i.e. $CC^{-}C=C$. If A is nonsingular and U and V are column vectors, then

$$(A+UV^T)^{-1} = A^{-1}-(A^{-1}U)(V^TA^{-1})(1+V^TA^{-1}U)^{-1} \quad \text{if} \quad 1+V^TA^{-1}U \neq 0, \tag{1}$$

$$(A+UV^T)^{-} = A^{-1} \quad \text{if} \quad 1+V^TA^{-1}U = 0. \tag{2}$$

(1) is given by Householder [2] and can be easily verified. Householder [2] also remarks that if $1+V^TA^{-1}=0$, then $A+UV^T$ is singular. Verification of (2) is as follows

$$\begin{aligned}(A+UV^T)(A+UV^T)^{-}(A+UV^T) &= (A+UV^T)A^{-1}(A+UV^T)\\ &= A+UV^T+UV^T+UV^TA^{-1}UV^T\\ &= A+UV^T+UV^T-UV^T\\ \text{since } 1+V^TA^{-1}U &= 0\\ &= A+UV^T\end{aligned}$$

The repeated use of (1) or (2) as the case may be to find an inverse of an $n\times n$ matrix B is known as the method of rank annihilation.

$$\text{Let } B = D+\sum_{i=1}^{n} U_iV_i^T, \tag{3}$$

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where D is a nonsingular matrix of known inverse. Thus, we can define a sequence of matrices $\{C_i\}$ such that

$$\begin{aligned} C_0 &= D \\ C_k &= D+\sum_{i=1}^{k} U_i V_i^T \qquad k = 1, 2, \ldots, n \\ &= C_{k-1}+U_k V_k^T. \end{aligned}$$

Then using (1) or (2) as the case may be we get a sequence of inverse matrices $\{E_i\}$ and

$$\begin{aligned} E_n &= B^{-1} \quad \text{if} \quad B \text{ is nonsingular} \\ &= B^{-} \quad \text{if} \quad B \text{ is singular} \end{aligned}$$

such that

$$BB^{-}B = B.$$

Let B be any $n\times n$ matrix to be inverted and let $U=B-I$, where I is $n\times n$ identity matrix. Partition U by column so that $U=\{U_1, \ldots, U_n\}$ and V_i is the i^{th} column vector of an $n\times n$ identity matrix. Then

$$B = I+\sum_{i=1}^{n} U_i V_i^T. \tag{4}$$

The advantage of this method has been discussed by Edelblute [1].

The applicability of the method, however, crucially depends on whether any of the intermediate matrices(¹) C_k turns out to be singular, or equivalently, whether any of the numbers $1+V_k^T C_{k-1}^{-1} U_k$ vanishes or not. If B is written as in (4) it can be easily verified that $1+V_k^T C_{k-1}^{-1} U_k$ is proportional to the determinant

$$\Delta_k \equiv \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{vmatrix}$$

which is the k-rowed principal minor of the matrix $B\equiv(b_{ij})$. Hence for C_k to be nonsingular it is necessary and sufficient that $\Delta_i\neq0$, for $i=1, 2, \ldots, k$. This requires, in particular, that $b_{11}\neq0$, $b_{11}b_{22}-b_{12}b_{21}\neq0$, etc. If $b_{11}=0$ the method can still be applied provided at least one of the diagonal elements of Δ_k is nonzero, say $b_{ii}\neq0$, $i\leq k$. This would mean $1+V_i^T U_i\neq0$ and so one would have to define the sequence $\{C_i\}$ in a slightly different order, for example,

$$\begin{gathered} C_0 = I, \qquad C_1 = I+U_i V_i^T, \qquad C_2 = C_1+U_1 V_1^T, \\ C_i = C_{i-1}+U_{i-1} V_{i-1}^T, \qquad C_{i+1} = C_i+U_{i+1} V_{i+1}^T, \qquad \ldots \\ C_n = C_{n-1}+U_n V_n^T. \end{gathered}$$

(¹) We would like to thank the referee for pointing out this important question about the applicability of the method.

To illustrate these points, one can see that the present method cannot be applied to invert the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ since both diagonal elements are zeros in this case, but we can invert $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ by this method.

Another important point concerns the rank $r(B)$, of the matrix B. If $r(B)=n-1$ and $C_1, \ldots, C_{n-1}$ are all nonsingular then the above method can be directly applied to obtain the singular inverse B^-. However, if $0 \leq r < n-1$, all Δ_k vanish for $r < k \leq n$ and hence $C_{r+1}, \ldots, C_{n-1}$ are singular matrices. But if $C_1^{-1}, C_2^{-1}, \ldots, C_r^{-1}$ all exist then $C_{r+1}^- = C_r^{-1}$ is one of the possible singular inverses of B, satisfying the Penrose condition. It must be realized that as the rank gets smaller, the degree of arbitrariness multiplies, as can be seen from the fact that if $r=0$, *any* arbitrary matrix B^- satisfies the equation

$$OB^-O = 0.$$

3. **Examples.**

(a) Let $B=\begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 6 & 6 & 9 \end{bmatrix}$ $\qquad r(B)=2$. Then

$$U_1 = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \qquad U_2 = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \qquad U_3 = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} [1 \quad 0 \quad 0] + \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} [0 \quad 1 \quad 0] + \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} [0 \quad 0 \quad 1]$$

$$C_0 = I \qquad E_0 = I$$

$$C_1 = I + U_1 V_1^T$$

$$E_1 = I - U_1 V_1^T (1 + V_1^T U_1)^{-1} = I - \tfrac{1}{2} U_1 V_1^T = I - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$C_2 = C_1 + U_2 V_2^T$$

$$E_2 = E_1 - E_1 U_2 (V_2^T E_1)(1 + V_2^T E_1 U_2)^{-1}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ -4 & 2 & 0 \\ -6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$C_3 = C_2 + U_3 V_3^T$$

Since

$$1 + V_3^T E_2 U_3 = 0$$

$$E_3 = E_2 = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix} = B^-$$

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It can be easily verified $BB^{-}B=B$.

(b) Let $B=\begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}$ $r(B)=1$. Here

$$U_1 = [1 \quad 2 \quad 4]^T, \qquad U_2 = [1 \quad 0 \quad 2]^T, \qquad U_3 = [3 \quad 3 \quad 5]^T,$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} [1 \quad 0 \quad 0] + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} [0 \quad 1 \quad 0] + \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} [0 \quad 0 \quad 1].$$

$$C_0 = I, \qquad E_0 = I,$$

$$C_1 = I + U_1 V_1^T,$$

$$E_1 = I - U_1 V_1^T (1 + V_1^T U_1)^{-1} = I - \tfrac{1}{2} U_1 V_1^T = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$C_2 = C_1 + U_2 V_2^T = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \qquad \therefore \quad |C_2| = 0.$$

It can be easily verified that

$$BE_1B = B.$$

References

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