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Invariant measures for $\mathscr{B}$-free systems revisited

AURELIA DYMEK, JOANNA KUŁAGA-PRZYMUS and DANIEL SELL

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland
(e-mail: aurbart@mat.umk.pl, joasiak@mat.umk.pl, dsell@mat.umk.pl)

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Abstract. For $\mathscr{B} \subseteq \mathbb{N}$, the $\mathscr{B}$-free subshift X_η is the orbit closure of the characteristic function of the set of $\mathscr{B}$-free integers. We show that many results about invariant measures and entropy, previously only known for the hereditary closure of X_η, have their analogues for X_η as well. In particular, we settle in the affirmative a conjecture of Keller about a description of such measures [G. Keller. Generalized heredity in $\mathcal{B}$-free systems. *Stoch. Dyn.* **21**(3) (2021), Paper No. 2140008]. A central assumption in our work is that η^* (the Toeplitz sequence that generates the unique minimal component of X_η) is regular. From this, we obtain natural periodic approximations that we frequently use in our proofs to bound the elements in X_η from above and below.

Key words: invariant measures, B-free dynamics, dynamical diagrams, topological entropy, intrinsic ergodicity, entropy density, tautness
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Contents

1. *Introduction*

1.1. *Background.* Given $\mathscr{B} \subseteq \mathbb{N}$, consider the corresponding set $\mathcal{M}_{\mathscr{B}} = \bigcup_{b\in\mathscr{B}} b\mathbb{Z}$ of the *multiples* of $\mathscr{B}$ and its complement $\mathcal{F}_{\mathscr{B}} = \mathbb{Z} \setminus \mathcal{M}_{\mathscr{B}}$, that is, the set of *$\mathscr{B}$-free integers*. We study the dynamics of $\eta = \mathbf{1}_{\mathcal{F}_{\mathscr{B}}} \in \{0, 1\}^{\mathbb{Z}}$, that is, of the orbit closure X_η of η under the left shift σ. The motivation for such studies goes back to the 1930s, when sets of multiples were investigated from the *number-theoretic perspective* by Besicovitch, Chowla, Erdős and others (see [12] and the references therein). In 2010, Sarnak [34] suggested to study the *dynamics* of the square-free system, i.e. X_η corresponding to $\mathscr{B}$ being the set of squares of all primes. In this case, $\eta|_{\mathbb{N}}$ is the square of the Möbius function μ, and the aim was to gain more knowledge about the Möbius function itself. Sarnak formulated a certain 'program' for μ^2 and indicated how to prove the statements about μ^2. Without going into details, there was a list of properties related both to measure-theoretic and topological dynamics of X_{μ^2}. A natural question arose whether analogous results are true for other sets $\mathscr{B}$. The dynamics of X_η was studied systematically for the first time in [9] in the Erdős case, that is, when $\mathscr{B}$ is infinite and pairwise coprime with $\sum_{b\in\mathscr{B}} 1/b < \infty$. In this case, the properties of X_η resemble the properties of X_{μ^2}. In particular, X_η is *hereditary*, that is, if $x \in X_\eta$ and $y \in \{0, 1\}^{\mathbb{Z}}$ is such that $y \le x$ coordinatewise then $y \in X_\eta$. In fact, we have

$$X_\eta = X_{\mathscr{B}} := \{x \in \{0, 1\}^{\mathbb{Z}} : |\mathrm{supp}\, x \bmod b| \le b - 1 \text{ for any } b \in \mathscr{B}\}$$

($X_{\mathscr{B}}$ is called the $\mathscr{B}$-admissible subshift). When we relax the assumptions on $\mathscr{B}$, various things can happen to X_η, in particular, it may no longer be hereditary. Thus, one often looks at its *hereditary closure* $\widetilde{X}_\eta$, that is, the smallest hereditary subshift containing X_η. Such general $\mathscr{B}$-free systems were studied in [7]. We may have $X_\eta \subsetneq \widetilde{X}_\eta \subsetneq X_{\mathscr{B}}$ (see [7] for various examples).

In this paper, we concentrate on invariant measures on X_η. Let us give now some more detailed background related to this. In the Erdős case, η turns out to be a generic point for the so-called *Mirsky measure* [9] denoted by ν_η:

$$\frac{1}{L}\sum_{\ell\le L} \delta_{\sigma^\ell\eta} \to \nu_\eta$$

(in this case, the above formula can be treated as the definition of ν_η). In other words, the frequencies of $0-1$ blocks on η exist (in the square-free case, they were first studied by Mirsky [30, 31], hence the name). In general, η might not be a generic point. However, it is quasi-generic along any sequence (ℓ_i) realizing the lower density of $\mathcal{M}_{\mathscr{B}}$ (that is, such that $\lim_{i\to\infty}(1/\ell_i)|\mathcal{M}_{\mathscr{B}} \cap [1, \ell_i]| = \liminf_{L\to\infty}(1/L)|\mathcal{M}_{\mathscr{B}} \cap [1, L]| =: \underline{d}(\mathcal{M}_{\mathscr{B}})$). This is a consequence of the deep number-theoretic result of Davenport and Erdős [3] that the logarithmic density of $\mathcal{M}_{\mathscr{B}}$, that is, $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) = \lim_{L\to\infty}(1/\ln L)\sum_{\ell\in\mathcal{M}_{\mathscr{B}}\cap[1,L]}(1/\ell)$ always exists and we have

$$\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) = \underline{d}(\mathcal{M}_{\mathscr{B}}) = \lim_{K\to\infty} d(\mathcal{M}_{\mathscr{B}_K}) \quad \text{where } \mathscr{B}_K = \{b \in \mathscr{B} : b \le K\} \tag{1}$$

($d(A)$ for $A \subseteq \mathbb{Z}$ stands for the natural density: $d(A) = \lim_{L\to\infty}(1/L)|A \cap [1, L]|$). Again, we call the resulting measure the Mirsky measure and denote it by ν_η:

$$\lim_{i\to\infty} \frac{1}{\ell_i} \sum_{\ell\leq\ell_i} \delta_{\sigma^\ell\eta} = \nu_\eta,$$

see [7]. The following problems, already asked by Sarnak in the square-free case, arise.

- Give a description of the set $\mathcal{P}(X_\eta)$ of all invariant measures on X_η.
- Compute the topological entropy $h(X_\eta)$ of X_η.
- Determine whether X_η is *intrinsically ergodic*, that is, whether it has only one measure of maximal entropy.

The solution to the second problem and the positive answer to the third one in the square-free case were given by Peckner [32]. However, the proof used the properties of the squares of primes and it was not clear if it can be extended to a more general setting. It turned out to be true:

$$\text{for any } \mathscr{B} \subseteq \mathbb{N}, \quad \widetilde{X}_\eta \text{ is intrinsically ergodic.} \tag{2}$$

This was proved in [22] in the Erdős case (where $X_\eta = \widetilde{X}_\eta$) and then, in [7], for all sets $\mathscr{B} \subseteq \mathbb{N}$. Moreover, the topological entropy $h(\widetilde{X}_\eta)$ of $\widetilde{X}_\eta$ is equal to the upper density of $\mathcal{F}_\mathscr{B}$:

$$h(\widetilde{X}_\eta) = \overline{d} := \overline{d}(\mathcal{F}_\mathscr{B}) \tag{3}$$

and

$$\text{the measure of maximal entropy on } \widetilde{X}_\eta \text{ is of the form } M_*(\nu_\eta \otimes B_{1/2,1/2}), \tag{4}$$

where $B_{1/2,1/2}$ is the symmetric Bernoulli measure on $\{0, 1\}^\mathbb{Z}$ and $M\colon (\{0, 1\}^\mathbb{Z})^2 \to \{0, 1\}^\mathbb{Z}$ stands for the coordinatewise multiplication (in each case, the proofs were given in the corresponding paper covering the intrinsic ergodicity in the same class). We also have

$$h(\widetilde{X}_\eta) = 0 \iff \mathcal{P}(\widetilde{X}_\eta) = \{\delta_{\mathbf{0}}\} \iff \widetilde{X}_\eta \text{ is uniquely ergodic} \tag{5}$$

(this is true, in general, for hereditary subshifts; for a proof, see [26]).

As for the set of invariant measures, it was shown in [22] that in the Erdős case,

$$\mathcal{P}(\widetilde{X}_\eta) = \{M_*(\nu_\eta \vee \kappa) : \kappa \in \mathcal{P}(\{0, 1\}^\mathbb{Z})\}, \tag{6}$$

where $\nu_\eta \vee \kappa$ stands for any joining of ν_η and κ, that is, any probability measure ρ on $(\{0, 1\}^\mathbb{Z})^2$ invariant under $\sigma^{\times 2}$ whose projection onto the first coordinate equals ν_η and the projection onto the second coordinate equals κ. Later, in [7], this result was extended to any set $\mathscr{B} \subseteq \mathbb{N}$.

Recall that a central role in the theory of $\mathscr{B}$-free systems is played by the notion of tautness [12]:

$$\mathscr{B} \subseteq \mathbb{N} \text{ is taut if for every } b \in \mathscr{B}, \text{ we have } \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\setminus\{b\}}) < \boldsymbol{\delta}(\mathcal{M}_\mathscr{B}).$$

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It was shown in [7] (see Theorem C therein) that for any $\mathscr{B} \subseteq \mathbb{N}$, there exists a unique taut set $\mathscr{B}' \subseteq \mathbb{N}$ with $\mathcal{M}_{\mathscr{B}} \subseteq \mathcal{M}_{\mathscr{B}'}$ and $\nu_\eta = \nu_{\eta'}$ (for more details about $\mathscr{B}'$, see §1.2.3). In fact, we have

$$\mathcal{P}(\widetilde{X}_\eta) = \mathcal{P}(\widetilde{X}_{\eta'}). \tag{7}$$

Moreover, in [7] (see Corollaries 4.35 and 9.1 therein), the following combinatorial result on taut sets was proved. Fix $\mathscr{B} \subseteq \mathbb{N}$ and a taut set $\mathscr{C} \subseteq \mathbb{N}$. Let $\eta_{\mathscr{C}} := \mathbf{1}_{\mathcal{F}_{\mathscr{C}}}$. Then the following are equivalent:

$$\begin{aligned}\text{for each } b \in \mathscr{B}, \text{ there exists } c \in \mathscr{C} \text{ such that } c \mid b &\iff \eta_{\mathscr{C}} \leq \eta \iff \widetilde{X}_{\eta_{\mathscr{C}}} \subseteq \widetilde{X}_\eta \\ &\iff \eta_{\mathscr{C}} \in \widetilde{X}_\eta \iff \nu_{\eta_{\mathscr{C}}} \in \mathcal{P}(\widetilde{X}_\eta) \iff \mathcal{P}(\widetilde{X}_{\eta_{\mathscr{C}}}) \subseteq \mathcal{P}(\widetilde{X}_\eta).\end{aligned} \tag{8}$$

In particular, an immediate consequence of this result is a list of conditions equivalent to $\mathscr{B} = \mathscr{C}$, whenever both $\mathscr{B}$ and $\mathscr{C}$ are taut, see [7, Theorem L].

Last but not least, let us mention some results related to the subset $\mathcal{P}^e(\widetilde{X}_\eta)$ of the ergodic measures on $\widetilde{X}_\eta$. It was shown in [23] that $\mathcal{P}(\widetilde{X}_\eta)$ is a Poulsen simplex (that is, a non-trivial simplex with dense subset of ergodic measures with respect to the weak-star topology) whenever $h(\widetilde{X}_\eta) > 0$. Recall that the density of ergodic measures implies the arcwise connectedness of the set of invariant measures [29] (the latter property was proved to hold in a hereditary setting wider than just $\mathscr{B}$-free systems in [20]). Recently, a yet stronger result was obtained by Konieczny, Kupsa and Kwietniak [21]: namely,

$$\text{the subset } \mathcal{P}^e(\widetilde{X}_\eta) \text{ of ergodic measures on } \widetilde{X}_\eta \text{ is entropy-dense in } \mathcal{P}(\widetilde{X}_\eta), \tag{9}$$

that is, for any $\mu \in \mathcal{P}(\widetilde{X}_\eta)$, there exist $\mu_n \in \mathcal{P}^e(\widetilde{X}_\eta)$ such that $\mu_n \to \mu$ weakly and the measure-theoretic entropies $h(\widetilde{X}_\eta, \sigma, \mu_n)$ of $(\widetilde{X}_\eta, \sigma, \mu_n)$ converge to the measure-theoretic entropy $h(\widetilde{X}_\eta, \sigma, \mu)$ of $(\widetilde{X}_\eta, \sigma, \mu)$.

Clearly, if X_η is hereditary, all of the above results apply to $X_\eta = \widetilde{X}_\eta$. We study analogous questions and prove the analogues of (2)–(9) for X_η in the non-hereditary case. For a summary of our results, see §1.3.

1.2. *Notation and main objects*

1.2.1. *Dynamics.* We say that (X, T) is a *topological dynamical system* if T is a homeomorphism of a compact metric space X. We equip X with the Borel sigma-algebra. The set of all probability Borel T-invariant measures will be denoted by $\mathcal{P}(X, T)$ (or just $\mathcal{P}(X)$ if T is clear from the context). The subset of *ergodic measures* will be denoted by $\mathcal{P}^e(X, T)$ or $\mathcal{P}^e(X)$. For each choice of $\mu \in \mathcal{P}(X)$, the triple (X, T, μ) is called a *measure-theoretic dynamical system*. Given two measure-theoretic dynamical systems (X, T, μ) and (Y, S, ν), we say that (Y, S, ν) is a *factor* of (X, T, μ) whenever there exists a measurable map $\pi : X \to Y$ (defined μ-almost everywhere (a.e.)) such that the image $\pi_*(\mu)$ of μ via π equals ν and $\pi \circ T = S \circ \pi$ μ-a.e.

Both in the measure-theoretic and in the topological setting, there is a notion of *entropy* that describes the complexity of a given system. The measure-theoretic entropy of (X, T, μ) is denoted by $h(X, T, \mu)$. We skip its lengthy definition and refer the reader, for example, to [5]. For a topological dynamical system (X, T), the topological entropy is

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denoted by $h(X)$ or $h(X, T)$. We will mostly deal with 0–1 *subshifts*, that is, (X, σ), where $X \subseteq \{0, 1\}^{\mathbb{Z}}$ is closed and invariant under the *left shift* $\sigma\colon \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$. In this case, the *topological entropy* is easy to define: if $p_n(X)$ is the number of distinct blocks of length n appearing on X, then $h(X) = \lim_{n\to\infty}(1/n) \log_2 p_n(X)$. If X is the orbit closure of $x \in \{0, 1\}^{\mathbb{Z}}$, we will write $p_n(x)$ instead of $p_n(X)$. There is the following *variational principle* (valid in general, not only for subshifts): $h(X, T) = \sup_{\mu\in\mathcal{P}(X,T)} h(X, T, \mu)$. In the case of subshifts, there is always at least one measure μ realizing the supremum from the variational principle. If this measure is unique, we say that X is *intrinsically ergodic*.

Given a topological dynamical system (X, T) and a point $x \in X$, we say that x is a *generic point* for $\mu \in \mathcal{P}(X, T)$ if $(1/L) \sum_{\ell\le L} \delta_{T^\ell x} \to \mu$ weakly. We say that $x \in X$ is *quasi-generic for μ along (ℓ_i)* if $(1/\ell_i) \sum_{\ell\le\ell_i} \delta_{T^\ell x} \to \mu$ weakly.

Given two measure-theoretic dynamical systems (X_i, T_i, μ_i), $i = 1, 2$, we say that $\rho \in \mathcal{P}(X_1 \times X_2, T_1 \times T_2)$ (with $X_1 \times X_2$ equipped with the product sigma-algebra) is a *joining* of (X_1, T_1, μ_1) and (X_2, T_2, μ_2), whenever $\mu_i = (\pi_i)_*(\rho)$ for $i = 1, 2$ (π_i will always denote the projection onto the ith coordinate, we will also use similar notation for projections onto more than one coordinate). We write then $\rho = \mu_1 \vee \mu_2$ or $\rho \in J((X_1, T_1, \mu_1), (X_2, T_2, \mu_2))$. We always have $\mu_1 \otimes \mu_2 \in J((X_1, T_1, \mu_1), (X_2, T_2, \mu_2))$. In fact, if (Y_i, S_i, ν_i) is a factor of (X_i, T_i, μ_i) via a factor map F_i, $i = 1, 2$ and $\rho = \nu_1 \vee \nu_2$, then there exists $\hat{\rho} \in J((X_1, T_1, \mu_1), (X_2, T_2, \mu_2))$ such that $(Y_1 \times Y_2, S_1 \times S_2, \rho)$ is a factor of $(X_1 \times X_2, T_1 \times T_2, \hat{\rho})$ via $F_1 \times F_2$ (for example, the so-called *relatively independent extension* of ρ has such a property). Last but not least, for $S\colon (X_1, T_1, \mu_1) \to (X_2, T_2, \mu_2)$, we will denote by $\triangle_S$ the graph joining of (X_2, T_2, μ_2) and (X_1, T_1, μ_1) given by $\triangle_S(A_2 \times A_1) = \mu_1(S^{-1}A_2 \cap A_1)$ for any measurable $A_1 \subseteq X_1$, $A_2 \subseteq X_2$. (Note that usually $\triangle_S$ stands for the joining of T_1 and T_2 where the coordinates are written in the opposite order.) For more information on joinings, we refer the reader to [11].

1.2.2. *Toeplitz systems.* A sequence $x \in \{0, 1\}^{\mathbb{Z}}$ is called *Toeplitz* if for each $i \in \mathbb{Z}$, there exists $s \in \mathbb{N}$ such that $x(i + sk) = x(i)$ for all $k \in \mathbb{Z}$. A *Toeplitz subshift* is the orbit closure of a Toeplitz sequence under the left shift. Any Toeplitz subshift is *minimal* [35] (the orbit of each point is dense). For each symbol $a \in \{0, 1\}$ and any $s \in \mathbb{N}$, we set

$$\mathrm{Per}(x, a, s) := \{i \in \mathbb{Z} \ :\ x(i + sk) = a \text{ for all } k \in \mathbb{Z}\}.$$

The *s-periodic part* of x is defined to be the set of positions

$$\mathrm{Per}(x, s) := \mathrm{Per}(x, 0, s) \cup \mathrm{Per}(x, 1, s).$$

A Toeplitz sequence x is called *regular* if

$$\lim_{r\to\infty} d\bigg(\bigcup_{s\le r} \mathrm{Per}(x, s)\bigg) = 1.$$

(Notice that this is equivalent to the usual definition via the so-called period structure.) The remaining Toeplitz sequences are called *irregular*. For any regular Toeplitz sequence, the corresponding Toeplitz subshift is uniquely ergodic, see [13, Theorem 5]. For more information on Toeplitz sequences, we refer the reader, for example, to the survey [4].

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1.2.3. *$\mathscr{B}$-free systems.* Since the notation differs a bit between various papers related to $\mathscr{B}$-free systems that are crucial for this work, we need to make certain adjustments.

Subshifts. First, let us recall the main subshifts that are of our interest. Given $\mathscr{B} \subseteq \mathbb{N}$, we consider

$$X_\eta = \overline{\{\sigma^k \eta : k \in \mathbb{Z}\}} \subseteq X_{\mathscr{B}} = \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp}\, x \bmod b| \le b-1 \text{ for each } b \in \mathscr{B}\},$$

where $\text{supp}\, x = \{n \in \mathbb{Z} : x(n) = 1\}$ stands for the support of x. They are called the *$\mathscr{B}$-free subshift* X_η and the *$\mathscr{B}$-admissible subshift* $X_{\mathscr{B}}$ corresponding to the set $\mathscr{B}$. Moreover, the so-called *hereditary closure* $\widetilde{X}_\eta$ of X_η is given by $\widetilde{X}_\eta = M(X_\eta \times \{0,1\}^{\mathbb{Z}})$, where $M\colon (\{0,1\}^{\mathbb{Z}})^2 \to \{0,1\}^{\mathbb{Z}}$ stands for the coordinatewise multiplication of sequences (this is equivalent to defining $\widetilde{X}_\eta$ as the smallest hereditary subshift containing X_η). Since $X_{\mathscr{B}}$ is hereditary, we have

$$X_\eta \subseteq \widetilde{X}_\eta \subseteq X_{\mathscr{B}}.$$

Usually, we will assume that $\mathscr{B}$ is *primitive*, that is, for any $b, b' \in \mathscr{B}$, if $b \mid b'$, then $b = b'$. This assumption has no influence on the studied dynamics since $\mathcal{M}_{\mathscr{B}} = \mathcal{M}_{\mathscr{B}^{\text{prim}}}$, where by $\mathscr{B}^{\text{prim}}$, we will denote the maximal primitive subset of $\mathscr{B}$.

In fact, there are also some other interesting subshifts of $X_{\mathscr{B}}$ that we will discuss in a later paragraph. Let us now give an overview of the most important classes of sets $\mathscr{B}$ appearing in the literature. We say that $\mathscr{B} \subseteq \mathbb{N}$ is:

- *Erdős* if $\mathscr{B}$ is infinite, pairwise coprime and $\sum_{b\in\mathscr{B}}(1/b) < \infty$;
- *Besicovitch* if $d(\mathcal{M}_{\mathscr{B}})$ exists;
- *taut* if for every $b \in \mathscr{B}$, we have $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\setminus\{b\}}) < \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})$;
- *Behrend* if $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) = 1$.

Recall (see [7, Theorem 3.7]) that any non-trivial Behrend set contains an infinite pairwise coprime subset. Moreover, $\mathscr{B}$ is taut if and only if $c\mathcal{A} \not\subseteq \mathscr{B}$ for any Behrend set $\mathcal{A}$ and any $c \in \mathbb{N}$, see [12].

Given $\mathscr{B} \subseteq \mathbb{N}$, we can now define the following.

- $\mathscr{B}' := (\mathscr{B} \cup C)^{\text{prim}}$, where

  $$C = \{c \in \mathbb{N} : c\mathcal{A} \subseteq \mathscr{B} \text{ for some Behrend set } \mathcal{A}\}.$$

  The set $\mathscr{B}'$ is called the *tautification of* $\mathscr{B}$, and it is the unique taut set such that $\nu_\eta = \nu_{\eta'}$ (see [7, 6] for more details about $\mathscr{B}'$).
- $\mathscr{B}^* := (\mathscr{B} \cup D)^{\text{prim}}$, where

  $$D = \{d \in \mathbb{N} : d\mathcal{A} \subseteq \mathscr{B} \text{ for some infinite pairwise coprime set } \mathcal{A}\}.$$

  The set $\mathscr{B}^*$ corresponds to the unique minimal subshift X_{η^*} of X_η (see [17, Corollary 5]). By [17, Lemma 3(c)], $\mathscr{B}^*$ does not contain a scaled copy of an infinite pairwise coprime subset. Thus, $\mathscr{B}^*$ does not contain a scaled copy of a Behrend set and, hence, $\mathscr{B}^*$ is taut (for another proof, see [14, Lemma 3.7]). Moreover, η^* is a Toeplitz sequence (see [14, Theorem B]) with a subsequence of $(\text{lcm}(\mathscr{B}^*_K))_{K\ge 1}$ being

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its period structure, which in particular means that η^* is a regular Toeplitz sequence if and only if

$$\lim_{K\to\infty} d(\mathbb{Z} \setminus \mathrm{Per}(\eta^*, \mathrm{lcm}(\mathscr{B}^*_K)) = 0. \tag{10}$$

(We will not need the notion of a period structure of a Toeplitz sequence, so let us skip it here and refer the reader to [4].)

We have

$$X_{\eta^*} \subseteq X_{\eta'} \subseteq X_\eta, \tag{11}$$

see [6, Remark 3.22] for the first inclusion, and [24, (27)] for the second one. Note also that it was shown earlier that $X_{\eta^*} \subseteq X_\eta$, see [17, Corollary 1.5]. We have

$$(\mathscr{B}')^* = \mathscr{B}^*. \tag{12}$$

Indeed, $X_{(\eta')^*}$ is the unique minimal subshift of $X_{\eta'}$, while X_{η^*} is the unique minimal subshift of X_η. Hence, since $X_{\eta'} \subseteq X_\eta$, it follows that $X_{(\eta')^*} = X_{\eta^*}$. This is equivalent to (12) by [7, Theorem L], cf. (8).

Basic algebraic objects. There are also certain important objects of algebraic nature related to $\mathscr{B}$:

- the product group $G := \prod_{b\in\mathscr{B}} \mathbb{Z}/b\mathbb{Z}$;
- the canonical embedding $\Delta\colon \mathbb{Z} \to G$ given by $\Delta(n) = (n, n, \ldots)$;
- the subgroup $H := \overline{\Delta(\mathbb{Z})}$;
- the rotation $R = R_{\Delta(1)}\colon H \to H$ given by $R(h) = h + \Delta(1)$;
- the window associated to $\mathscr{B}$, given by $W := \{h \in H : h_b \neq 0 \text{ for each } b \in \mathscr{B}\}$, and the closure of its interior, which we denote by $\underline{W} := \overline{\mathrm{int}(W)}$;
- the coding function $\varphi_A\colon H \to \{0, 1\}^{\mathbb{Z}}$ for $A \subseteq H$, given by $\varphi_A(h)(n) = 1 \iff h + \Delta(n) \in A$; note that $\varphi_A \circ R = \sigma \circ \varphi_A$; in particular, we will use
$$\varphi := \varphi_W \quad \text{and} \quad \underline{\varphi} := \varphi_{\underline{W}};$$
note that $\eta = \varphi(\Delta(0))$;
- the subset of admissible sequences with only one residue class mod each $b \in \mathscr{B}$ missing:
$$Y := \{x \in \{0, 1\}^Z : |\mathrm{supp}\, x \bmod b| = b - 1 \text{ for each } b \in \mathscr{B}\} \subseteq X_{\mathscr{B}};$$
- the function $\theta\colon Y \to G$ 'reading' the (unique) missing residue class mod each $b \in \mathscr{B}$, which is given by $\theta(y) = h \iff \mathrm{supp}\, y \cap (b\mathbb{Z} - h_b) = \emptyset$ for $b \in \mathscr{B}$.

All these objects can be defined just as well for $\mathscr{B}'$ and $\mathscr{B}^*$. We will use the superscripts $'$ and * to indicate which of them we mean. For example, we have $H' = \overline{\Delta'(\mathbb{Z})}$ where $\Delta'\colon \mathbb{Z} \to G'$ and similarly

$$W' := \{h \in H' : h_b \neq 0 \bmod b \text{ for each } b \in \mathscr{B}'\} \quad \text{and} \quad \underline{W'} = \overline{\mathrm{int}(W')}.$$

Also, we will write φ' for $\varphi'_{W'}$ and $\underline{\varphi'}$ for $\varphi'_{\underline{W'}}$.

Remark 1.1. Notice that the meaning of W' differs from the one used in [17]: Keller used W' for $\overline{\mathrm{int}(W)}$, which we denote as $\underline{W}$.

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Group homomorphisms. By [17, Lemma 1.2], there is a continuous surjective group homomorphism

$$\Gamma_{H,H^*} \colon H \to H^*$$

given by the unique continuous extension of the map $\Delta(n) \mapsto \Delta^*(n)$ to H. In fact, the following lemma provides a direct formula for Γ_{H,H^*} (by the definition of $\mathscr{B}^*$, for each b^*, there exists $b \in \mathscr{B}$ such that $b^* \mid b$).

LEMMA 1.2. *Let $h \in H$. Then $\Gamma_{H,H^*}(h)_{b^*} = h_b \bmod b^*$ for any $b \in \mathscr{B}$ and any $b^* \in \mathscr{B}^*$ such that $b^* \mid b$. In particular, $\Gamma_{H,H^*}(h)_b = h_b$ for any $b \in \mathscr{B} \cap \mathscr{B}^*$.*

Proof. Let $(n_k)_{k\geq 1}$ be such that $\lim_{k\to\infty} \Delta(n_k) = h$. Fix $b \in \mathscr{B}$. Then $n_k \bmod b = h_b$ for sufficiently large $k \geq 1$. Therefore, $n_k \bmod b^* = h_b \bmod b^*$ for any b^* such that $b^* \mid b$. The assertion follows by the continuity of Γ_{H,H^*}. □

Moreover, it was shown in [17, Lemma 1.4] that

$$\Gamma_{H,H^*}(\underline{W}) = W^* \quad \text{and} \quad \Gamma_{H,H^*}(H \setminus \underline{W}) = H^* \setminus W^*. \tag{13}$$

It follows that

$$\underline{\varphi}(h) = \varphi^*(\Gamma_{H,H^*}(h)). \tag{14}$$

Indeed, $\varphi^*(\Gamma_{H,H^*}(h))(n) = 1$ if and only if $\Gamma_{H,H^*}(h) + \Delta^*(n) = \Gamma_{H,H^*}(h + \Delta(n)) \in W^*$, which is equivalent to $h + \Delta(n) \in \underline{W}$ due to (13).

More subshifts. We will also need:

$$X_\varphi := \overline{\varphi(H)}$$

and

$$[\underline{\varphi}, \varphi] := \{x \in \{0,1\}^{\mathbb{Z}} : \underline{\varphi}(h) \leq x \leq \varphi(h) \text{ for some } h \in H\}. \tag{15}$$

The subshift X_φ first appeared in [19] (under the name $\mathcal{M}_W^G$) and was later studied in [14]. The set $[\underline{\varphi}, \varphi]$ that may not be a subshift (it is σ-invariant, but is not necessarily closed) was introduced in [17]. If $\mathscr{B}$ is primitive, then $\varphi(H) \subseteq [\underline{\varphi}, \varphi] \subseteq X_\varphi$ by [17, Theorem 1.1], so in particular,

$$X_\varphi = \overline{[\underline{\varphi}, \varphi]}. \tag{16}$$

Moreover, if $\mathscr{B}$ is taut, then by [17, Corollary 1.2], we have

$$X_\eta = X_\varphi = \overline{[\underline{\varphi}, \varphi]}. \tag{17}$$

Similar notation to that in (15) will be used for sequences. Given $w, x \in \{0,1\}^{\mathbb{Z}}$, we set

$$[w, x] := \{\sigma^m y \in \{0,1\}^{\mathbb{Z}} : w \leq y \leq x,\ m \in \mathbb{Z}\}.$$

Again, this may not be a subshift; one can consider its closure $\overline{[w, x]}$ if necessary.

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Remark 1.3. Let us comment here on the codomain of θ. Since θ is defined on whole Y, in general, we cannot say much more than that $\theta(y) \in G$. It was shown in [7, Remark 2.45] that $\theta(Y \cap \widetilde{X}_\eta) \subseteq H$. However, this is not sufficient for us: we need to think of θ as of a function from (at least) $Y \cap [\underline{\varphi}, \varphi]$ to H. We will show that

$$\theta(Y \cap \widetilde{X_\varphi}) = \theta(Y \cap X_\varphi) \subseteq H.$$

In the first equality, '$\supseteq$' follows from $\widetilde{X}_\varphi \supseteq X_\varphi$. For the converse inclusion, consider $y \in Y \cap \widetilde{X}_\varphi$ and $x \in X_\varphi = \overline{\varphi(H)}$ with $y \leq x$. Notice that $\overline{\varphi(H)} \subseteq X_{\mathscr{B}}$ since supp $\varphi(h)$ misses the residue class $-h_b$ modulo b for each $b \in \mathscr{B}$ and $X_{\mathscr{B}}$ is closed. Thus, supp x misses at least one residue class for each $b \in \mathscr{B}$. Due to $y \in Y$ and $y \leq x$, the support of x misses exactly one residue class for each b, namely the same as supp y. This yields $\theta(y) = \theta(x) \in \theta(Y \cap X_\varphi)$.

To see $\theta(Y \cap X_\varphi) \subseteq H$, we fix $b \in \mathscr{B}$ and $x \in X_\varphi = \overline{\varphi(H)}$. Then there exists a sequence $(\varphi(h_n))$ which converges to x, and (by definition) we have $\varphi(h_n)|_{-(h_n)_b+b\mathbb{Z}} = 0$. Since H is compact, we can assume that (h_n) has a limit $h \in H$. In particular, there exists $n_0 \in \mathbb{N}$ with $(h_n)_b = h_b$ for all $n \geq n_0$. This yields $\varphi(h_n)|_{-h_b+b\mathbb{Z}} = 0$ for all $n \geq n_0$ and thus $x|_{-h_b+b\mathbb{Z}} = 0$. For $x \in Y \cap X_\varphi$, it follows that $-h_b$ is the unique residue class modulo b that supp x misses. Since $b \in \mathscr{B}$ was arbitrary, we obtain $\theta(x) = h \in H$.

1.2.4. *Dynamical diagrams.* The aim of this section is to introduce a certain language related to diagrams involving dynamical systems and factoring maps between them. It will allow us to summarize some of our results on diagrams, which, in turn, can help to understand the structure of some more complicated proofs since the diagrams are easier to 'glue together' than the assertions written in the form of sentences. We will use the language of category theory. Namely, we consider the category where:

- the *objects* are triples of the form $(X, T, \mathcal{P}_X)$, where (X, T) is a topological dynamical system and $\emptyset \neq \mathcal{P}_X \subseteq \mathcal{P}(X)$; if $\mathcal{P}_X = \mathcal{P}(X)$, we skip it and write (X, T) instead of $(X, T, \mathcal{P}(X))$;
- a *morphism* from $(X, T, \mathcal{P}_X)$ to $(Y, S, \mathcal{P}_Y)$ is a map $f\colon (X, T, \mathcal{P}_X) \to (Y, S, \mathcal{P}_Y)$ such that there exist $X_0 \subseteq X$ where X_0 is T-invariant with $\mu(X_0) = 1$ for any $\mu \in \mathcal{P}_X$, $f\colon X_0 \to Y$, $f_*(\mathcal{P}_X) \subseteq \mathcal{P}_Y$ and $S \circ f = f \circ T$ on X_0.

Any graph whose vertices are the above-defined objects and whose arrows denote morphisms is called a *dynamical diagram*.

Remark 1.4. We identify two morphisms $f, g\colon (X, T, \mathcal{P}_X) \to (Y, S, \mathcal{P}_Y)$, whenever f and g agree on a subset $X_0 \subseteq X$ that is of full measure for every measure $\mu \in \mathcal{P}_X$.

Definition 1.5. We define the *composition* of morphisms $f\colon (X, T, \mathcal{P}_X) \to (Y, S, \mathcal{P}_Y)$ and $g\colon (Y, S, \mathcal{P}_Y) \to (Z, R, \mathcal{P}_Z)$ as the composition $g \circ f$. Notice that such a definition is correct in view of Remark 1.4. Indeed, let $f\colon X_0 \to Y$ and $g\colon Y_0 \to Z$, where $\mu(X_0) = 1$ for every $\mu \in \mathcal{P}_X$ and $\nu(Y_0) = 1$ for every $\nu \in \mathcal{P}_Y$. Then the composition $g \circ f$ is defined on $X_0 \cap f^{-1}(Y_0)$ and $\mu(X_0 \cap f^{-1}(Y_0)) = 1$ for any $\mu \in \mathcal{P}_X$.

Definition 1.6. We will say that a dynamical diagram *commutes* if for any choice of $(X, T, \mathcal{P}_X)$ and $(Y, S, \mathcal{P}_Y)$ in this diagram, the composition of morphisms along any

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path connecting $(X, T, \mathcal{P}_X)$ with $(Y, S, \mathcal{P}_Y)$ does not depend on the choice of the path, including the trivial (zero) path.

Remark 1.7. In the definition of commutativity, we implicitly assume that our diagram includes, for each vertex $(X, T, \mathcal{P}_X)$, the identity map $\mathrm{id}\colon (X, T, \mathcal{P}_X) \to (X, T, \mathcal{P}_X)$. To increase the readability of the diagrams, we will skip the corresponding arrow in our figures. Notice that this means in particular that whenever a dynamical diagram of the form

$$(X, T, \mathcal{P}_X) \underset{g}{\overset{f}{\rightleftarrows}} (Y, S, \mathcal{P}_Y)$$

is commutative, then $g \circ f = \mathrm{id}_X$ a.e. with respect to any $\mu \in \mathcal{P}_X$ and $f \circ g = \mathrm{id}_Y$ a.e. with respect to any $\nu \in \mathcal{P}_Y$. Note that usually, diagrams with loops do not appear in the context of commutative diagrams in category theory—they will however appear in the present paper.

Remark 1.8. In a commutative diagram for any pair of its vertices $(X, T, \mathcal{P}_X)$, $(Y, S, \mathcal{P}_Y)$, there is at most one morphism $f\colon (X, T, \mathcal{P}_X) \to (Y, S, \mathcal{P}_Y)$. Notice also that any linear dynamical diagram is automatically commutative (by a linear diagram, we mean any diagram whose underlying undirected graph consists of vertices arranged in a line). The same applies to any dynamical diagram that is of the form of a directed tree (a graph whose underlying undirected graph is a tree, that is, a connected acyclic undirected graph).

Definition 1.9. We will say that a morphism $f\colon (X, T, \mathcal{P}_X) \to (Y, S, \mathcal{P}_Y)$ is *surjective* if $f_*(\mathcal{P}_X) = \mathcal{P}_Y$. We will say that a dynamical diagram is *surjective* if every morphism in this diagram is surjective. If $(X, T, \mathcal{P}_X) \xrightarrow{f} (Y, S, \mathcal{P}_Y)$ is surjective, we will sometimes just say that (the morphism) f is surjective. Notice that this notion is not the same as the surjectivity of the map $f\colon X \to Y$.

Remark 1.10. (Cf. Remark 1.7) Any commutative dynamical diagram that is a loop is automatically surjective. For example, if

$$(X, T, \mathcal{P}_X) \underset{g}{\overset{f}{\rightleftarrows}} (Y, S, \mathcal{P}_Y)$$

is a commutative dynamical diagram, then it is surjective. Indeed, $\mathcal{P}_X = \mathrm{id}_*(\mathcal{P}_X) = g_*(f_*(\mathcal{P}_X)) \subseteq g_*(\mathcal{P}_Y) \subseteq \mathcal{P}_X$, so, in fact, $\mathcal{P}_X = g_*(\mathcal{P}_Y)$. By the same token, $\mathcal{P}_Y = f_*(\mathcal{P}_X)$.

Example 1.11

(1) Suppose that $\mathscr{B} \subseteq \mathbb{N}$ is taut. Then $\nu_\eta \in \mathcal{P}(X_\eta \cap Y)$ by [7, Theorem H], so $\mathcal{P}(X_\eta \cap Y) \neq \emptyset$. Thus,

$$(X_\eta \cap Y, \sigma) \xrightarrow{\theta} (H, R) \xrightarrow{\varphi} (X_\eta, \sigma)$$

is a dynamical diagram. Its subdiagram $(X_\eta \cap Y, \sigma) \xrightarrow{\theta} (H, R)$ is surjective (by the unique ergodicity of (H, R), we have $\theta_*(\nu) = m_H \in \mathcal{P}(H)$ for any $\nu \in \mathcal{P}(X_\eta \cap Y)$),

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while $(H, R) \xrightarrow{\varphi} (X_\eta, \sigma)$ is not surjective unless X_η is uniquely ergodic (cf. Corollary G in §1.3.1).

(2) The dynamical diagram

$$(\{0,1\}^{\mathbb{Z}}, \sigma) \underset{\sigma}{\overset{\sigma}{\rightleftarrows}} (\{0,1\}^{\mathbb{Z}}, \sigma)$$

does not commute: indeed, $\sigma \circ \sigma \neq \mathrm{id}$ (except at the four fixed points of σ^2). Notice, however, that if we equip each vertex with $\emptyset \neq \mathcal{P} \subseteq \{\delta_{\mathbf{0}}, \frac{1}{2}(\delta_{\ldots 10101\ldots} + \delta_{\ldots 01010\ldots}), \delta_{\mathbf{1}}\}$, then

$$(\{0,1\}^{\mathbb{Z}}, \sigma, \mathcal{P}) \underset{\sigma}{\overset{\sigma}{\rightleftarrows}} (\{0,1\}^{\mathbb{Z}}, \sigma, \mathcal{P})$$

becomes a commutative dynamical diagram (and thus it is surjective by Remark 1.10).

(3) If $\mathscr{B} \subseteq \mathbb{N}$ is taut, then

$$(X_\eta, \sigma, \{\nu_\eta\}) \underset{\varphi}{\overset{\varphi^{-1}}{\rightleftarrows}} (H, R)$$

is a commutative dynamical diagram (and thus it is surjective by Remark 1.10). Indeed, $\varphi : (H, R, m_H) \to (X_\eta, \sigma, \nu_\eta)$ is a measure-theoretic isomorphism, see [9] for the Erdős case and [7, Theorem F] for the taut case. The map φ^{-1} can be replaced with θ (recall that for taut $\mathscr{B}$, we have $\nu_\eta(X_\eta \cap Y) = 1$, so θ is well defined ν_η-a.e. on X_η).

(4) The diagram

$$((\{0,1\}^{\mathbb{Z}})^2, \sigma^{\times 2}, \{\nu_\eta \vee \kappa : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\}) \xrightarrow{M} (\widetilde{X}_\eta, \sigma)$$

is clearly a dynamical diagram (as $M \circ (\sigma \times \sigma) = \sigma \circ M$ everywhere). It is linear, hence commutative. Moreover, it is surjective by (6).

1.3. *Summary.* In this section, we present our main results. They are divided into three groups:

- results about invariant measures;
- combinatorial results related to the notion of tautness;
- entropy results.

We also discuss how to interpret some of them in terms of dynamical diagrams and indicate the main steps in their proofs.

1.3.1. *Main results: invariant measures.* In [17], Keller formulated a conjecture on the form of $\mathcal{P}(X_\eta)$. Let us restate it using our notation.

Conjecture 1. [17, Conjecture 1] Let $\mathscr{B} \subseteq \mathbb{N}$ be such that η^* is a regular Toeplitz sequence. Then for any $\nu \in \mathcal{P}(X_\varphi)$, there exists $\rho \in \mathcal{P}(H \times \{0,1\}^{\mathbb{Z}}, R \times \sigma)$ such that for any measurable $A \subseteq X_\varphi$,

$$\nu(A) = \int_{H \times \{0,1\}^{\mathbb{Z}}} \mathbf{1}_A(\underline{\varphi}(h) + x \cdot (\varphi(h) - \underline{\varphi}(h)))\, d\rho(h, x).$$

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In other words, for each $\nu \in \mathcal{P}(X_\varphi)$, we have $\nu = (M_H)_*(\rho)$ for some $\rho \in \mathcal{P}(H \times \{0, 1\}^{\mathbb{Z}}, R \times \sigma)$, where $M_H \colon H \times \{0, 1\}^{\mathbb{Z}} \to [\underline{\varphi}, \varphi]$ is given by

$$M_H(h, x) = \underline{\varphi}(h) + x \cdot (\varphi(h) - \underline{\varphi}(h)).$$

Notice that each $\rho \in \mathcal{P}(H \times \{0, 1\}^{\mathbb{Z}}, R \times \sigma)$ is a joining of m_H with some measure $\kappa \in \mathcal{P}(\{0, 1\}^{\mathbb{Z}})$, that is, $\rho = m_H \vee \kappa$. Our motivation for writing this paper was to prove the above conjecture. In fact, we will prove not only that all σ-invariant measures on X_φ are of the form $(M_H)_*(m_H \vee \kappa)$, but also that the opposite inclusion holds and that $\mathcal{P}(X_\eta) = \mathcal{P}(X_\varphi)$. Thus, we not only settle Keller's conjecture, but also answer his question from [17] about the existence of invariant measures supported on $X_\varphi \setminus X_\eta$ (there are no such measures). The following theorem that captures all of this is our main result.

THEOREM A. *For any $\mathscr{B} \subseteq \mathbb{N}$ such that η^* is a regular Toeplitz sequence, we have*

$$\mathcal{P}(X_\eta) = \mathcal{P}(X_\varphi) = \{(M_H)_*(m_H \vee \kappa) : \kappa \in \mathcal{P}(\{0, 1\}^{\mathbb{Z}})\}.$$

An auxiliary result, used to prove Theorem A, but also interesting on its own, is another description of the set $\mathcal{P}(X_\eta) = \mathcal{P}(X_\varphi)$.

THEOREM B. (Cf. (6)) *For any $\mathscr{B} \subseteq \mathbb{N}$ such that η^* is a regular Toeplitz sequence, we have*

$$\mathcal{P}(X_\eta) = \mathcal{P}(X_\varphi) = \{N_*((\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa) : \kappa \in \mathcal{P}(\{0, 1\}^{\mathbb{Z}})\},$$

where $N \colon (\{0, 1\}^{\mathbb{Z}})^3 \to \{0, 1\}^{\mathbb{Z}}$ is the map given by $N(w, x, y) = w + y \cdot (x - w)$ and $\nu_{\eta^*} \triangle \nu_\eta$ is the joining of ν_{η^*} with ν_η for which the pair (η^*, η) is quasi-generic along any sequence (ℓ_i) realizing the lower density of $\mathcal{M}_{\mathscr{B}}$.

Remark 1.12. Note that it is non-trivial that (η^*, η) is quasi-generic under $\sigma \times \sigma$ along (ℓ_i) realizing the lower density of $\mathcal{M}_{\mathscr{B}}$—this will be shown in course of the proof of Theorem B. In fact, we will describe the limit measure, see Lemma 2.3. Notice also that once a pair $(x, y) \in \{0, 1\}^{\mathbb{Z}}$ is quasi-generic under $\sigma \times \sigma$ for some measure ρ, then ρ is $(\sigma \times \sigma)$-invariant. Moreover, x and y are quasi-generic (along the same subsequence) for the marginals of ρ and thus, ρ is a joining of its marginals.

THEOREM C. (Cf. (7), recall also (11)) *For any $\mathscr{B} \subseteq \mathbb{N}$ such that η^* is a regular Toeplitz sequence, we have $\mathcal{P}(X_\eta) = \mathcal{P}(X_{\eta'})$.*

1.3.2. *Main results: tautness and combinatorics*

PROPOSITION D. (Cf. (8)) *Let $\mathscr{B} \subseteq \mathbb{N}$. Suppose that $\mathscr{C} \subseteq \mathbb{N}$ is taut. Then the following are equivalent:*

(a) $(\forall_{b\in\mathscr{B}}\, \exists_{c\in\mathscr{C}}\, c \mid b)$ *and* $(\forall_{c\in\mathscr{C}}\, \exists_{b^*\in\mathscr{B}^*} b^* \mid c)$;
(b) $\eta^* \leq \eta_{\mathscr{C}} \leq \eta$;
(c) $X_{\eta_{\mathscr{C}}} \subseteq X_\eta$;
(d) $\eta_{\mathscr{C}} \in X_\eta$;

(a′) $(\forall_{b'\in\mathscr{B}'}\, \exists_{c\in\mathscr{C}}\, c \mid b')$ *and* $(\forall_{c\in\mathscr{C}}\, \exists_{b^*\in\mathscr{B}^*}\, b^* \mid c)$;
(b′) $\eta^* \leq \eta_{\mathscr{C}} \leq \eta'$;
(c′) $X_{\eta_{\mathscr{C}}} \subseteq X_{\eta'}$;
(d′) $\eta_{\mathscr{C}} \in X_{\eta'}$;

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(e) $\nu_{\eta_{\mathscr{C}}} \in \mathcal{P}(X_\eta)$; (e′) $\nu_{\eta_{\mathscr{C}}} \in \mathcal{P}(X_{\eta'})$;
(f) $\mathcal{P}(X_{\eta_{\mathscr{C}}}) \subseteq \mathcal{P}(X_\eta)$; (f′) $\mathcal{P}(X_{\eta_{\mathscr{C}}}) \subseteq \mathcal{P}(X_{\eta'})$.

Given $\mathscr{B} \subseteq \mathbb{N}$, let

$$Taut(\mathscr{B}) := \{\mathscr{C} \subseteq \mathbb{N} : \mathscr{C} \text{ is taut and } X_{\eta_{\mathscr{C}}} \subseteq X_\eta\}.$$

Consider the following partial order $\prec$ on $Taut(\mathscr{B})$:

$$\mathscr{C}_1 \prec \mathscr{C}_2 \iff X_{\eta_{\mathscr{C}_1}} \subseteq X_{\eta_{\mathscr{C}_2}}.$$

Clearly, $\mathscr{B}', \mathscr{B}^* \in Taut(\mathscr{B})$. Moreover, $\mathscr{B}^*$ is the smallest element of $Taut(\mathscr{B})$. Indeed, if $\mathscr{C} \in Taut(\mathscr{B})$, then $X_{\eta^*} \subseteq X_{\eta_{\mathscr{C}}}$ since X_{η^*} is the unique minimal subset of X_η. As an immediate consequence of Proposition D (more precisely, by $(c) \iff (c')$), we obtain the following.

COROLLARY E. *For any $\mathscr{B} \subseteq \mathbb{N}$, $\mathscr{B}'$ is the largest element of $Taut(\mathscr{B})$ with respect to $\prec$.*

1.3.3. *Main results: entropy.* Last, but not least, we prove some results on the entropy of $\mathscr{B}$-free systems.

THEOREM F. (Cf. (3)) *For any $\mathscr{B} \subseteq \mathbb{N}$, we have $h(X_\eta) \geq \overline{d} - \overline{d}^*$, where $\overline{d} = \overline{d}(\mathcal{F}_{\mathscr{B}})$ and $\overline{d}^* = \overline{d}(\mathcal{F}_{\mathscr{B}^*})$. If additionally X_{η^*} is uniquely ergodic (in particular, if η^* is a regular Toeplitz sequence), then $h(X_\eta) = \overline{d} - d^*$, where $d^* = d(\mathcal{F}_{\mathscr{B}^*})$.*

COROLLARY G. (Cf. (5)) *For any $\mathscr{B} \subseteq \mathbb{N}$ such that η^* is a regular Toeplitz sequence, we have*

$$h(X_\eta) = 0 \iff \mathcal{P}(X_\eta) = \{\nu_\eta\} \iff X_\eta \text{ is uniquely ergodic}$$

(note that if the above holds, then $\nu_\eta = \nu_{\eta^*}$).

Remark 1.13. In Corollary G, the second equivalence is true without any assumption on η^*. It seems open whether there exists $\mathscr{B}$ such that η^* is an irregular Toeplitz sequence with $h(X_\eta) = 0$ and $\mathcal{P}(X_\eta)$ being not a singleton, cf. Remark 1.14 below.

THEOREM H. (Cf. (2) and (4)) *For any $\mathscr{B} \subseteq \mathbb{N}$ such that η^* is a regular Toeplitz sequence, the subshift X_η is intrinsically ergodic. The measure of maximal entropy equals $N_*((\nu_{\eta^*} \triangle \nu_\eta) \otimes B_{1/2,1/2})$.*

THEOREM I. (Cf. (9)) *For any $\mathscr{B} \subseteq \mathbb{N}$ such that η^* is a regular Toeplitz sequence, the ergodic measures are entropy-dense in $\mathcal{P}(X_\eta)$.*

1.3.4. *Dynamical diagrams viewpoint.* In this section, we present a dynamical diagrams viewpoint on Theorems B, C, A and H. The first three of these results can be formulated in terms of dynamical diagrams and the structure of the proofs also relies on this notion. As for Theorem H, the dynamical diagrams serve as a tool in the proof.

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On Theorems B and C. These two results can be proved separately (Theorem C is then a consequence of Theorem B); however, there is a nice way to treat them together, which has the additional advantage of slightly shortening the proofs. Recall that by (11), (16) and (17), we have

$$\overline{[\underline{\varphi'}, \varphi']} = X_{\varphi'} = X_{\eta'} \subseteq X_\eta \subseteq X_\varphi = \overline{[\underline{\varphi}, \varphi]}.$$

Moreover, under the extra assumption that η^* is a regular Toeplitz sequence, by [17, Remark 1.4] and observing (12), we have

$$\mathcal{P}(X_\varphi) = \mathcal{P}([\underline{\varphi}, \varphi]) \quad \text{and} \quad \mathcal{P}(X_{\varphi'}) = \mathcal{P}([\underline{\varphi'}, \varphi']). \tag{18}$$

Remark 1.14. The first equality in (18) is actually the main reason for the extra assumption on η^* in Keller's conjecture from [17] (see Remark 1.4 therein). In fact, this goes deeper. If η^* is a regular Toeplitz sequence, then $\mathcal{P}(X_{\eta^*}) = \{\nu_{\eta^*}\}$, while when we drop the assumption on η^*, various things can happen to $\mathcal{P}(X_{\eta^*})$: it can be a singleton consisting only of ν_{η^*}, see [18, Theorem 2] (even if η^* is an irregular Toeplitz sequence!), but it can also contain some positive entropy measure, see [18, Theorem 1]. Thus, since $X_{\eta^*} \subseteq X_\eta$, we cannot expect to obtain a consistent description of $\mathcal{P}(X_\eta)$ without imposing any restrictions on η^*. We will use the fact that the Toeplitz sequence η^* is regular very frequently in our proofs.

Continuing our argument from (18), we obtain for a regular Toeplitz sequences η^* that

$$\begin{aligned}\mathcal{P}([\underline{\varphi'}, \varphi']) &= \mathcal{P}(\overline{[\underline{\varphi'}, \varphi']}) = \mathcal{P}(X_{\varphi'}) = \mathcal{P}(X_{\eta'}) \\ &\subseteq \mathcal{P}(X_\eta) \subseteq \mathcal{P}(X_\varphi) = \mathcal{P}(\overline{[\underline{\varphi}, \varphi]}) = \mathcal{P}([\underline{\varphi}, \varphi]).\end{aligned} \tag{19}$$

Therefore, the assertions of Theorems B and C are equivalent to the following two inclusions:

$$\mathcal{P}([\underline{\varphi}, \varphi]) \subseteq \{N_*((\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa) : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\} \subseteq \mathcal{P}(X_{\varphi'}). \tag{20}$$

Consider the following diagram:

$$\begin{array}{c} ((\{0,1\}^{\mathbb{Z}})^3, \sigma^{\times 3}, \{(\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\}) \\ \Big\downarrow N \\ ([\underline{\varphi}, \varphi], \sigma) \\ \Big\downarrow \mathrm{id} \\ (X_{\varphi'}, \sigma) \end{array} \tag{$\mathbf{D}_{B,C}$}$$

and notice that the assertions of Theorems B and C are equivalent to ($\mathbf{D}_{B,C}$) being a surjective commutative dynamical diagram. Indeed:

- ($\mathbf{D}_{B,C}$) is a dynamical diagram if and only if the maps N and id are morphisms, which implies the second inclusion in (20) holds.

Notice that by (19), the map id is a morphism if and only if $\mathcal{P}([\underline{\varphi}, \varphi]) = \mathcal{P}(X_{\varphi'})$. Therefore, id is then automatically surjective.

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- ($\mathbf{D}_{B,C}$) is in addition surjective if and only if the morphism N is surjective, which implies the first inclusion in equation (20) holds.

Moreover, if both inclusions in (20) hold, then by (19), we obtain the equality $\{N_*((\nu_{\eta^*}\triangle\nu_\eta)\vee\kappa) : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\} = \mathcal{P}([\underline{\varphi},\varphi]) = \mathcal{P}(X_{\varphi'})$, which implies that N and id are surjective morphisms.

On Theorem A. Having proved Theorem B first, to prove Theorem A, we will only need to show that

$$\{N_*((\nu_{\eta*}\triangle\nu_\eta)\vee\kappa) : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\} = \{(M_H)_*(m_H\vee\kappa) : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\}.$$

Let $\underline{\varphi}\otimes\varphi\colon H\to(\{0,1\}^{\mathbb{Z}})^2$ and $(\underline{\varphi}\otimes\varphi)(h) = (\underline{\varphi}(h),\varphi(h))$. Consider the following diagram:

$$\begin{tikzcd}
(H\times\{0,1\}^{\mathbb{Z}}, R\times\sigma) \arrow[d, "(\underline{\varphi}\otimes\varphi)\times\mathrm{id}"] \arrow[dd, bend left, "M_H"] \\
((\{0,1\}^{\mathbb{Z}})^3, \sigma^{\times 3}, \{(\nu_{\eta^*}\triangle\nu_\eta)\vee\kappa : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\}) \arrow[d, "N"] \\
(X_\eta, \sigma)
\end{tikzcd} \tag{$\mathbf{D}_A$}$$

Then:

- if ($\mathbf{D}_A$) is a commutative diagram, then

$$\{(M_H)_*(m_H\vee\kappa) : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\} \subseteq \{N_*((\nu_{\eta*}\triangle\nu_\eta)\vee\kappa) : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\}$$

  (indeed, by the commutativity, 'travelling' via M_H is the same as 'travelling' first via $(\underline{\varphi}\otimes\varphi)\times\mathrm{id}$ and then via N);
- if ($\mathbf{D}_A$) is surjective, then

$$\{N_*((\nu_{\eta*}\triangle\nu_\eta)\vee\kappa) : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\} \subseteq \{(M_H)_*(m_H\vee\kappa) : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\}$$

  (indeed, we can travel up from $N_*((\nu_{\eta*}\triangle\nu_\eta)\vee\kappa)$ to $(\nu_{\eta*}\triangle\nu_\eta)\vee\kappa$ via N, then again up by $(\underline{\varphi}\otimes\varphi)\times\mathrm{id}$, that is, use the surjectivity of $(\underline{\varphi}\otimes\varphi)\times\mathrm{id}$ and finally use that $M_H = N\circ((\underline{\varphi}\otimes\varphi)\times\mathrm{id})$ as ($\mathbf{D}_A$) commutes).

In other words, the assertion of Theorem A follows from Theorem B and the commutativity and the surjectivity of ($\mathbf{D}_A$).

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In fact, Theorems A, B and C can be summarized using a single diagram, namely:

$$\begin{tikzcd}
(H \times \{0,1\}^{\mathbb{Z}}, R \times \sigma) \arrow[d, "(\underline{\varphi} \otimes \varphi) \times \mathrm{id}"] \arrow[dd, bend left=60, "M_H"] \\
((\{0,1\}^{\mathbb{Z}})^3, \sigma^{\times 3}, \{(\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\}) \arrow[d, "N"] \\
([\underline{\varphi}, \varphi], \sigma) \arrow[d, "\mathrm{id}"] \\
(X_{\varphi'}, \sigma)
\end{tikzcd}$$

Notice that if we prove that the above diagram is a commutative and surjective dynamical diagram, then indeed we get:

- $\mathcal{P}(X_{\varphi'}) = \mathcal{P}(X_{\eta'}) = \mathcal{P}(X_\eta) = \mathcal{P}([\underline{\varphi}, \varphi])$;
- $\mathcal{P}(X_\eta) = \{N_*((\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa) : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\}$
  $= \{(M_H)_*(m_H \vee \kappa) : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\}$.

On Theorem H. The main idea of the proof of Theorem H is to equip the diagram

$$(H \times \{0,1\}^{\mathbb{Z}}, R \times \sigma) \xrightarrow{M_H} (X_\eta, \sigma)$$

(which is surjective by Theorem A) with an 'intermediate' vertex:

$$(H \times \{0,1\}^{\mathbb{Z}}, R \times \sigma) \xrightarrow{\Psi} (H \times \{0,1\}^{\mathbb{Z}}, \widetilde{R}) \xrightarrow{\Phi} (X_\eta, \sigma),$$

where $\widetilde{R}$ is a certain skew product over $R \colon H \to H$ and the maps Φ and Ψ are morphisms defined later. We prove then that $h(H \times \{0,1\}^{\mathbb{Z}}, \widetilde{R}) = \overline{d} - d^*$ (which equals to $h(X_\eta)$ by Theorem F) and prove the intrinsic ergodicity of $(H \times \{0,1\}^{\mathbb{Z}}, \widetilde{R})$. For the details, see §4.2.

2. *Invariant measures*

Before we begin working on the description of $\mathcal{P}(X_\eta)$, let us concentrate on X_η itself. Keller [17] proved that for any taut set $\mathscr{B}$, the subshift X_η is in a way 'hereditary'. We rephrase his result in the following way.

Proposition 2.1. *For any $\mathscr{B} \subseteq \mathbb{N}$, we have $X_\eta \subseteq \overline{[\eta^*, \eta]} \subseteq X_\varphi$. In particular, if $\mathscr{B}$ is taut, $X_\eta = \overline{[\eta^*, \eta]} = X_\varphi$.*

Proof. Clearly, $\varphi(\Delta(0)) = \eta \in [\eta^*, \eta]$. Moreover, by [17, Corollary 1.4], we have $\eta^* = \underline{\varphi}(\Delta(0))$. Thus, $[\eta^*, \eta] \subseteq [\underline{\varphi}, \varphi]$. This yields

$$X_\eta \subseteq \overline{[\eta^*, \eta]} \subseteq \overline{[\underline{\varphi}, \varphi]} = X_\varphi.$$

By [17, Corollary 1.2], if $\mathscr{B}$ is taut, we have $X_\eta = X_\varphi$, which completes the proof. $\square$

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2.1. *Proof of Theorems B and C*

2.1.1. *($\mathbf{D}_{B,C}$) is a (commutative) dynamical diagram.* We will need a certain lemma from [2] about 'lifting' quasi-generic points to joinings. We formulate it here for $\mathbb{Z}$-actions, while the original version is more general (the result is true for actions of countable cancellative semigroups and arbitrary Følner sequences).

THEOREM 2.2. [2, Theorem 5.16] *Let $\mathcal{A}_1, \mathcal{A}_2$ be finite alphabets. If $x \in \mathcal{A}_1^{\mathbb{Z}}$ is quasi-generic for ν along (ℓ_i) and $\nu \vee \kappa \in \mathcal{P}(\mathcal{A}_1^{\mathbb{Z}} \times \mathcal{A}_2^{\mathbb{Z}}, \sigma \times \sigma)$, then there exists $y \in \mathcal{A}_2^{\mathbb{Z}}$ such that the pair (x, y) is quasi-generic for $\nu \vee \kappa$ along some subsequence of (ℓ_i).*

Let (ℓ_i) be a sequence realizing the lower density of $\mathcal{M}_{\mathscr{B}}$ and suppose that X_{η^*} is uniquely ergodic (in particular, this happens if η^* is a regular Toeplitz sequence). If the pair (η^*, η) is quasi-generic along a subsequence (ℓ_{i_j}) of (ℓ_i) for some measure, then this limit measure must be a joining of ν_{η^*} and ν_η. In fact, we have the following lemma which we will prove in a moment.

LEMMA 2.3. *Let $\mathscr{B} \subseteq \mathbb{N}$ be such that X_{η^*} is uniquely ergodic. Let (ℓ_i) be any sequence realizing the lower density of $\mathcal{M}_{\mathscr{B}}$. Then the point (η^*, η) is quasi-generic along (ℓ_i) for $(\underline{\varphi} \otimes \varphi)_*(m_H)$.*

Remark 2.4. Instead of $(\underline{\varphi} \otimes \varphi)_*(m_H)$, we will usually write $\nu_{\eta^*} \triangle \nu_\eta$. In this subsection, we will only use that (η^*, η) is quasi-generic along (ℓ_i), while the specific form of the limit measure will be used later. Let us justify here our notation $\nu_{\eta^*} \triangle \nu_\eta$ and show that this is a certain off-diagonal joining with marginals ν_{η^*} and ν_η. Indeed, by (14), we have

$$\nu_{\eta^*} \triangle \nu_\eta = ((\varphi^* \circ \Gamma_{H,H^*}) \otimes \varphi)_*(m_H).$$

Notice that

$$S := \varphi^* \circ \Gamma_{H,H^*} \circ \varphi^{-1} \colon (\{0,1\}^{\mathbb{Z}}, \sigma, \nu_\eta) \to (\{0,1\}^{\mathbb{Z}}, \sigma, \nu_{\eta^*})$$

is a factoring map. Moreover, for any measurable sets $A, B \subseteq \{0,1\}^{\mathbb{Z}}$, we have

$$\begin{aligned}\triangle_S(A \times B) &= \nu_\eta((\varphi^* \circ \Gamma_{H,H^*} \circ \varphi^{-1})^{-1}(A) \cap B) \\ &= m_H((\varphi^* \circ \Gamma_{H,H^*})^{-1}(A) \cap \varphi^{-1}(B)) = ((\varphi^* \circ \Gamma_{H,H^*}) \otimes \varphi)_*(m_H)(A \times B) \\ &= \nu_{\eta^*} \triangle \nu_\eta(A \times B).\end{aligned}$$

Recall also that it was shown in [7] that

$$\eta \text{ and } \eta' \text{ differ along } (\ell_i)_{i \geq 1} \text{ on a subset of zero density,} \tag{21}$$

where $(\ell_i)_{i \geq 1}$ is any sequence realizing the lower density of $\mathcal{M}_{\mathscr{B}'}$. Moreover (see the proof of [7, Lemma 4.11]), any sequence (ℓ_i) realizing the lower density of $\mathcal{M}_{\mathscr{B}'}$ is also realizing the lower density of $\mathcal{M}_{\mathscr{B}}$. For any such $(\ell_i)_{i \geq 1}$, also

$$(\eta^*, \eta') \text{ is quasi-generic for } \nu_{\eta^*} \triangle \nu_\eta \text{ along } (\ell_i), \tag{22}$$

whenever X_{η^*} is uniquely ergodic.

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Let us now prove that ($\mathbf{D}_{B,C}$) is indeed a dynamical diagram (and since it is linear, it is then commutative by Remark 1.8). Notice that it suffices to show that for any measure of the form $N_*(\rho)$, where $\rho = (\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa$ with $\kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})$, we have $N_*(\rho)(X_{\varphi'}) = 1$. To see that this is indeed the case, fix such a measure ρ. It follows by Theorem 2.2 and by (22) that ρ has a quasi-generic point of the form (η^*, η', y) with $y \in \{0,1\}^{\mathbb{Z}}$. Therefore, $z := N(\eta^*, \eta', y)$ is quasi-generic for $N_*(\rho)$ and thus $N_*(\rho)(X_z) = 1$. It remains to notice that $\eta^* \leq z \leq \eta'$. Thus,

$$X_z \subseteq \overline{[\eta^*, \eta']} = X_{\varphi'},$$

where the last equality follows from Proposition 2.1.

Proof of Lemma 2.3. Fix (ℓ_i) which realizes the lower density of $\mathcal{M}_{\mathscr{B}}$. By a pure measure theory argument (see the proof of [9, Theorem 4.1]), we only need to prove that

$$\frac{1}{\ell_i} \sum_{n \leq \ell_i} \delta_{(\sigma^n \underline{\varphi}(\Delta(0)), \sigma^n \varphi(\Delta(0)))}(\underline{A} \times A) \to (\underline{\varphi} \otimes \varphi)_*(m_H)(\underline{A} \times A)$$

for

$$\underline{A} = \{x \in \{0,1\}^{\mathbb{Z}} : x|_{\underline{S}} \equiv 0\} \quad \text{and} \quad A = \{x \in \{0,1\}^{\mathbb{Z}} : x|_S \equiv 0\},$$

with $\underline{S}, S \subseteq \mathbb{Z}$ being arbitrary finite sets. By $\sigma \circ \underline{\varphi} = \underline{\varphi} \circ R$ and $\sigma \circ \varphi = \varphi \circ R$, this is equivalent to proving that

$$\lim_{i \to \infty} \frac{1}{\ell_i} \sum_{n \leq \ell_i} \mathbf{1}_{\underline{\varphi}^{-1}(\underline{A}) \times \varphi^{-1}(A)}(R^n(\Delta(0)), R^n(\Delta(0))) = (\underline{\varphi} \otimes \varphi)_*(m_H)(\underline{A} \times A).$$

The main underlying idea is to approximate $\underline{\varphi}^{-1}(\underline{A})$ and $\varphi^{-1}(A)$ by clopen sets, so that we can use the ergodicity properties of rotations. We will begin with the right-hand side, as it is easier (the approximation of the left-hand side requires the use of the Davenport–Erdős theorem, i.e. (1)).

Approximation of the right-hand side. We have

$$C := \varphi^{-1}(A) = \bigcap_{s \in S} R^{-s} W^c.$$

Let, for $K \geq 1$,

$$W_K := \{h \in H : h_b \neq 0 \text{ for all } b \in \mathscr{B}_K\}.$$

Let

$$C_K := \bigcap_{s \in S} R^{-s} W_K^c. \tag{23}$$

Each W_K is clopen and $W_K \searrow W$ when $K \to \infty$. Thus, given $\varepsilon > 0$, for K large enough, we have

$$m_H(C \triangle C_K) < \varepsilon. \tag{24}$$

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Recall from (14) that $\underline{\varphi} = \varphi^* \circ \Gamma_{H,H^*}$ and let

$$\underline{C} := \underline{\varphi}^{-1}(\underline{A}) = \Gamma^{-1}_{H,H^*}((\varphi^*)^{-1}(\underline{A})) = \Gamma^{-1}_{H,H^*} \bigcap_{s\in\underline{S}} (R^*)^{-s}(W^*)^c = \bigcap_{s\in\underline{S}} \Gamma^{-1}_{H,H^*}(R^*)^{-s}(W^*)^c.$$

Define W^*_K in a similar way as W_K:

$$W^*_K = \{h^* \in H^* : h^*_{b^*} \neq 0 \text{ for all } b^* \in \mathscr{B}^*_K\}.$$

Finally, let

$$\underline{C}_K := \bigcap_{s\in\underline{S}} \Gamma^{-1}_{H,H^*}(R^*)^{-s}(W^*_K)^c. \tag{25}$$

Then, for K large enough,

$$m_H(\underline{C} \triangle \underline{C}_K) < \varepsilon. \tag{26}$$

Notice that

$$(\underline{\varphi} \otimes \varphi)_*(m_H)(\underline{A} \times A) = m_H(\underline{\varphi}^{-1}(\underline{A}) \cap \varphi^{-1}(A)) = m_H(\underline{C} \cap C).$$

Thus, it follows by (24) and (26) that

$$|(\underline{\varphi} \otimes \varphi)_*(m_H)(\underline{A} \times A) - m_H(\underline{C}_K \cap C_K)| \leq 2\varepsilon$$

for K sufficiently large.

Approximation of the left-hand side. Let $(\ell_i)_{i\geq 1}$ be a sequence realizing the lower density of $\mathcal{M}_{\mathscr{B}}$. By definition,

$$\frac{1}{\ell_i} \sum_{n\leq \ell_i} \mathbf{1}_{\underline{A}\times A}(\underline{\varphi}(R^n(\Delta(0))), \varphi(R^n(\Delta(0)))) = \frac{1}{\ell_i} \sum_{n\leq \ell_i} \mathbf{1}_{\underline{C}}(R^n(\Delta(0)))\mathbf{1}_C(R^n(\Delta(0))).$$

Moreover, for K large enough,

$$\begin{aligned}
&\lim_{i\to\infty}\left|\frac{1}{\ell_i}\sum_{n\leq \ell_i} \mathbf{1}_{\underline{C}}(R^n(\Delta(0)))\mathbf{1}_C(R^n(\Delta(0))) - \frac{1}{\ell_i}\sum_{n\leq \ell_i} \mathbf{1}_{\underline{C}}(R^n(\Delta(0)))\mathbf{1}_{C_K}(R^n(\Delta(0)))\right| \\
&\quad \leq \lim_{i\to\infty} \frac{1}{\ell_i}\sum_{n\leq \ell_i} \mathbf{1}_{C\setminus C_K}(R^n(\Delta(0))) \leq \lim_{i\to\infty} \sum_{s\in S} \frac{1}{\ell_i}\sum_{n\leq \ell_i} \mathbf{1}_{W^c\setminus W^c_K}(R^{n+s}(\Delta(0))) \\
&\quad = |S| \cdot \lim_{i\to\infty} \frac{1}{\ell_i}\sum_{n\leq \ell_i} \mathbf{1}_{W^c\setminus W^c_K}(R^n(\Delta(0))) = |S| \cdot \lim_{i\to\infty} \frac{1}{\ell_i} |[1,\ell_i] \cap (\mathcal{M}_{\mathscr{B}} \setminus \mathcal{M}_{\mathscr{B}_K})| < \varepsilon,
\end{aligned}$$

where the second inequality follows from

$$C \setminus C_K \subseteq \bigcup_{s\in S} R^{-s}(W^c \setminus W^c_K),$$

the last equality from

$$R^n(\Delta(0)) \in W^c \setminus W^c_K \iff n \in \mathcal{M}_{\mathscr{B}} \setminus \mathcal{M}_{\mathscr{B}_K}$$

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and the last inequality is a consequence of the Davenport–Erdős theorem (i.e. (1))—we use that $(\ell_i)_{i\geq 1}$ is a specific sequence and that K is large only for this last inequality.

We will now use similar arguments for C_K and $\underline{C}_K$ instead of C and $\underline{C}$. We have

$$\begin{aligned}
&\lim_{i\to\infty}\bigg|\frac{1}{\ell_i}\sum_{n\leq\ell_i}\mathbf{1}_{\underline{C}}(R^n(\Delta(0)))\mathbf{1}_{C_K}(R^n(\Delta(0)))-\frac{1}{\ell_i}\sum_{n\leq\ell_i}\mathbf{1}_{\underline{C}_K}(R^n(\Delta(0)))\mathbf{1}_{C_K}(R^n(\Delta(0)))\bigg|\\
&\quad\leq\lim_{k\to\infty}\frac{1}{\ell_i}\sum_{n\leq\ell_i}\mathbf{1}_{\underline{C}\setminus\underline{C}_K}(R^n(\Delta(0)))\\
&\quad\leq|\underline{S}|\cdot\lim_{i\to\infty}\frac{1}{\ell_i}\sum_{n\leq\ell_i}\mathbf{1}_{\Gamma^{-1}_{H,H^*}((W^*)^c\setminus(W^*_K)^c)}(R^n(\Delta(0)))\\
&\quad=|\underline{S}|\cdot\lim_{i\to\infty}\frac{1}{\ell_i}\sum_{n\leq\ell_i}\mathbf{1}_{(W^*)^c\setminus(W^*_K)^c}((R^*)^n(\Delta^*(0)))\\
&\quad=|\underline{S}|\cdot d(\mathcal{M}_{\mathscr{B}^*}\setminus\mathcal{M}_{\mathscr{B}^*_K})<\varepsilon,
\end{aligned}$$

where in the second inequality, we used

$$\underline{C}\setminus\underline{C}_K\subseteq\bigcup_{s\in\underline{S}}R^{-s}\Gamma^{-1}_{H,H^*}((W^*)^c\setminus(W^*_K)^c),$$

the first equality follows from $\Gamma_{H,H^*}(R^n\Delta(0))=(R^*)^n\Gamma_{H,H^*}(\Delta(0))=(R^*)^n(\Delta^*(0))$, the second equality is a consequence of

$$(R^*)^n(\Delta^*(0))\in(W^*)^c\setminus(W^*_K)^c\iff n\in\mathcal{M}_{\mathscr{B}^*}\setminus\mathcal{M}_{\mathscr{B}^*_K}$$

and the last inequality follows by (1), i.e. the Davenport–Erdős theorem (notice that we use here that X_{η^*} is uniquely ergodic, so, in particular, $\mathscr{B}^*$ is Besicovitch and thus the density of $\mathcal{M}_{\mathscr{B}^*}$ along (ℓ_i) is just its natural density).

Convergence for clopen sets. After the above reductions, it remains to prove that

$$\lim_{i\to\infty}\frac{1}{\ell_i}\sum_{n\leq\ell_i}\mathbf{1}_{\underline{C}_K}(R^n(\Delta(0)))\mathbf{1}_{C_K}(R^n(\Delta(0)))=m_H(\underline{C}_K\cap C_K).$$

However, both C_K and $\underline{C}_K$ are clopen (recall (23) and (25)) and thus the claim follows directly by the unique ergodicity of R. □

2.1.2. *($\mathbf{D}_{B,C}$) is surjective.* This part of the proof relies mostly on certain natural periodic approximations of η and η^*. More precisely, we will need a periodic approximation of η from above and of η^* from below.

For each $K\geq 1$, we set $\mathscr{B}_K:=\{b\in\mathscr{B}:b\leq K\}$ and $\mathscr{B}^*_K=\{b^*\in\mathscr{B}^*:b^*\leq K\}$. We define $\varphi_K: H\to\{0,1\}^{\mathbb{Z}}$ by

$$\varphi_K(h)(n)=1\iff(R^nh)_b\neq 0\quad\text{for all } b\in\mathscr{B}_K.$$

Recall that there is a continuous group homomorphism $\Gamma_{H,H^*}: H \to H^*$ with $\underline{\varphi}(h) = \varphi^*(\Gamma_{H,H^*}(h))$, see (14). We define $\underline{\varphi}_K : H \to \{0,1\}^{\mathbb{Z}}$ by

$$\underline{\varphi}_K(h)(n) = \begin{cases} \underline{\varphi}(h)(n) & \text{if } n \in \operatorname{Per}(\underline{\varphi}(h), \operatorname{lcm}(\mathscr{B}_K^*)), \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} 1 & \text{if } n \in \operatorname{Per}(\varphi^*(\Gamma_{H,H^*}(h)), 1, \operatorname{lcm}(\mathscr{B}_K^*)), \\ 0 & \text{otherwise.} \end{cases}$$

Note that for every $h \in H$, we have

$$\underline{\varphi}_K(h) \le \underline{\varphi}(h) \le \varphi(h) \le \varphi_K(h). \tag{27}$$

LEMMA 2.5. *For any $K \ge 1$, the functions φ_K and $\underline{\varphi}_K$ depend on a finite number of coordinates. In particular, they are continuous.*

Proof. For φ_K, the assertion is clear by the very definition. Let us now turn to $\underline{\varphi}_K$. To shorten notation, we will write $h^* = \Gamma_{H,H^*}(h)$ and $s^* = \operatorname{lcm}(\mathscr{B}_K^*)$. We will show that $\operatorname{Per}(\varphi^*(h_1^*), 1, s^*) = \operatorname{Per}(\varphi^*(h_2^*), 1, s^*)$ whenever h_1^* and h_2^* agree on $\mathscr{B}_K^*$. Since there exists $L \in \mathbb{N}$ such that every $b^* \in \mathscr{B}_K^*$ divides some $b \in \mathscr{B}_L$, by Lemma 1.2, it then follows that $\underline{\varphi}_K(h)$ is determined by $(h_b)_{b\in\mathscr{B}_L}$. To see that $\operatorname{Per}(\varphi^*(h^*), 1, s^*)$ depends only on $\mathscr{B}_K^*$, take $h_1^*, h_2^* \in H^* = \overline{\Delta^*(\mathbb{Z})}$ with $(h_2^* - h_1^*)_{b^*} = 0$ for all $b^* \in \mathscr{B}_K^*$. Then there exists a sequence (n_k) with $\Delta^*(n_k) \to h_2^* - h_1^*$ and $\operatorname{lcm}(\mathscr{B}_K^*) = s^* \mid n_k$. We notice that

$$\operatorname{Per}(\varphi^*(h_1^* + \Delta^*(n_k)), 1, s^*) = \operatorname{Per}(\sigma^{n_k}\varphi^*(h_1^*), 1, s^*) = \operatorname{Per}(\varphi^*(h_1^*), 1, s^*) - n_k$$
$$= \operatorname{Per}(\varphi^*(h_1^*), 1, s^*),$$

since $\operatorname{Per}(\varphi^*(h_1^*), 1, s^*)$ is an s^*-periodic set. In particular, for every $j \in \operatorname{Per}(\varphi^*(h_1^*), 1, s^*)$, we get $\varphi^*(h_1^* + \Delta^*(n_k))(j) = 1$ for all k. Since $h_1^* + \Delta^*(n_k)$ converges to h_2^*, and φ^* is coordinatewise upper semicontinuous, this yields $\varphi^*(h_2^*)(j) = 1$ for all $j \in \operatorname{Per}(\varphi^*(h_1^*), 1, s^*)$ and hence $\operatorname{Per}(\varphi^*(h_1^*), 1, s^*) \subseteq \operatorname{Per}(\varphi^*(h_2^*), 1, s^*)$. By the symmetry between h_1^* and h_2^*, the converse inclusion follows from the same argument, thus proving the claim. □

Similar to $[\underline{\varphi}, \varphi]$, we define $[\underline{\varphi}_K, \varphi_K] := \{x \in \{0,1\}^Z : \underline{\varphi}_K(h) \le x \le \varphi_K(h)$ for some $h \in H\}$.

LEMMA 2.6. *The set $[\underline{\varphi}_K, \varphi_K] \subseteq \{0,1\}^{\mathbb{Z}}$ is a subshift.*

Proof. That $[\underline{\varphi}_K, \varphi_K]$ is closed follows immediately from the continuity of φ_K and $\underline{\varphi}_K$. In addition, it is σ-invariant as

$$\varphi_K \circ R = \sigma \circ \varphi_K \text{ and } \underline{\varphi}_K \circ R = \sigma \circ \underline{\varphi}_K.$$

Indeed, the first equality holds as φ_K is a coding of orbits of points in H with respect to $\{h \in H : h_b \ne 0$ for all $b \in \mathscr{B}_K\}$. The second equality follows from the definition of $\underline{\varphi}_K$ in terms of $\underline{\varphi}$, the equality $\underline{\varphi} \circ R = \sigma \circ \underline{\varphi}$ (since $\underline{\varphi}$ is a coding) and $\operatorname{Per}(\sigma x, s) = \operatorname{Per}(x, s) - 1$ for x in the orbit closure of a Toeplitz sequence. □

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We set $\eta_K := \varphi_K(\Delta(0)) = \mathbf{1}_{\mathcal{F}_{\mathscr{B}_K}}$ and $\underline{\eta}_K := \underline{\varphi}_K(\Delta(0))$. Then $\underline{\eta}_K \leq \eta^* \leq \eta \leq \eta_K$ (this is a special case of (27) for $h = \Delta(0)$).

LEMMA 2.7. *Let $\mathscr{B} \subseteq \mathbb{N}$ and suppose that η^* is a regular Toeplitz sequence. Let (ℓ_i) be a sequence realizing the lower density of $\mathcal{M}_{\mathscr{B}}$. Then*

$$\lim_{K\to\infty} \overline{d}_{(\ell_i)}(\{n \in \mathbb{N} : (\underline{\eta}_K(n), \eta_K(n)) \neq (\eta^*(n), \eta(n))\}) = 0.$$

Proof. It suffices to notice that

$$\lim_{K\to\infty} \overline{d}_{(\ell_i)}(\{n \in \mathbb{N} : \eta_K(n) \neq \eta(n)\}) = 0$$

and

$$\lim_{K\to\infty} \overline{d}_{(\ell_i)}(\{n \in \mathbb{N} : \underline{\eta}_K(n) \neq \eta^*(n)\}) = 0.$$

The first assertion follows by the Davenport–Erdős theorem (that is, by (1)). For the second, notice that $\underline{\eta}_K(n) \neq \eta^*(n)$ implies that $n \notin \mathrm{Per}(\eta^*, \mathrm{lcm}(\mathscr{B}^*_K))$. Thus,

$$\lim_{K\to\infty} \overline{d}(\{n \in \mathbb{N} : \underline{\eta}_K(n) \neq \eta^*(n)\}) \leq \lim_{K\to\infty} d(\mathbb{Z} \setminus \mathrm{Per}(\eta^*, \mathrm{lcm}(\mathscr{B}^*_K))) = 0 \tag{28}$$

as η^* is a regular Toeplitz sequence, cf. (10). □

We will also need the following well-known fact related to quasi-generic points and the corresponding invariant measures (we skip its proof and refer the reader e.g. to [28, Appendix C], see also [27]).

PROPOSITION 2.8. *Let $\mathcal{A}$ be a finite alphabet and suppose that $(\ell_i) \subseteq \mathbb{N}$ is an increasing sequence and that $x_K \in \mathcal{A}^{\mathbb{Z}}$ for $K \geq 1$ and $x \in \mathcal{A}^{\mathbb{Z}}$ are such that*

$$\lim_{K\to\infty} \overline{d}_{(\ell_i)}(\{n \in \mathbb{N} : x_K(n) \neq x(n)\}) = 0.$$

Suppose additionally that x_K, $K \geq 1$, and x are quasi-generic along (ℓ_i) for measures ν_K, $K \geq 1$, and ν, respectively. Then $\nu_K \to \nu$ in the weak topology.

Last but not least, we will need the following result to pass from the description of ergodic measures to that of all invariant measures on X_η.

PROPOSITION 2.9. *Suppose that for a subshift $X \subseteq \{0, 1\}^{\mathbb{Z}}$, we have $\mathcal{P}^e(X) \subseteq \{N_*((\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa) : \kappa \in \mathcal{P}(\{0, 1\}^{\mathbb{Z}})\}$. Then $\mathcal{P}(X) \subseteq \{N_*((\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa) : \kappa \in \mathcal{P}(\{0, 1\}^{\mathbb{Z}})\}$.*

We skip the proof—it is a repetition (with obvious changes such as replacing the map M by N and the Mirsky measure ν_η by the joining $\nu_{\eta^*} \triangle \nu_\eta$) of the proof of an analogous part of [28, Theorem 4.1.23] (more specifically, see p. 66 therein). The main tool there is the ergodic decomposition and the Arsenin–Kunungui theorem on measurable selection (see, e.g. [15, Theorem 18.18]). A more general result (with a shorter proof) will be published in [25].

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Now, fix $\nu \in \mathcal{P}^e(\overline{[\underline{\varphi}, \varphi]}) = \mathcal{P}^e([\underline{\varphi}, \varphi])$, see (18). Since ν is ergodic, since $[\underline{\varphi}, \varphi] \subseteq [\underline{\varphi}_K, \varphi_K]$ and the latter set is a subshift by Lemma 2.6, there exists a generic point $u_K \in [\underline{\varphi}_K, \varphi_K]$ for ν. Without loss of generality, we can assume

$$\underline{\eta}_K \leq u_K \leq \eta_K.$$

(Indeed, since $u_K \in [\underline{\varphi}_K, \varphi_K]$, there exists $h \in H$ with $\underline{\varphi}_K(h) \leq u_K \leq \varphi_K(h)$. If $j \in \mathbb{N}$ is such that $h_b + j = 0 \bmod b$ for all b in a sufficiently large, finite subset of $\mathscr{B}$, then Lemma 2.5 shows $\underline{\varphi}_K(\Delta(0)) \leq \sigma^j u_K \leq \varphi_K(\Delta(0))$, where $\sigma^j u_K$ is generic for ν.) Thus, there exits $y_K \in \{0,1\}^{\mathbb{Z}}$ such that $u_K = N(\underline{\eta}_K, \eta_K, y_K)$. Notice that $(\underline{\eta}_K, \eta_K, y_K)$ is quasi-generic for some measure ρ_K. Using the periodicity of $\underline{\eta}_K$ and η_K, we hence obtain that $(\underline{\eta}_K, \eta_K)$ is generic for $(\pi_{1,2})_*(\rho_K)$. In addition, (η^*, η) is quasi-generic along (ℓ_i) for $\nu_{\eta^*} \triangle \nu_\eta$ by (21) and (22). Thus, Proposition 2.8 and Lemma 2.7 yield

$$(\pi_{1,2})_*(\rho_K) \to \nu_{\eta^*} \triangle \nu_\eta.$$

If ρ is a limit of ρ_K, it follows that $(\pi_{1,2})_*(\rho) = \nu_{\eta^*} \triangle \nu_\eta$, so ρ is of the form $\rho = (\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa$ for some $\kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})$. Finally, since $(\underline{\eta}_K, \eta_K, y_K)$ is quasi-generic for ρ_K, it follows that $u_K = N(\underline{\eta}_K, \eta_K, y_K)$ is quasi-generic for $N_*(\rho_K)$. However, by assumption, u_K is also generic for ν, which yields $\nu = N_*(\rho_K)$ for all $K \in \mathbb{N}$, and thus $\nu = N_*(\rho) \in \{N_*((\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa) : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\}$. This proves $\mathcal{P}^e([\underline{\varphi}, \varphi]) \subseteq \{N_*((\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa) : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\}$. To complete the proof of the surjectivity of $\mathbf{D}_{B,C}$, we use Proposition 2.9.

2.2. *Proof of Theorem A.*

2.2.1. *($\mathbf{D}_A$) is a commutative dynamical diagram.* The proof that ($\mathbf{D}_A$) is a commutative dynamical diagram uses two ingredients. The first of them is that the following is a dynamical diagram:

$$\begin{array}{c} (H \times \{0,1\}^{\mathbb{Z}}, R \times \sigma) \\ \Big\downarrow{\scriptstyle (\underline{\varphi} \otimes \varphi) \times \mathrm{id}} \\ ((\{0,1\}^{\mathbb{Z}})^3, \sigma^{\times 3}, \{(\nu_{\eta^*} \triangle \nu_\eta) \vee \kappa : \kappa \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})\}) \end{array}$$

which is a consequence of Lemma 2.3.

The second ingredient that we need to prove the commutativity of ($\mathbf{D}_A$) is the following equality (that holds everywhere):

$$N \circ ((\underline{\varphi} \otimes \varphi) \times \mathrm{id}) = M_H,$$

which can be checked in a direct one-line calculation.

2.2.2. *($\mathbf{D}_A$) is surjective.* Recall that by (19) and by the surjectivity of ($\mathbf{D}_{B,C}$), the diagram

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$$\begin{array}{c}((\{0,1\}^{\mathbb{Z}})^3, \sigma^{\times 3}, \{(\nu_{\eta^*}\triangle\nu_\eta)\vee\kappa : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\})\\ \big\downarrow N \\ (X_\eta, \sigma)\end{array}$$

is surjective. Thus (using also the commutativity of ($\mathbf{D}_A$)), to prove the surjectivity of ($\mathbf{D}_A$), it suffices to prove that

$$\begin{array}{c}(H\times\{0,1\}^{\mathbb{Z}}, R\times\sigma)\\ \big\downarrow (\underline{\varphi}\otimes\varphi)\times \mathrm{id} \\ ((\{0,1\}^{\mathbb{Z}})^3, \sigma^{\times 3}, \{(\nu_{\eta^*}\triangle\nu_\eta)\vee\kappa : \kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}})\})\end{array} \tag{29}$$

is surjective. However, by Lemma 2.3 (cf. Remark 2.4), $((\{0,1\}^{\mathbb{Z}})^2, \sigma^{\times 2}, \nu_{\eta^*}\triangle\nu_\eta)$ is a factor of (H, R, m_H) via $\underline{\varphi}\otimes\varphi$, so given any joining $(\nu_{\eta^*}\triangle\nu_\eta)\vee\kappa$, it suffices to take its relatively independent extension to a joining of m_H with κ to conclude that (29) is surjective.

3. *Tautness and combinatorics*

Proof of Proposition D. We first show that the conditions (a′)–(f′) are all equivalent. We then pass to proving that, in fact, they are also equivalent to each of conditions (a)–(f).

Note that the implications (d′) $\implies$ (c′) $\implies$ (f′) $\implies$ (e′) are immediate. Next we show (e′) $\implies$ (a′). It was shown in [16] that for taut sets, the corresponding Mirsky measure is of full support in the corresponding $\mathscr{B}$-free subshift. Applying this to $\mathscr{C}$, we conclude that each block that appears on $X_{\eta_{\mathscr{C}}}$ is of positive $\nu_{\eta_{\mathscr{C}}}$-measure. Thus, it follows from condition (e′) that $X_{\eta_{\mathscr{C}}}\subseteq X_{\eta'}$, and hence $\widetilde{X}_{\eta_{\mathscr{C}}}\subseteq\widetilde{X}_{\eta'}$. Using (8), we obtain that $\forall_{b'\in\mathscr{B}'}\exists_{c\in\mathscr{C}}\, c\mid b'$. Moreover $X_{\eta_{\mathscr{C}}}\subseteq X_\eta$ implies $X_{\eta^*}\subseteq X_{\eta_{\mathscr{C}}}$, since X_{η^*} is the unique minimal subset of X_η. This yields $\widetilde{X}_{\eta^*}\subseteq\widetilde{X}_{\eta_{\mathscr{C}}}$. Using (8) again, we obtain that $\forall_{c\in\mathscr{C}}\,\exists_{b^*\in\mathscr{B}^*}\, b^*\mid c$, which proves condition (a′). Next we note that (a′) $\implies$ (b′) by the very definition of η', $\eta_{\mathscr{C}}$ and η^*. To finish the first part, it only remains to notice that by Proposition 2.1 and tautness of $\mathscr{B}'$, it follows that $X_{\eta'}=\overline{[\eta^*,\eta']}$, which yields (b′) $\implies$ (d′).

Since the proof of (b′) $\implies$ (d′) was the only place where we used the tautness of $\mathscr{B}'$, the same arguments as above show also that (d) $\implies$ (c) $\implies$ (f) $\implies$ (e) $\implies$ (a) $\implies$ (b). We now prove (b) $\implies$ (b′). As (b′) $\implies$ (d′) was already shown, and as (d′) $\implies$ (d) follows directly from $X_{\eta'}\subseteq X_\eta$, this will finish the proof. Thus, suppose that $\eta_{\mathscr{C}}\le\eta$. It follows then by (8) that $\nu_{\eta_{\mathscr{C}}}\in\mathcal{P}(\widetilde{X}_\eta)=\mathcal{P}(\widetilde{X}_{\eta'})$. Applying again (8), we obtain $\eta_{\mathscr{C}}\le\eta'$, and hence condition (b′). □

4. *Entropy*

4.1. *Entropy of X_η: proof of Theorem F and of Corollary G*

Remark 4.1. If X_η is uniquely ergodic, then the Mirsky measure ν_η (whose entropy is zero) is the unique invariant measure and it follows immediately by the variational principle that $h(X_\eta)=0$.

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Proof of Theorem F. To show the inequality $h(X_\eta) \geq \overline{d} - \overline{d}^*$, we first assume that $\mathscr{B}$ is taut and consider the following block: $B = \eta[1, n] \in \{0, 1\}^n$. Then $B(\ell) = 0$ for any $\ell \in \mathcal{M}_{\mathscr{B}} \cap [1, n]$ and $B(\ell) = 1$ for any $\ell \in \mathcal{F}_{\mathscr{B}} \cap [1, n]$ (so, in particular, for any $\ell \in \mathcal{F}_{\mathscr{B}^*} \cap [1, n]$). It follows by Proposition 2.1 that any block $C \in \{0, 1\}^n$ that agrees with B on the positions belonging to $\mathcal{M}_{\mathscr{B}} \cup \mathcal{F}_{\mathscr{B}^*}$ also appears in X_η. There are

$$2^{n-|(\mathcal{M}_{\mathscr{B}} \cup \mathcal{F}_{\mathscr{B}^*}) \cap [1,n]|} = 2^{|\mathcal{F}_{\mathscr{B}} \cap [1,n]| - |\mathcal{F}_{\mathscr{B}^*} \cap [1,n]|}$$

such blocks C (they are pairwise distinct). Thus,

$$2^{|\mathcal{F}_{\mathscr{B}} \cap [1,n]| - |\mathcal{F}_{\mathscr{B}^*} \cap [1,n]|} \leq p_n(\eta).$$

It follows that

$$\begin{aligned} \overline{d} - \overline{d}^* &= \limsup_{n\to\infty} \frac{|\mathcal{F}_{\mathscr{B}} \cap [1, n]|}{n} - \limsup_{n\to\infty} \frac{|\mathcal{F}_{\mathscr{B}^*} \cap [1, n]|}{n} \\ &\leq \limsup_{n\to\infty} \frac{|\mathcal{F}_{\mathscr{B}} \cap [1, n]| - |\mathcal{F}_{\mathscr{B}^*} \cap [1, n]|}{n} \leq \lim_{n\to\infty} \frac{\log_2 p_n(\eta)}{n} = h(X_\eta). \end{aligned} \tag{30}$$

For general (not necessarily taut) $\mathscr{B}$, we apply (30) to the tautification $\mathscr{B}'$. We use $X_{\eta'} \subseteq X_\eta$, $(\mathscr{B}')^* = \mathscr{B}^*$ and $\overline{d}' = \overline{d}$ (see (11), (12) and (21), respectively) to obtain

$$h(X_\eta) \geq h(X_{\eta'}) \geq \overline{d}' - (\overline{d}')^* = \overline{d} - \overline{d}^*.$$

Now, assume additionally that X_{η^*} is uniquely ergodic. Fix $K \geq 1$ and let $n \in (\prod_{b\in\mathscr{B}_K} b)\mathbb{N}$. Since $\eta^* \leq \eta \leq \eta_K$, it follows that

$$p_n(\eta) \leq p_n(\overline{[\eta^*, \eta_K]}).$$

For any block $B \in \{0, 1\}^n$ which appears in $\overline{[\eta^*, \eta_K]}$, there exists $M \in \mathbb{Z}$ such that

$$\eta^*[M + 1, M + n] \leq B \leq \eta_K[M + 1, M + n].$$

Clearly, if $\eta^*(M + \ell) = 1$, then $B(\ell) = 1$ and there are

$$|\mathrm{supp}\ \eta^*[M + 1, M + n]|$$

such ‘mandatory’ 1s on B coming from η^*. Moreover, if $\eta_K(M + \ell) = 0$, then also $B(\ell) = 0$ and there are

$$n - |\mathrm{supp}\ \eta_K[M + 1, M + n]|$$

such ‘mandatory’ 0s on B coming from η_K. All the other positions on B can be altered arbitrarily, without loosing the property that B appears in $\overline{[\eta^*, \eta_K]}$. The number of such ‘free’ positions equals

$$|\mathrm{supp}\ \eta_K[M + 1, M + n]| - |\mathrm{supp}\ \eta^*[M + 1, M + n]|.$$

Each choice of 0s and 1s on the ‘free’ positions yields a different block of length n from $\overline{[\eta^*, \eta_K]}$. Thus, for each choice of M, we obtain

$$2^{|\mathrm{supp}\ \eta_K[M+1,M+n]| - |\mathrm{supp}\ \eta^*[M+1,M+n]|}$$

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blocks and it follows that

$$p_n(\eta)\leq p_n(\overline{[\eta^*,\eta_K]})\leq p_n(\eta^*)\cdot p_n(\eta_K)\cdot 2^{\sup_{M\in\mathbb{Z}}(|\mathrm{supp}\ \eta_K[M+1,M+n]|-|\mathrm{supp}\ \eta^*[M+1,M+n]|)}. \tag{31}$$

Since η_K is $\mathrm{lcm}(\mathscr{B}_K)$-periodic and $\mathrm{lcm}(\mathscr{B}_K)\mid n$, we have

$$|\mathrm{supp}\ \eta_K[M+1,M+n]|=nd(\mathcal{F}_{\mathscr{B}_K}). \tag{32}$$

By the uniform ergodicity of X_{η^*}, for any $\varepsilon>0$ and for large enough n, we have

$$n(d^*-\varepsilon)\leq|\mathrm{supp}\ \eta^*[M+1,M+n]|\leq n(d^*+\varepsilon) \tag{33}$$

for every $M\in\mathbb{Z}$. Using (31), (32) and (33), we conclude that

$$p_n(\eta)\leq p_n(\eta^*)\cdot p_n(\eta_K)\cdot 2^{nd(\mathcal{F}_{\mathscr{B}_K})-n(d^*-\varepsilon)}.$$

Hence,

$$h(X_\eta)\leq h(X_{\eta^*})+h(X_{\eta_K})+d(\mathcal{F}_{\mathscr{B}_K})-d^*.$$

By Remark 4.1, we have $h(X_{\eta^*})=0$ and $h(X_{\eta_K})=0$ since η_K is periodic. Recall also that by the Davenport–Erdős theorem (i.e. (1)), $\lim_{K\to\infty}d(\mathcal{F}_{\mathscr{B}_K})=\overline{d}$. This yields

$$h(X_\eta)\leq\overline{d}-d^*,$$

which completes the proof of Theorem F. □

For the proof of Corollary G, we will need the following lemma.

LEMMA 4.2. *For any $\mathscr{B}\subseteq\mathbb{N}$ such that $\eta'\neq\eta^*$, we have $\overline{d}>\overline{d}^*$.*

Proof. Since $\overline{d}=\overline{d}':=\overline{d}(\mathcal{F}_{\mathscr{B}'})$, we can assume without loss of generality that $\mathscr{B}$ is taut. Let (ℓ_i) be a sequence realizing the lower density of $\mathcal{M}_{\mathscr{B}^*}$. It follows by the result of Davenport and Erdős (that is, by (1)) that

$$\begin{aligned}\overline{d}^*=\overline{d}(\mathcal{F}_{\mathscr{B}^*})&=\lim_{i\to\infty}\frac{1}{\ell_i}|\mathcal{F}_{\mathscr{B}^*}\cap[1,\ell_i]|=\liminf_{i\to\infty}\frac{1}{\ell_i}|\mathcal{F}_{\mathscr{B}^*}\cap[1,\ell_i]|\\&\leq\liminf_{i\to\infty}\frac{1}{\ell_i}|\mathcal{F}_{\mathscr{B}}\cap[1,\ell_i]|\leq\limsup_{i\to\infty}\frac{1}{\ell_i}|\mathcal{F}_{\mathscr{B}}\cap[1,\ell_i]|\leq\overline{d}(\mathcal{F}_{\mathscr{B}})=\overline{d}.\end{aligned}$$

If $\overline{d}=\overline{d}^*$, then all inequalities in the above formula become equalities. In particular,

$$\lim_{i\to\infty}\frac{1}{\ell_i}|\mathcal{F}_{\mathscr{B}}\cap[1,\ell_i]|\quad\text{exists and equals }\overline{d}=\overline{d}^*,$$

so that η is generic along (ℓ_i) for ν_η. Since $\eta^*\leq\eta$ (by the construction of $\mathscr{B}^*$), it follows that $(1/\ell_i)|\{n\in[1,\ell_i]:\eta(n)\neq\eta^*(n)\}|\to 0$. Thus, since η^* is generic along (ℓ_i) for ν_{η^*}, it follows immediately that η has to be generic along (ℓ_i) for the very same measure, that is, ν_{η^*}. However, we know that the Mirsky measure ν_η is the unique measure of maximal density, that is, the invariant measure of the greatest value for the cylinder $\{x\in X_\eta:x(0)=1\}$, in each $\mathscr{B}$-free subshift (see, e.g. [**19**, Theorem 4 and Corollary 4], cf. also [**1**, Ch. 7]), which gives us $\nu_\eta=\nu_{\eta^*}$. Now it suffices to use [**7**, Corollary 9.2] which says (in particular) that the latter condition is equivalent to $\eta=\eta^*$ (cf. (8)). □

Proof of Corollary G. By Remark 4.1, if X_η is uniquely ergodic, then $h(X_\eta) = 0$.

Suppose that $h(X_\eta) = 0$. Then, by Theorem F, we have $\overline{d} = d^*$ which implies $\eta' = \eta^*$ by Lemma 4.2. The latter condition is equivalent to $\mathscr{B}' = \mathscr{B}^*$ by [7, Theorem L], as both $\mathscr{B}'$ and $\mathscr{B}^*$ are taut (cf. (8)). It follows immediately that $X_{\eta'}$ must be uniquely ergodic as it is equal to X_{η^*} and the latter subshift is uniquely ergodic since η^* is assumed to be a regular Toeplitz sequence. It suffices to use Theorem C to complete the proof. □

4.2. *Intrinsic ergodicity of X_η: proof of Theorem H.* Consider first the case when $\eta' = \eta^*$. It follows by Theorem C that

$$\mathcal{P}(X_\eta) = \mathcal{P}(X_{\eta'}) = \mathcal{P}(X_{\eta^*}).$$

Thus, if X_{η^*} is uniquely ergodic, then X_η is also uniquely ergodic. Moreover, the pair (η^*, η) is quasi-generic for $\nu_{\eta^*} \triangle \nu_\eta$ along (ℓ_i) realizing the lower density of $\mathcal{M}_{\mathscr{B}}$. It follows by Remark 4.1 that

$$\{\nu_\eta\} = \mathcal{P}(X_\eta) = \mathcal{P}(X_{\eta^*}) = \{\nu_{\eta^*}\}.$$

Thus, $\nu_{\eta^*} \triangle \nu_\eta$ is the diagonal joining of two copies of ν_η. Let (x, x, y) be a generic point for $(\nu_{\eta^*} \triangle \nu_\eta) \otimes B_{1/2,1/2}$. Then $N(x, x, y) = x$ is a generic point for $N_*((\nu_{\eta^*} \triangle \nu_\eta) \otimes B_{1/2,1/2})$. It follows immediately that $N_*((\nu_{\eta^*} \triangle \nu_\eta) \otimes B_{1/2,1/2}) = \nu_\eta$.

Assume now that $\eta' \neq \eta^*$. We will study the following diagram:

$$\begin{tikzcd}
(H \times \{0,1\}^{\mathbb{Z}}, R \times \sigma) \arrow[d, "\Psi"] \arrow[dd, bend left=60, "M_H"] \\
(H \times \{0,1\}^{\mathbb{Z}}, \widetilde{R}) \arrow[d, "\Phi"] \\
([\underline{\varphi}, \varphi], \sigma)
\end{tikzcd} \tag{34}$$

Let us now introduce all maps appearing in this diagram. We define $\widetilde{R} \colon H \times \{0,1\}^{\mathbb{Z}} \to H \times \{0,1\}^{\mathbb{Z}}$ by

$$\widetilde{R}(h, x) = \begin{cases} (Rh, x) & \text{if } \underline{\varphi}(h)(0) = \varphi(h)(0), \\ (Rh, \sigma x) & \text{if } 0 = \underline{\varphi}(h)(0) < \varphi(h)(0) = 1. \end{cases}$$

Let

$$Z_\infty := \{z \in \{0,1\}^{\mathbb{Z}} : |\text{supp}\, z \cap (-\infty, 0]| = |\text{supp}\, z \cap [0, \infty)| = \infty\}.$$

Given $x \in \{0,1\}^{\mathbb{Z}}$ and $z \in Z_\infty$, let $\hat{x}_z$ be the sequence obtained by reading consecutive coordinates of x which are in the support of z and such that

$$\hat{x}_z(0) = x(\min\{k \geq 0 : z(k) = 1\}).$$

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Now, let

$$H_\infty := \{h \in H : R^n h \in W \setminus \underline{W} \text{ infinitely often both in the future and in the past}\}$$

and define $\Psi\colon H_\infty \times \{0,1\}^{\mathbb{Z}} \to H_\infty \times \{0,1\}^{\mathbb{Z}}$ by

$$\Psi(h,x) = (h, \widehat{x}_{\varphi(h)-\underline{\varphi}(h)})$$

(notice that $\varphi(h)(n) - \underline{\varphi}(h)(n) = 1 \iff R^n h \in W \setminus \underline{W}$, so for $h \in H_\infty$, we have $\varphi(h) - \underline{\varphi}(h) \in Z_\infty$). It remains to define Φ. Let $\Phi\colon H_\infty \times \{0,1\}^{\mathbb{Z}} \to [\underline{\varphi}, \varphi]$ be defined by mapping (h,x) to the unique element in $[\underline{\varphi}, \varphi]$ such that

$$\underline{\varphi}(h) \le \Phi(h,x) \le \varphi(h) \quad \text{and} \quad (\Phi(h,x))\widehat{\;}_{\varphi(h)-\underline{\varphi}(h)} = x.$$

We will show that the diagram in (34) commutes (it will then follow by Theorem A that the maps M_H and Φ are surjective morphisms).

LEMMA 4.3. *For any $\mathscr{B} \subseteq \mathbb{N}$, we have $m_H(W \setminus \underline{W}) = \overline{d} - \overline{d}^*$. Moreover, if $\eta' \neq \eta^*$, we have $m_H(W \setminus \underline{W}) > 0$.*

Before we begin the proof of this lemma, recall some results from [17] that we already mentioned in the introduction: there is a continuous surjective group homomorphism $\Gamma_{H,H^*}\colon H \to H^*$, which maps $\Delta(n)$ to $\Delta^*(n)$. In addition, it has the following property, see (13):

$$\Gamma_{H,H^*}(\underline{W}) = W^* \quad \text{and} \quad \Gamma_{H,H^*}(H \setminus \underline{W}) = H^* \setminus W^*.$$

Recall also that it was shown in [14, Lemma 4.1] that

$$m_H(W) = \overline{d}(\mathcal{F}_{\mathscr{B}}) = \overline{d}. \tag{35}$$

Proof of Lemma 4.3. We have

$$\begin{aligned} m_H(W \setminus \underline{W}) &= m_H(W) - m_H(\underline{W}) \\ &= m_H(W) - m_H(\Gamma_{H,H^*}^{-1}(W^*)) \\ &= m_H(W) - (\Gamma_{H,H^*})_*(m_H)(W^*) \\ &= m_H(W) - m_{H^*}(W^*) \end{aligned}$$

(the second equality follows from (13) and the fourth equality follows by the unique ergodicity of R^*). It remains to use (35) to deduce that $m_H(W \setminus \underline{W}) = \overline{d} - \overline{d}^*$ and Lemma 4.2 to conclude that $m_H(W \setminus \underline{W}) > 0$ whenever $\eta' \neq \eta^*$. □

It follows now from Lemma 4.3 and from the ergodicity of (H, R, m_H) that $m_H(H_\infty) = 1$. Thus, to conclude that (34) commutes, it remains to check whether for every $h \in H_\infty$ and every $x \in \{0,1\}^{\mathbb{Z}}$, we have the commutativity relations

$$\begin{aligned} M_H(h,x) &= (\Phi \circ \Psi)(h,x), \\ (\widetilde{R} \circ \Psi)(h,x) &= (\Psi \circ (R \times \sigma))(h,x), \\ (\sigma \circ \Phi)(h,x) &= (\Phi \circ \widetilde{R})(h,x). \end{aligned} \tag{36}$$

The first equality in (36) is immediate by the definition of the maps, while the second and third follow from

$$\widehat{\sigma x}_{\sigma z} = \begin{cases} \hat{x}_z & \text{if } z(0) = 0, \\ \sigma \hat{x}_z & \text{if } z(0) = 1, \end{cases} \tag{37}$$

(the proof of (37) consists of a straightforward but lengthy calculation; an analogous property is proved in [22]).

Notice that it follows by Theorem A that the morphism M_H is surjective. Thus, the morphism Φ is also surjective.

Now, we are ready to complete the proof of the intrinsic ergodicity of X_η. The main ideas come from [22]. We will present the sketch of the proof only (similarly as in [7] for $\widetilde{X}_\eta$). Clearly, any point from $H_\infty \cap (W \setminus \underline{W})$ returns to $W \setminus \underline{W}$ infinitely often under R and $m_H(H_\infty \cap (W \setminus \underline{W})) = m_H(W \setminus \underline{W})$. Recall that

$$\widetilde{R}(h, x) = \begin{cases} (Rh, x) & \text{if } \underline{\varphi}(h)(0) = \varphi(h)(0), \\ (Rh, \sigma x) & \text{if } 0 = \underline{\varphi}(h)(0) < \varphi(h)(0) = 1. \end{cases}$$

Then every point from $(H_\infty \cap (W \setminus \underline{W})) \times \{0, 1\}^{\mathbb{Z}}$ returns to $(W \setminus \underline{W}) \times \{0, 1\}^{\mathbb{Z}}$ infinitely often under $\widetilde{R}$ and $\nu((H_\infty \cap (W \setminus \underline{W})) \times \{0, 1\}^{\mathbb{Z}}) = \nu((W \setminus \underline{W}) \times \{0, 1\}^{\mathbb{Z}})$ for every $\nu \in \mathcal{P}(H \times \{0, 1\}^{\mathbb{Z}}, \widetilde{R})$. Thus, the induced transformation $\widetilde{R}_{(W\setminus\underline{W})\times\{0,1\}^{\mathbb{Z}}}$ is well defined, that is, $\widetilde{R}_{(W\setminus\underline{W})\times\{0,1\}^{\mathbb{Z}}}(h, x) = \widetilde{R}^{n(h,x)}(h, x)$ for ν-almost every $(h, x) \in (W \setminus \underline{W}) \times \{0, 1\}^{\mathbb{Z}}$, where $n(h, x) := \min\{n \geq 1 : \widetilde{R}^n(h, x) \in (W \setminus \underline{W}) \times \{0, 1\}^{\mathbb{Z}}\}$. It follows that $\widetilde{R}_{(W\setminus\underline{W})\times\{0,1\}^{\mathbb{Z}}} = R_{(W\setminus\underline{W})} \times \sigma$ a.e. for any $\widetilde{R}$-invariant measure.

We will show now that $\widetilde{R}$ has a unique measure of maximal (measure-theoretic) entropy. Since $m_H(W \setminus \underline{W}) > 0$ whenever $\eta' \neq \eta^*$ (see Lemma 4.3) and since $\kappa(W \setminus \underline{W} \times \{0, 1\}^{\mathbb{Z}}) = m_H(W \setminus \underline{W}) > 0$ for any $R_{W\setminus\underline{W}} \times \sigma$-invariant probability measure κ, in view of the Abramov formula, it suffices to show that $\widetilde{R}_{(W\setminus\underline{W})\times\{0,1\}^{\mathbb{Z}}} = R_{(W\setminus\underline{W})} \times \sigma$ has a unique measure of maximal entropy. For any $R_{(W\setminus\underline{W})} \times \sigma$-invariant measure κ, by the Pinsker formula, we have

$$\begin{aligned} &h(\{0, 1\}^{\mathbb{Z}}, \sigma, (\pi_2)_*(\kappa)) \leq h((W \setminus \underline{W}) \times \{0, 1\}^{\mathbb{Z}}, R_{W\setminus\underline{W}} \times \sigma, \kappa) \\ &\leq h(W \setminus \underline{W}, R_{W\setminus\underline{W}}, (\pi_1)_*(\kappa)|_{(W\setminus\underline{W})}) + h(\{0, 1\}^{\mathbb{Z}}, \sigma, (\pi_2)_*(\kappa)) = h(\{0, 1\}^{\mathbb{Z}}, \sigma, (\pi_2)_*(\kappa)), \end{aligned}$$

where $h(W \setminus \underline{W}, R_{W\setminus\underline{W}}, (\pi_1)_*(\kappa)|_{(W\setminus\underline{W})})$ vanishes by the Abramov formula as $R_{W\setminus\underline{W}}$ is an induced map coming from a rotation. Since $(\pi_2)_*(\kappa)$ can be arbitrary, it follows that a measure κ has the maximal entropy among all $R_{W\setminus\underline{W}} \times \sigma$-invariant measures if and only if $h((W \setminus \underline{W}) \times \{0, 1\}^{\mathbb{Z}}, R_{W\setminus\underline{W}} \times \sigma, \kappa) = h(\{0, 1\}^{\mathbb{Z}}, \sigma)$. Moreover, κ is a measure of maximal entropy for $R_{W\setminus\underline{W}} \times \sigma$ if and only if $(\pi_2)_*(\kappa)$ is the measure of maximal entropy for σ, that is, when $(\pi_2)_*(\kappa)$ is the Bernoulli measure $B_{1/2,1/2}$, that is, when κ is a joining of the unique invariant measure for $R_{W\setminus\underline{W}}$ and $B_{1/2,1/2}$. Since the unique invariant measure for $R_{W\setminus\underline{W}}$ is of zero entropy, it follows from the disjointness of K-automorphisms with zero entropy automorphisms [10] that κ is the product measure. In particular, κ is unique.

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The last step to conclude the intrinsic ergodicity of X_η is to show that

$$h(H \times \{0,1\}^{\mathbb{Z}}, \widetilde{R}) = m_H(W \setminus \underline{W}) = \overline{d} - d^* = h(X_\eta). \tag{38}$$

Let us justify each of the equalities above. By the variational principle, by the Abramov formula and by the Pinsker formula, we have

$$\begin{aligned} h(H \times \{0,1\}^{\mathbb{Z}}, \widetilde{R}) &= \sup_\rho \{h(H \times \{0,1\}^{\mathbb{Z}}, \widetilde{R}, \rho)\} \\ &= m_H(W \setminus \underline{W}) \cdot \sup_\rho \{h((W \setminus \underline{W}) \times \{0,1\}^{\mathbb{Z}}, R_{W \setminus \underline{W}} \times \sigma, \rho)\} \\ &= m_H(W \setminus \underline{W}) \cdot \sup_\rho \{h(\{0,1\}^{\mathbb{Z}}, \rho)\} = m_H(W \setminus \underline{W}), \end{aligned}$$

where the suprema are taken over all Borel probability invariant measures for the corresponding maps. This yields the first equality in (38). Moreover, the middle equality in (38) follows by Lemma 4.3, while the last one follows by Theorem F.

4.3. *Entropy density of X_η: proof of Theorem I.* The idea of the proof of Theorem I is the same as that of the analogous result for $\widetilde{X}_\eta$ in [**21**]. Let us introduce the necessary tools and notation. Given $x, y \in \{0,1\}^{\mathbb{Z}}$, consider the following *premetric*:

$$\underline{d}(x, y) := \liminf_{n \to \infty} \frac{1}{n} |\{1 \le i \le n : x(i) \neq y(i)\}|$$

(being a premetric means that $\underline{d}$ is a real-valued, non-negative, symmetric function on $(\{0,1\}^{\mathbb{Z}})^2$ vanishing on the diagonal; the triangle inequality for $\underline{d}$ fails). As a premetric, $\underline{d}$ induces a Hausdorff pseudometric $\underline{d}^H$ on the space of all non-empty subsets of $\{0,1\}^{\mathbb{Z}}$ in the following way:

$$\underline{d}(x, Y) := \inf_{y \in Y} \underline{d}(x, y) \quad \text{and} \quad \underline{d}^H(X, Y) := \max\{\sup_{x \in X} \underline{d}(x, Y), \sup_{y \in Y} \underline{d}(y, X)\}$$

for any $\emptyset \neq X, Y \subseteq \{0,1\}^{\mathbb{Z}}$ and $x \in X$, $y \in Y$.

Let us now recall some results from [**21**] (we formulate them for 0–1 shifts, however, they are valid for shifts over any finite alphabet).

PROPOSITION 4.4. [**21**, Proposition 26] *Let $x \in \{0,1\}^{\mathbb{Z}}$ be a periodic point under σ. Then the hereditary closure of the orbit of x is a transitive sofic shift.*

Remark 4.5. We skip here the definition of a sofic shift as it is quite technical and this notion serves here as a tool only. Namely, in any sofic transitive shift, the ergodic measures are entropy dense (more general results are known, see [**8**, **33**]). The following modification of Proposition 4.4 holds. Let $w, x \in \{0,1\}^{\mathbb{Z}}$ be periodic, such that $w \le x$. Then $[w, x]$ is a transitive sofic shift. The proof is a straightforward adjustment of the proof of [**21**, Proposition 26].

PROPOSITION 4.6. [**21**, Corollary 20] *Let $(X_K)_{K \ge 1} \subseteq \{0,1\}^{\mathbb{Z}}$ be a sequence of transitive sofic shifts. If $X \subseteq \{0,1\}^{\mathbb{Z}}$ is a subshift such that $\underline{d}^H(X_K, X) \to 0$, then ergodic measures are entropy-dense in $\mathcal{P}(X)$.*

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Proof of Theorem I. Since $\mathcal{P}(X_\eta) = \mathcal{P}(X_{\eta'})$ by Theorem C, we can assume without loss of generality that $\mathscr{B}$ is taut. Hence, $X_\eta = \overline{[\eta^*, \eta]}$ by Proposition 2.1, and in view of Remark 4.5 and Proposition 4.6, it suffices to prove that $\underline{d}^H([\underline{\eta}_K, \eta_K], \overline{[\eta^*, \eta]}) \to 0$. To do so, we show that

$$\underline{d}^H([\underline{\eta}_K, \eta_K], \overline{[\eta^*, \eta]}) \le \overline{d}(\underline{\eta}_K, \eta^*) + \underline{d}(\eta, \eta_K),$$

where the right-hand side tends to zero by the regularity of η^* (cf. (28)) and the Davenport–Erdős theorem (i.e. (1)). Fix now $K \ge 1$. We claim that for $\underline{\eta}_K \le x \le \eta_K$, there exists $y \in [\eta^*, \eta]$ with

$$\underline{d}(x, y) \le \overline{d}(\underline{\eta}_K, \eta^*) + \underline{d}(\eta, \eta_K). \tag{39}$$

Indeed, set $y := N(\eta^*, \eta, x) = \eta^* + x(\eta - \eta^*)$. Then $y(n) \ne x(n)$ implies that $\eta^*(n) = \eta(n)$ and $\underline{\eta}_K(n) \ne \eta_K(n)$ (recall that $\underline{\eta}_K \le \eta^* \le \eta \le \eta_K$). Thus, for every $n \in \mathbb{Z}$ with $y(n) \ne x(n)$, we have either that $\underline{\eta}_K(n) \ne \eta^*(n)$ or that $\eta(n) \ne \eta_K(n)$, that is,

$$\{n \in \mathbb{N} : x(n) \ne y(n)\} \subseteq \{n \in \mathbb{N} : \underline{\eta}_K(n) \ne \eta^*(n)\} \cup \{n \in \mathbb{N} : \eta(n) \ne \eta_K(n)\}.$$

This yields (39). To finish the proof, note that for every $x \in [\underline{\eta}_K, \eta_K]$, there exists $m \in \mathbb{Z}$ such that $\underline{\eta}_K \le \sigma^m x \le \eta_K$. By the above construction, for every $x \in [\underline{\eta}_K, \eta_K]$, there exists therefore $y \in [\eta^*, \eta]$ with $\underline{d}(x, y) \le \underline{d}(\underline{\eta}_K, \eta^*) + \underline{d}(\eta, \eta_K)$. In addition, we have $\overline{[\eta^*, \eta]} \subseteq [\underline{\eta}_K, \eta_K]$. □

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