WEAKLY COMPACT SETS IN ORLICZ SPACES

TSUYOSHI ANDÔ

1. Introduction. The purpose of this paper is to characterize weak compactness in Orlicz spaces. Though an Orlicz space is a Banach space, it will be viewed from the standpoint of the theory of Köthe spaces. Considering that a norm-bounded subset is not weakly compact in general, we shall give some criteria for weak compactness in terms of the functional defining an Orlicz space. Because weak compactness is closely connected with the continuity of the semi-norms on the conjugate space, at the same time some properties of continuous semi-norms on Orlicz spaces will be brought to light.

The first characterization (Theorem 1) is concerned with degree of smoothness of the functional at the origin. In Theorem 2 a connection between weak compactness and boundedness (by another functional) is obtained. In Theorem 3 the result in Theorem 2 is stated as a proposition about continuous semi-norms.

2. Preliminaries. When $M(\xi)$ is a real-valued convex function such that $M(0) = 0$, $M(\xi) = M(-\xi)$ and $M(\xi)/\xi \to \infty$ as $\xi \to \infty$, it is called an *N-function.* On account of convexity, it admits the non-decreasing right-hand derivative $p(\xi)$, and the function

$$N(\xi) = \int_0^{|\xi|} q(\eta)d\eta$$

(where $q(\eta)$ is the right-inverse of $p(\xi)$) is also an N-function. $M(\xi)$ and $N(\xi)$ are called mutually *complementary* to each other. By Young's inequality (see (**3**, § 2; **4**, II-1))

$$|\xi\cdot\eta| \leqslant M(\xi) + N(\eta). \tag{1}$$

Let G be an abstract set and μ be a fixed (non-negative) countably additive measure on the σ-algebra $\boldsymbol{B}$ of subsets of G. We assume $0 < \mu(G) < \infty$. For an N-function $M(\xi)$ we shall consider an (extended) real valued functional $\boldsymbol{M}(f)$ defined, on the class of all measurable functions $f(t)$ on G, by

$$\boldsymbol{M}(f) \equiv \int M(f(t))d\mu$$

(where $\int d\mu$ denotes the integral on G). The functional $\boldsymbol{M}(f)$ is called the *modular* defined by the N-function $M(\xi)$, in accordance to the terminology of Nakano (**5**).

The Orlicz space $L_M{}^*$, defined by an N-function $M(\xi)$, is the class of all

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measurable functions $f(t)$ such that $\boldsymbol{M}(\alpha f) < \infty$ for some $\alpha = \alpha(f) > 0$. $L_M{}^*$ is a linear space with the usual addition and scalar multiplication. Moreover, it is a Banach space with the norm (see **(3**, § 9; **4**, II-2**)**)

$$|||f|||_M = \inf(1/|\xi|) \qquad \text{where } \boldsymbol{M}(\xi f) \leqslant 1. \tag{2}$$

Our starting point is in the fact that $L_M{}^*$ is exactly the class of all measurable functions $f(t)$ such that

$$\int |f(t) \cdot g(t)| d\mu < \infty$$

for all $g(t) \in L_N{}^*$ (where $N(\xi)$ is complementary to $M(\xi)$) (see **(3**, § 9; **4**, II-2**)**); in other words it is a *Köthe space* with the adjoint $L_N{}^*$ in the sense of Dieudonné **(2)**. The norm is known (see **(3**, § 9; **4**, II-2; **5**, § 40**)**)

$$|||f|||_M \leqslant \sup_g \int |f(t) \cdot g(t)| d\mu \leqslant 2 \cdot |||f|||_M \tag{3}$$

where $\boldsymbol{N}(g) \leqslant 1$.

In this paper, the *weak topology* in question is not that as a Banach space, but that defined by $L_N{}^*$, that is, $\sigma(L_M{}^*, L_N{}^*)$. This is also the standpoint of Nakano **(5)**. When $\boldsymbol{B}$ is atomless, the Banach weak topology and $\sigma(L_M{}^*, L_N{}^*)$ coincide with each other if and only if $M(\xi)$ satisfies the following condition (Δ_2): $\limsup_{\xi\to\infty} M(2\xi)/M(\xi) < \infty$ (see **(3**, § 10; **4**, II-2**)**). General norm-bounded linear functionals are not manageable, as is seen in **(1)**. Even if a subset A of $L_M{}^*$ is bounded by the modular $\boldsymbol{M}(f)$, it is not necessarily (relatively) weakly compact. A known criterion for weak compactness in terms of linear functionals can be summarized as follows (see **(2**; **5**, § 28**)**).

LEMMA. *A subset A of $L_M{}^*$ is (relatively) weakly compact, if and only if it is weakly bounded and equi-continuous in the following sense:*

$$\sup_{f\in A} \int_E |f(t) \cdot g(t)| d\mu \to 0 \quad \text{as} \quad \mu(E) \to 0 \;\; (g(t) \in \overset{*}{L_N}).$$

This lemma reveals a connection between weak compactness and a property of semi-norms on the adjoint. A semi-norm $||f||$ considered in this paper will always be assumed to have the following property: $|f(t)| \leqslant |g(t)|$ almost everywhere implies $||f|| \leqslant ||g||$. A semi-norm is called *continuous*, if $||f_E|| \to 0$ as $\mu(E) \to 0$ where $f_E(t) = f(t)$ or $= 0$ according as $t \in E$ or $t \notin E$. The lemma shows that A is (relatively) weakly compact if and only if the semi-norm (on $L_N{}^*$) defined by

$$||f|| = \sup_{g\in A} \int |f(t) \cdot g(t)| d\mu$$

is continuous. The definition of continuity suggests the following notion (cf. **(3)**): a semi-norm $||f||$ is called *uniformly continuous on* $L_N{}^*$, if $||f_E|| \to 0$ (as $\mu(E) \to 0$) uniformly with respect to $f(t)$ with $\boldsymbol{N}(f) \leqslant 1$.

On account of (3) and the lemma, if boundedness by modular $\boldsymbol{M}(f)$ implies (relatively) weak compactness, the norm $|||f|||_N$ on L_N^* is continuous. When $\boldsymbol{B}$ is atomless, this in turn is equivalent to the (Δ_2) property for $N(\xi)$ (see (**3**, § 10; **4**, II-3)). Further, it is known (see (**3**, § 4)) that $N(\xi)$ has (Δ_2) if and only if $M(\xi)$ has the following (∇_2) property: $\liminf_{\xi\to\infty} M(\eta\xi)/M(\xi) \geqslant 2\eta$ for some $\eta > 0$. Thus when $\boldsymbol{B}$ is atomless, boundedness by modular $\boldsymbol{M}(f)$ implies (relatively) weak compactness, if and only if $M(\xi)$ has (∇_2).

3. Weak compactness. In order to obtain equi-continuity, Young's inequality will be useful. In fact, from (1) it follows that

$$\int |f(t)\cdot g(t)|d\mu \leqslant \boldsymbol{M}(f) + \boldsymbol{N}(g)$$

for all $f(t)$ and $g(t)$, hence for all $\xi, \eta > 0$, and $E \in \boldsymbol{B}$

$$\int_E |f(t)\cdot g(t)|d\mu \leqslant \boldsymbol{M}(\xi f)/\xi\eta + \boldsymbol{N}(\eta\cdot g_E)/\xi\eta.$$

For $\eta > 0$ with $\boldsymbol{N}(\eta g) < \infty$ by the absolute continuity of indefinite integrals it results that $\boldsymbol{N}(\eta g_E) \to 0$ as $\mu(E) \to 0$. Accordingly if $\sup_{f\in A}\boldsymbol{M}(\xi f)/\xi \to 0$ as $\xi \to 0$, then A is equi-continuous, hence (relatively) weakly compact.

Conversely let A be weakly compact and $M(\xi)$ satisfies the condition: $M(\xi)/\xi \to 0$ as $\xi \to 0$. Suppose that $\sup_{f\in A}\boldsymbol{M}(\xi f)/\xi \geqslant 3\epsilon > 0$ for some $\epsilon > 0$ and for all $\xi > 0$. Then there exist a sequence of positive numbers (ξ_k) and a sequence $(f_k) \subset A$ such that

$$\tfrac{1}{2} > \xi_1 > \xi_2 > \ldots, \sum_k \xi_k < \infty$$

and

$$M(k\xi_k)\,.\,\mu(G) \leqslant \epsilon\cdot\xi_k, \quad \boldsymbol{M}(\xi_k\cdot f_k) \geqslant 3\epsilon\xi_k$$

for all k. Since weak compactness implies boundedness by the norm, it may be assumed that $\sup_k \boldsymbol{M}(f_k) \leqslant 1$. Since $2\cdot\xi_k \leqslant 1$ and $M(\xi)$ is convex, $\boldsymbol{M}(2\xi_k\cdot f_k) \leqslant 2\xi_k\boldsymbol{M}(f_k) \leqslant 2\xi_k$. We will prove that there exists a sequence (g_k) in L_N^* such that

$$\boldsymbol{M}(\xi_k\cdot f_{kE}) + \boldsymbol{N}(g_{kE}) = \xi_k\cdot\int_E |f_k(t)\cdot g_k(t)|d\mu \tag{4}$$

for all $E \in \boldsymbol{B}$. In fact, since $M(\xi) + N(p(\xi)) = p(\xi)\xi$ where $p(\xi)$ is the right-hand derivative of $M(\xi)$ (see (**3**, § 2)), we may take $g_k(t) \equiv p(\xi_k|f_k(t)|)$. Then $\boldsymbol{N}(g_k) \leqslant \boldsymbol{M}(2\xi_k\cdot f_k) \leqslant 2\xi_k$ for all k, because $p(\xi)\xi \leqslant M(2\xi) - M(\xi) \leqslant M(2\xi)$ for $\xi \geqslant 0$ by convexity of $M(\xi)$. Defining $g(t) \equiv \sup_k |g_k(t)|$, it can be proved by Fatou's Lemma that

$$\boldsymbol{N}(g) \leqslant \sum_k \boldsymbol{N}(g_k) \leqslant 2\sum_k \xi_k < \infty,$$

that is, $g \in L_N^*$. Writing $E_k = \{t; |f_k(t)| \geqslant k\}$ it follows that $M(k)\cdot\mu(E_k)$

$\leqslant \boldsymbol{M}(f_k) \leqslant 1$, hence $\mu(E_k) \leqslant 1/M(k) \to 0$ as $k \to 0$. By the lemma there exists j with

$$\int_{E_j} |f_j(t) \cdot g(t)| \cdot d\mu < \epsilon,$$

then from (4)

$$\boldsymbol{M}(\xi_j f_j) \leqslant M({}_j\xi_j)\mu(G) + \int_{E_j} M(\xi_j \cdot f_{j(t)}) d\mu \leqslant \epsilon\xi_j + \xi_j . \int_{E_j} |f_j(t) \cdot g(t)| d\mu \leqslant 2\epsilon\xi_j.$$

This contradicts the properties of (ξ_k) and (f_k). Thus if A is weakly compact and $M(\xi)/\xi \to 0$ as $\xi \to 0$, then

$$\sup_{f \epsilon A} \boldsymbol{M}(\xi f)/\xi \to 0 \quad \text{as} \quad \xi \to 0.$$

Now we shall consider the general case. Since $M(\xi)/\xi$ is non-decreasing for $\xi > 0$, there exists the limit $\alpha \equiv \lim_{\xi \downarrow 0} M(\xi)/\xi$. Then the function $M_1(\xi) \equiv M(\xi) - \alpha|\xi|$ is also an N-function. Since $M_1(\xi) \leqslant M(\xi) \leqslant M_1(\xi) + \alpha|\xi|$ and $\mu(G) < \infty$, it follows $L_M^* = L_{M_1}^*$. $M_1(\xi)$ satisfies the condition: $M_1(\xi)/\xi \to 0$ as $\xi \to 0$. Thus A is (relatively) weakly compact in L_M^*, if and only if $\sup_{f \epsilon A} \boldsymbol{M}_1(\xi f)/\xi \to 0$ as $\xi \to 0$. In the language of the modular $\boldsymbol{M}(f)$, it can be stated as follows:

THEOREM 1. *A subset A of L_M^* is (relatively) weakly compact, if and only if*

$$\boldsymbol{M}(\xi f)/\xi \to \alpha \int |f(t)| d\mu \quad as \quad \xi \downarrow 0$$

uniformly with respect to $f(t) \in A$, where $\alpha = \lim_{\xi \to 0} M(\xi)/\xi$.

There arises the question whether boundedness by another modular implies (relatively) weak compactness. We shall attempt to find the class of N-functions which has the property that (relatively) weak compactness in L_M^* is equivalent to boundedness by the modular defined by an N-function in the class.

An N-function $\Phi(\xi)$ is said to *increase more rapidly* than another $M(\xi)$, if for any $\eta > 0$ there exist $\rho, \xi_0 > 0$ such that

(5) $$\Phi(\rho\xi) \geqslant \rho\eta M(\xi) \qquad \text{for } \xi \geqslant \xi_0.$$

Remark by the way that $M(\xi)$ has (∇_2), if and only if it increases more rapidly than itself. The condition (5) may be stated in the following form: for any $\epsilon > 0$ there exist $\delta, \xi_0 > 0$ such that $\epsilon\Phi(\xi) \geqslant M(\delta\xi)/\delta$ for $\xi \geqslant \xi_0$. Now if a subset A of L_M^* is bounded by the modular $\boldsymbol{\Phi}(f)$ (where $\Phi(\xi)$ increases more rapidly than $M(\xi)$), it is not hard to see that A satisfies the condition in Theorem 1, and consequently it is (relatively) weakly compact.

Conversely let A be weakly compact in L_M^*. We distinguish between two cases. First if there exists $\gamma > 0$ such that ess. $\sup|f(t)| \leqslant \gamma$ for all $f(t) \in A$, the function $\Phi(\xi) \equiv M(\xi)^2$ is an N-function increasing more rapidly than $M(\xi)$,

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and $\sup_{f\in A}\mathbf{\Phi}(f) \leqslant \Phi(\gamma)\cdot\mu(G) < \infty$, that is, A is bounded by modular $\mathbf{\Phi}(f)$. Next consider the case $\sup_{f\in A}\gamma(f) = \infty$ where $\gamma(f) = \text{ess.}\sup|f(t)|$. Writing $M_1(\xi) = M(\xi) - \alpha|\xi|$ (where $\alpha = \lim_{\xi\downarrow 0}M(\xi)/\xi$) by Theorem 1 there can be found a sequence of positive numbers (ξ_k) such that $\xi_1 > \xi_2 > \ldots$ and $\sup_{f\in A}\mathbf{M}_1(\xi_k f)/\xi_k \leqslant 1/2^{2k}$ for all k. Then writing

$$\Phi(\xi) = \sum_{\kappa} 2^k M_1(\xi_k\cdot\xi)/\xi_k,$$

by Fatou's theorem it results in

$$\int\Phi(f(t))d\mu \leqslant \sum_{\kappa} 2^k\mathbf{M}_1(\xi_k f)/\xi_k \leqslant \sum_{\kappa} 1/2^k \leqslant 1$$

consequently $\Phi(f(t)) < \infty$ almost everywhere for each $f(t)$. Since $\Phi(\xi)$ is non-decreasing for $\xi > 0$, this implies $\Phi(\xi) < \infty$ for all $\xi < \gamma(f)$, finally for all $\xi > 0$ because $\sup_{f\in A}\gamma(f) = \infty$. Then it is clear that $\Phi(\xi)$ is an N-function increasing more rapidly than $M(\xi)$ and A is bounded by the modular $\mathbf{\Phi}(f)$. Thus we obtain

THEOREM 2. *A subset A of $L_M{}^*$ is (relatively) weakly compact, if and only if it is bounded by the modular defined by an N-function (depending on A) $\Phi(\xi)$ increasing more rapidly than $M(\xi)$.*

3. Continuous semi-norms. Let $N(\xi)$ and $\Psi(\xi)$ be complementary to $M(\xi)$ and $\Phi(\xi)$ respectively. It is not hard to see that $\Psi(\xi)$ increases more rapidly than $N(\xi)$ if and only if $\Phi(\xi)$ is *completely weaker* than $M(\xi)$ in the following sense: for any $\eta > 0$ there exist ρ, $\xi_0 > 0$ such that

$$\Phi(\eta\xi) \leqslant \rho M(\xi) \qquad \text{for } \xi \geqslant \xi_0.$$

Remark by the way that $M(\xi)$ satisfies (Δ_2) if and only if it is completely weaker than itself.

Let $||f||$ be a continuous semi-norm on $L_M{}^*$ (with which $L_M{}^*$ is not necessarily complete). On account of the lemma, the subset

$$A \equiv \left\{g(t);\ \int|f(t)\cdot g(t)|d\mu \leqslant 1 \text{ for all } f(t) \text{ with } ||f|| \leqslant 1\right\}$$

is weakly compact in $L_N{}^*$. On the other hand, it is easily seen that $||f|| = \sup_{g\in A}\int|f(t)\cdot g(t)|d\mu$. By Theorem 2 A is bounded by the modular defined by an N-function $\Psi(\xi)$ increasing more rapidly than $N(\xi)$, say $\sup_{g\in A}\Psi(2g) \leqslant 1$. Then as is stated above, $\Phi(\xi)$, complementary to $\Psi(\xi)$, is completely weaker than $M(\xi)$. From (3) it follows that $||f|| \leqslant |||f|||_\Phi$ for all $f(t) \in L_M{}^*$. On the other hand, the norm $|||f|||_\Phi$ is continuous on $L_M{}^*$ (cf. (**3**, § 10)).

THEOREM 3. *A semi-norm $||f||$ on $L_M{}^*$ is continuous, if and only if it is majorated by the norm defined by an N-function (depending on $||f||$) $\Phi(\xi)$ completely weaker than $M(\xi)$.*

Finally we shall consider uniformly continuous semi-norms on $L_M{}^*$. Let $||f||$ be uniformly continuous. It is not hard to see that in this case the set A in the proof of Theorem 3 has the following property: $\sup_{f\in A}|||f_E|||_N \to 0$ as $\mu(E) \to 0$. On account of (2), this in turn means that $\sup_{f\in A} N(kf_E) \to 0$ as $\mu(E) \to 0$ for all k, consequently $\sup_{f\in A} \mathbf{N}(kf) < \infty$ for all k.

If, in general, a subset B of $L_N{}^*$ has the property that $\sup_{f\in B} N(f_E) \to 0$ as $\mu(E) \to 0$, there exists a sequence of positive numbers (α_k) such that $\alpha_1 > \alpha_2 > \ldots$ and $\sup_{f\in B}\mathbf{N}(f_E) \leqslant 1/2^{2k}$ for all $E \in \mathbf{B}$ with $\mu(E) \leqslant \alpha_\kappa$. Here we may assume that $\sup_{f\in B} N(f) \leqslant 1$. Consider an N-function $\Psi_0(\xi)$ whose right-hand derivative is equal to 2^k on each interval $[1/\alpha_k, 1/\alpha_{k+1})$. By definition $\Psi_0(\xi) \leqslant 2^k\xi$ on each such interval. For a fixed $f(t) \in B$, writing $E_k \equiv \{t; \beta_k \leqslant |f(t)| < \beta_{k+1}\}$ where $\beta_k = N^{-1}(1/\alpha_k)$, it follows that

$$\int \Psi_0[N(f(t))]d\mu \leqslant 2N(\beta_1)\mu(G) + \sum_\kappa 2^k \int_{Ek} N(f(t))d\mu$$
$$\leqslant 2N(\beta_1)\mu(G) + \sum_\kappa 1/2^k \leqslant 2N(\beta_1)\mu(G) + 1,$$

because $\mu(E_k)N(\beta_k) \leqslant \mathbf{N}(f) \leqslant 1$, hence $\mu(E_k) \leqslant 1/N(\beta_k) = \alpha_k$. Thus

$$\sup_{f\in B} \int \Psi_0[N(f(t))]d\mu < \infty.$$

Returning to the subject, in this way there exists a sequence of N-functions $(\Psi_k(\xi))$ such that

$$\sup_{f\in A} \int \Psi_k[N(2kf(t))]d\mu \leqslant 1/2^k$$

for all k. We shall treat only the case $\sup_{f\in A}\gamma(f) = \infty$ where $\gamma(f) =$ ess. $\sup|f(t)|$. Then, as in the proof of Theorem 2, an N-function is obtained by setting $\Psi(\xi) = \sum_\kappa \Psi_k[N(k\xi)]$, and from the properties of $(\Psi_k(\xi))$ it follows that $\sup_{f\in A}\Psi(2f) \leqslant 1$, hence by (3) $||f|| \leqslant |||f|||_\Phi$ for all $f(t) \in L_M{}^*$ where $\Phi(\xi)$ is complementary to $\Psi(\xi)$. $\Psi(\xi)$ is *essentially stronger* than $N(\xi)$ in the sense of Krasnoselskii and Ruttickii (**3**, § 13), that is,

$$\lim_{\xi\to\infty} N(k\xi)/\Psi(\xi) = 0 \qquad \text{for all } k.$$

On the other hand, $\Phi(\xi)$ is *essentially weaker* than $M(\xi)$, that is, $\lim_{\xi\to\infty}\Phi(k\xi)/M(\xi) = 0$ for all k, and the norm $|||f|||_\Phi$ is uniformly continuous on $L_M{}^*$ (see (**3**, § 13)).

THEOREM 4. *A semi-norm $||f||$ on $L_M{}^*$ is uniformly continuous, if and only if it is majorated by the norm defined by an N-function (depending on $||f||$) $\Phi(\xi)$ essentially weaker than $M(\xi)$.*

Analogous problems about Orlicz sequence spaces will be treated in another place.

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Research Institute of Applied Electricity
Hokkaidô University, Sapporo, Japan