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Oscillation theorems for semilinear hyperbolic and ultrahyperbolic operators

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The oscillation property of the semilinear hyperbolic or ultra-hyperbolic operator L defined by

$$L[u] \equiv \Delta_x u - \sum_{i,j=1}^{m} \frac{\partial}{\partial y_i}\left(a_{ij}(x, y)\frac{\partial u}{\partial y_j}\right) + f(x, y, u)$$

is studied. Sufficient conditions are provided for all solutions of $uL[u] \leq 0$ satisfying certain boundary conditions to be oscillatory. The basis of our results is the non-existence of positive solutions of the associated differential inequalities.

Oscillation criteria for linear hyperbolic differential equations have been obtained by Kahane [1], Kreith [2, 3], Pagan [7], Travis [8], and Young [9]. More recently, the author and Yoshida [5] established oscillation theorems for linear ultrahyperbolic operators. The purpose of this paper is to study the oscillation property of a class of nonlinear hyperbolic or ultrahyperbolic equations and inequalities. Use is made of some of the techniques and results developed by Naito and Yoshida [4] and Noussair and Swanson [6].

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ denote points in R^n and R^m, respectively. Let H be an unbounded domain in R^n defined by

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$$H = \{x = (x_1, \ldots, x_n) : 0 < x_i < \infty, i = 1, \ldots, n\} ,$$

and let G be a bounded domain in R^m with piecewise smooth boundary.

The partial differential operator to be considered in this paper is

$$L[u] \equiv \Delta_x u - \sum_{i,j=1}^{m} \frac{\partial}{\partial y_i} \left(a_{ij}(x, y) \frac{\partial u}{\partial y_j}\right) + f(x, y, u) ,$$

where Δ_x denotes the laplacian in R^n ; that is, $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$.

The coefficients $a_{ij}(x, y)$ are real-valued functions of class $C^1(\overline{H \times G})$, $(i, j = 1, \ldots, m)$, and $f(x, y, \xi)$ is a real-valued function of class $C^0(\overline{H \times G} \times R^1)$. The matrix (a_{ij}) is assumed to be symmetric and positive definite in $H \times G$. The domain D_L of L is the set of all real-valued functions of class $C^2(H \times G) \cap C^1(\overline{H \times G})$.

For each $u \in D_L$ we define the function $g(x)$ by

(1) $$g(x) = \frac{1}{\kappa} \int_G u(x, y)dy , \quad \left(\kappa = \int_G dy\right) .$$

LEMMA 1. *Assume that:*

(i) $f(x, y, \xi) \geq p(x)\phi(\xi)$ *for all* $(x, y) \in H \times G$ *and for all* $\xi > 0$ *, where* p *is continuous and non-negative in* H *and* ϕ *is continuous, non-negative, and convex in* $(0, \infty)$ *;*

(ii) $u(x, y) \in D_L$ *is a positive solution of the inequality* $L[u] \leq 0$ *in* $H \times G$ *and satisfies the boundary condition* $u = 0$ *on* $H \times \partial G$ *.*

Then the function $g(x)$ *given by* (1) *satisfies the differential inequality*

(2) $$\Delta_x g + p(x)\phi(g) \leq 0 , \quad x \in H .$$

Proof. Since $\Delta_x g(x) = \frac{1}{\kappa} \int_G \Delta_x u dy$, it follows from Green's formula that

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$$\Delta_x g(x) \le \frac{1}{\kappa}\int_G \sum_{i,j=1}^{m} \frac{\partial}{\partial y_i}\left(a_{ij}(x, y)\frac{\partial u}{\partial y_j}\right)dy - \frac{1}{\kappa}\int_G f(x, y, u)dy$$
$$= \frac{1}{\kappa}\int_{\partial G}\frac{\partial u}{\partial \nu}\,d\tau - \frac{1}{\kappa}\int_G f(x, y, u)dy ,$$

where $\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{m} a_{ij}(x, y)\nu_i \frac{\partial}{\partial y_j}$, $\nu = (\nu_1, \ldots, \nu_m)$ being the unit exterior normal vector to ∂G , and τ denotes the measure on ∂G . In view of the fact that $u > 0$ in $H \times G$ and $u = 0$ on $H \times \partial G$, $\frac{\partial u}{\partial \nu}$ must be non-positive. Therefore, using hypothesis *(i)* and Jensen's inequality applied to $\phi(u)$ over G , we get

$$\Delta_x g(x) \le -\frac{p(x)}{\kappa}\int_G \phi(u)dy$$
$$\le -p(x)\phi\left[\frac{1}{\kappa}\int_G u(x, y)dy\right] ,$$

which is the desired inequality (2).

We shall use the notation

$$H_r = H \cap \{x \in R^n : |x| > r\} , \quad r > 0 .$$

DEFINITION. A function $u(x, y) \in D_L$ which satisfies

(3) $\qquad uL[u] \le 0$ in $H \times G$ and $u = 0$ on $H \times \partial G$

is said to be *oscillatory* in $H \times G$ if it has a zero in $H_r \times G$ for every $r > 0$.

PROPOSITION 1. *Every solution of* (3) *is oscillatory in* $H \times G$ *if in addition to hypothesis (i) of Lemma* 1 *the following conditions are satisfied:*

(i) $f(x, y, -\xi) = -f(x, y, \xi)$ *for all* $(x, y) \in H \times G$ *and for all* $\xi > 0$ *;*

(ii) the differential inequality (2) *has no solution which is positive in* H_r *for any* $r > 0$.

Proof. Suppose to the contrary that there exists a solution $u(x, y)$

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of (3) which has no zero in $H_{r'} \times G$ for some $r' > 0$. If $u > 0$ in $H_{r'} \times G$, then $L[u] \le 0$ in $H_{r'} \times G$, and by Lemma 1, the function $g(x)$ defined by (1) is a positive solution of (2) in $H_{r'}$, contradicting the hypothesis *(ii)*.

Likewise, u cannot be negative in $H_{r'} \times G$, or else $-u$ would be a positive solution of (3).

In the case when $n = 1$, the operator L reduces to a hyperbolic operator and the inequality (2) becomes the ordinary differential inequality

$$\frac{d^2g}{dx^2} + p(x)\phi(g) \le 0 \ , \quad x > 0 \ . \tag{4}$$

Sufficient conditions for the non-existence of eventually positive solutions of (4) have recently been established by Naito and Yoshida [4] and Noussair and Swanson [6]. Here we present an oscillation criterion for the semilinear hyperbolic operator L $(n = 1)$ which follows from Proposition 1 combined with a result of [4, Theorem 2.1].

THEOREM 1. *Assume that the following conditions are satisfied:*

(I) $f(x, y, \xi) \ge p(x)\phi(\xi)$ *for all* $(x, y) \in (0, \infty) \times G$ *and for all* $\xi > 0$, *where* p *is continuous and non-negative in* $(0, \infty)$ *and* ϕ *is continuous, non-negative, and convex in* $(0, \infty)$;

(II) $f(x, y, -\xi) = -f(x, y, \xi)$ *for all* $(x, y) \in (0, \infty) \times G$ *and for all* $\xi > 0$;

(III) *there exist positive continuous functions* ϕ_1 *and* ϕ_2 *in* $(0, \infty)$ *such that*

(i) $\phi(\xi) \ge \phi_1(\xi)\phi_2(\xi)$ *for all* $\xi > 0$,

(ii) ϕ_1 *is non-increasing and* ϕ_2 *is non-decreasing for all* $\xi > 0$,

(iii) $\int_\varepsilon^\infty \frac{d\xi}{\phi_2(\xi)} < \infty$ *for some* $\varepsilon > 0$,

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$$(iv) \quad \int^{\infty} \xi p(\xi)\phi_1(k\xi)d\xi = \infty \quad \textit{for all} \quad k > 0 .$$

Then every solution of (3) $(n = 1)$ *is oscillatory in* $(0, \infty) \times G$.

COROLLARY 1. *Consider the semilinear hyperbolic equation*

$$(5) \qquad \frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^{m} \frac{\partial^2 u}{\partial y_i^2} + c(x)u^{\gamma} = 0 ,$$

where $c(x)$ *is a non-negative continuous function in* $(0, \infty)$ *and* $\gamma > 1$ *is the quotient of odd integers. Every solution* u *of* (5) *satisfying the boundary condition* $u = 0$ *on* $(0, \infty) \times \partial G$ *is oscillatory in* $(0, \infty) \times G$ *if*

$$\int^{\infty} xc(x)dx = \infty .$$

Next we consider the case $n \geq 2$. Letting (r, θ) denote hyper-spherical coordinates for R^n , H can be rewritten as

$$H = \{(r, \theta) : 0 < r < \infty, \theta \in \Theta\} ,$$

where Θ is the domain defined by

$$\Theta = \{\theta = (\theta_1, \ldots, \theta_{n-1}) : 0 < \theta_i < \pi/2, i = 1, \ldots, n-1\} .$$

The following notation will be used:

$$S_r = \{x \in R^n : |x| = r\} ,$$

$$H(r) = H \cap S_r ,$$

$$H(s, t) = \{x \in H : s < |x| < t\} .$$

The measure on S_r and S_1 will be denoted by σ and ω , respectively. The unit exterior normal vector to ∂H will be denoted by η .

Associated with every function $u \in D_L$, we define a function $h(r)$ in $(0, \infty)$ by the equation

$$(6) \qquad h(r) = \frac{1}{\sigma_r} \int_{H(r)} g(x)d\sigma ,$$

where $g(x)$ is the function given by (1) and σ_r denotes the area of

$H(r)$.

By employing the technique of Noussair and Swanson [6], we obtain the following principal tool.

LEMMA 2. *Assume that the hypotheses (i) and (ii) of Lemma* 1 *hold and, moreover, that*

(i) $p(x) \geq q(|x|)$ *in* H_{r_0} *for some* $r_0 > 0$ *, where* q *is continuous and non-negative in* $[r_0, \infty)$ *;*

(ii) $\frac{\partial g}{\partial \eta} \geq 0$ *on* ∂H_{r_0} *, where* g *is given by* (1).

Then the function $h(r)$ *defined by* (6) *satisfies the ordinary differential inequality*

$$\frac{d}{dr}\left(r^{n-1}\frac{dh}{dr}\right) + r^{n-1}q(r)\phi(h) \leq 0\ , \quad r \geq r_0\ . \tag{7}$$

Proof. Green's formula yields the integral identity

$$\int_{H(r_0,r)} \Delta_x g\,dx = \int_{H(r)} \frac{\partial g}{\partial r}\,d\sigma - r_0^{n-1}\int_{H(1)} \frac{\partial g}{\partial r}\,d\omega + \int_{r_0}^{r} d\rho \int_{\partial\Theta} \frac{\partial g}{\partial \eta}(\rho,\ \theta)\,d\mu \tag{8}$$

for any $r \geq r_0$, where μ denotes the measure on $\partial\Theta$. Since the following identities hold,

$$\frac{d}{dr}\left(\int_{H(r_0,r)} \Delta_x g\,dx\right) = \int_{H(r)} \Delta_x g\,d\sigma\ ,$$

$$\frac{d}{dr}\left(\int_{H(r)} \frac{\partial g}{\partial r}\,d\sigma\right) = \omega_1 \frac{d}{dr}\left(r^{n-1}\frac{dh}{dr}\right)\ ,$$

where ω_1 denotes the area of $H(1)$, differentiating (8) with respect to r and using condition *(ii)*, we obtain

$$\int_{H(r)} \Delta_x g\,d\sigma \geq \omega_1 \frac{d}{dr}\left(r^{n-1}\frac{dh}{dr}\right)\ . \tag{9}$$

On the other hand, applying Jensen's inequality to $\phi(g)$ over $H(r)$ and

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using condition *(i)*, we find

$$(10) \qquad \omega_1 r^{n-1} q(r)\phi(h) \le \int_{H(r)} p\phi(g)d\sigma \ .$$

The conclusion (7) now follows from Lemma 1, (9), and (10). This completes the proof.

PROPOSITION 2. *Let the following conditions hold.*

(i) $f(x, y, \xi) \ge q(|x|)\phi(\xi)$ *in* $H_{r_0} \times G \times (0, \infty)$ *for some* $r_0 > 0$, *where* q *is continuous and non-negative in* $[r_0, \infty)$ *and* ϕ *is continuous, non-negative, and convex in* $(0, \infty)$.

(ii) $f(x, y, -\xi) = -f(x, y, \xi)$ *for all* $(x, y) \in H_{r_0} \times G$ *and for all* $\xi > 0$.

(iii) The ordinary differential inequality (7) *has no positive solution in* $[r, \infty)$ *for any* $r \ge r_0$.

Then every solution u *of* (3) *which satisfies*

$$(11) \qquad \frac{\partial u}{\partial \eta} - \lambda(x, y)u = 0$$

on $\partial H_{r_0} \times G$ *is oscillatory in* $H \times G$, *where* $\lambda(x, y)$ *is a non-negative continuous function on* $\partial H_{r_0} \times G$.

Proof. If u is a solution of (3) which satisfies (11) and is positive in $H_{r_1} \times G$ for some $r_1 \ge r_0$, then we find from (1) and (11) that

$$\frac{\partial g}{\partial \eta} = \frac{1}{\kappa}\int_G \frac{\partial u}{\partial \eta}\, dy$$
$$= \frac{1}{\kappa}\int_G \lambda(x, y)u\,dy \ge 0 \ .$$

Define the function $h(r)$ by (6). Then, proceeding as in the proof of Proposition 1 and using Lemma 2, we can show that $h(r)$ is a positive solution of (7) in $[r_1, \infty)$. But this is a contradiction.

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The above proposition together with the results of Naito and Yoshida [4, Theorems 2.1 and 2.4] yields the following oscillation criteria for the semilinear ultrahyperbolic operator L .

THEOREM 2. *Let* $n = 2$ *and assume that:*

(i) the hypotheses (i) and (ii) of Proposition 2 *are satisfied;*

(ii) there exist positive continuous functions ϕ_1 *and* ϕ_2 *in* $(0, \infty)$ *such that (i), (ii), and (iii) of Theorem* 1 (III) *hold, and*

$$\int^{\infty} \xi(\log \xi)q(\xi)\phi_1(k \log \xi)d\xi = \infty \quad \text{for all} \quad k > 0 .$$

Then every solution of (3) *satisfying* (11) *is oscillatory in* $H \times G$.

THEOREM 3. *Let* $n \geq 3$ *and suppose that:*

(i) the hypotheses (i) and (ii) of Proposition 2 *are satisfied;*

(ii) there exist positive continuous functions ϕ_1, ϕ_2, ϕ_3 , *and* ϕ_4 *in* $(0, \infty)$ *such that*

$$\phi(\xi) \geq \phi_1(\xi)\phi_2(\xi) \quad \text{for all} \quad \xi > 0 ,$$

ϕ_1 *is non-increasing and* ϕ_2 *is non-decreasing for all* $\xi > 0$,

$$\phi_2(\xi\zeta) \geq \phi_3(\xi)\phi_4(\zeta) \quad \text{for all} \quad \xi, \zeta \quad \text{such that} \quad 0 < \xi < 1/\zeta ,$$

$$\int_{\varepsilon}^{\infty} \frac{d\xi}{\phi_3(\xi)} < \infty \quad \text{for some} \quad \varepsilon > 0 ,$$

$$\int^{\infty} \xi^{n-1}q(\xi)\phi_4\left(\frac{\xi^{2-n}}{n-2}\right)d\xi = \infty .$$

Then every solution of (3) *satisfying* (11) *is oscillatory in* $H \times G$.

COROLLARY 2. *Consider the semilinear ultrahyperbolic equation*

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=1}^{m} \frac{\partial^2 u}{\partial y_j^2} + c(|x|)u^{\gamma} = 0 , \tag{12}$$

where c *is a non-negative continuous function in* $(0, \infty)$ *and* $\gamma > 1$ *is the quotient of odd integers. Every solution* u *of* (12) *satisfying* (11)

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and $u = 0$ *on* $H \times \partial G$ *is oscillatory in* $H \times G$ *if*

$$\int^{\infty} \psi_n(r)c(r)dr = \infty ,$$

where

$$\psi_n(r) = \begin{cases} r \log r & \text{if } n = 2 , \\ r^{n-1+\gamma(2-n)} & \text{if } n \geq 3 . \end{cases}$$

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