FREE INVERSE SEMIGROUPS

Dedicated to the memory of Hanna Neumann

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In an important recent paper H. E. Scheiblich gave a construction of free inverse semigroups that throws considerable light on their structure [1]. In this note we give an alternative description of free inverse semigroups. What Scheiblich did was to construct a free inverse semigroup as a semigroup of isomorphisms between principal ideals of a semilattice E, say, thus realising free inverse semigroups as inverse subsemigroups of the semigroup T_E, a kind of inverse semigroup introduced and exploited by W. D. Munn [2]. We go instead directly to canonical forms for the elements of a free inverse semigroup. The connexion between our construction and that of Scheiblich's will be clear. There are several alternative procedures possible to reach our construction on which we comment on the way.

1. Introduction

Let X be a non-empty set. Let $X^{-1} = \{x^{-1} \mid x \in X\}$ be a set disjoint from X, where $x \mapsto x^{-1}$ is a one-to-one mapping of X upon X^{-1}. Set $Y = X \cup X^{-1}$. Denote by $\mathscr{F}_Y$ the free semigroup on Y: the elements of $\mathscr{F}_Y$ are the non-empty words in the alphabet Y and the product of any two words u, v in $\mathscr{F}_Y$ is the word uv obtained by juxtaposition of u and v. $\mathscr{F}_Y^1$ denotes the semigroup with identity obtained from $\mathscr{F}_Y$ by adjoining an identity element 1 to $\mathscr{F}_Y$. We shall also call 1 a word.

A word in $\mathscr{F}_Y$ is said to be *reduced* if it does not contain a syllable xx^{-1} or a syllable $x^{-1}x$, $x \in X$, as a subword. The word 1 of $\mathscr{F}_Y^1$ is also said to be reduced. Any word of $\mathscr{F}_Y^1$ determines a unique reduced word obtained from it by deleting, in succession, any syllables xx^{-1} or $x^{-1}x$, $x \in X$. The set of all reduced words in $\mathscr{F}_Y^1$ we shall denote by G. G is then the free group on X under the product

$$\cdot : (u, v) \to u \cdot v, u, v \in G,$$

where $u \cdot v$ denotes the reduced word determined by uv.

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If g is an element of G we shall denote its inverse in G by g^{-1}. Set

$$F = \{g_1 g_1^{-1} \cdots g_k g_k^{-1} \big| k \geqq 1, \text{ each } g_i \in G \setminus \{1\}\},$$

and set $F^1 = F \cup \{1\}$. Then F is a subsemigroup of $\mathscr{F}_Y$ and F^1 is a subsemigroup of $\mathscr{F}_Y^1$. Denote by W^1 the set F^1G:

$$W^1 = \{fg \,|\, f \in F^1, g \in G\}.$$

The product fg is here to be evaluated in $\mathscr{F}_Y^1$. Finally set $W = W^1 \setminus \{1\}$.

There are now several possible ways of proceeding to construct I_X, the free inverse semigroup on X. We shall proceed by first constructing a semilattice E, say, that will form the semilattice of idempotents of I_X. We shall then form a semi-direct product* of E^1 and G. The resulting semigroup J_X^1 is not an inverse semigroup and indeed is not regular. We shall then define a congruence on J_X^1 such that each congruence class contains precisely one regular element. The quotient of J_X^1 modulo this congruence will be I_X^1. Finally $I_X^1 = I_X \setminus \{1\}$. Alternatively I_X^1 may be identified with the semigroup of regular elements of J_X^1; and this is perhaps the simpler approach.

It will be seen at the end that what has effectively been done is to introduce an equivalence relation on W and then to define a product on the equivalence classes to give I_X. Although this latter procedure is on the face of it more direct—we shall give the formal definitions required later —the verification that the constructed object I_X is an inverse semigroup involves much the same argument as in the procedure we have chosen to adopt.

Another possible procedure would be to consider the cartesian product $E^1 \times G$, define an equivalence on this, and then define a product on the resulting set of equivalence classes to give I_X^1 and so I_X. We shall not give the details of this approach.

For each of the approaches mentioned we can instead go directly to I_X and suppress the introduction of the 1. However the introduction of the 1 smoothes the computational path.

2. The construction of I_X

LEMMA 1. *Each element of F is uniquely expressible in the form $g_1 g_1^{-1} \cdots g_k g_k^{-1}$, where $k \geqq 1$, and $g_i \in G \setminus \{1\}, i = 1, 2, \cdots, k$.*

PROOF. Let $a = g_1 g_1^{-1} \cdots g_k g_k^{-1}$. Then, as an element of $\mathscr{F}_Y$, a is a unique product of elements of Y, $a = y_1 \cdots y_n$, say. Each y_j is itself a reduced word, and so belongs to G. Consider the product in G, $y_1 \cdot \cdots \cdot y_n$. As we successively evaluate $y_1, y_1 \cdot y_2, \cdots,$ it is clear that the first time we get the value 1 is when we have

* What we term a semi-direct product (see below) is a generalization of the usual concept.

evaluated the product as far as the initial segment $g_1g_1^{-1}$ of a. This determines g_1 uniquely. Similarly, the next time the value of the product is 1 is on reaching the end of the initial segment $g_1g_1^{-1}g_2g_2^{-1}$. This determines g_2 uniquely. A continuation of the argument shows that the remaining g_j are also uniquely determined.

Let us agree to write $g \leqq h$, for g, h in G, if g is an initial segment of h, i.e. if there exists g', say, in G, such that $g \cdot g' = gg' = h$. Here, to require that $g \cdot g' = gg'$ means requiring either that gg' is reduced as it stands or that one of g and g' equals 1. Hence, for example, $1 \leqq g$, for all g in G. Then $\leqq$ is a partial order on G.

If A and B are nonempty subsets of G, we mean by $A \leqq B$ that each element of A is less than some element of B. If A is finite then Max A will denote the set of its maximal elements.

Because of Lemma 1, if $f = g_1g_1^{-1} \cdots g_kg_k^{-1} \in F$, then the set $\{g_1, \cdots, g_k\}$ is uniquely determined by f. We call it the *domain*, Dom f, of f, and if $f = 1$, we define Dom $f = \{1\}$. We define the *carrier* of f to be Car f = Max Dom f.

We say that two elements e and f of F^1 are $\sim$ equivalent if they have the same carrier: $e \sim f$ if and only if Car e = Car f.

LEMMA 2. *$\sim$ is a congruence on F^1.*

PROOF. $\sim$ is clearly an equivalence relation; that it is a congruence follows from the fact that $\leqq$ is a partial order on G.

We shall denote $F^1/\sim$ by E^1. Note that 1 is the sole element of its $\sim$ equivalence class. Set $E = E^1 \backslash \{1\}$. Effectively also $E = F/\sim$.

It will frequently be convenient to denote any element $f\sim$ of E^1 simply by f. We shall consequently allow ourselves when following this convention to replace f at any stage by any word of F^1 $\sim$ equivalent to it.

LEMMA 3. *E is a semilattice and E^1 is a semilattice obtained from E by the adjunction of an identity element* 1.

Relative to the natural order of the semilattice, $e \leqq f$ if and only if Car $f \leqq$ Car e, *for $e, f \in E^1$. Hence the set of maximal elements relative to its natural order, of E, is*

$$\{xx^{-1} \mid x \in X\} \cup \{x^{-1}x \mid x \in X\}.$$

PROOF. Clearly, if $e, f \in E^1$, then

$$\text{Car } e^2 = \text{Car } e,$$

and

$$\text{Car } ef = \text{Car } fe.$$

Thus E^1 is a semilattice.

Now $e \leqq f$ in the semilattice E^1, relative to its natural order, means that $ef = e$. But $ef = e$, i.e. $ef \sim e$, if and only if Car(ef) = Car e. Since Car(ef) = Max(Car $e \cup$ Car f), therefore Car(ef) = Car e if and only if Car $f \leqq$ Car e.

Each element x of G determines a transformation ε_x of E^1 defined as follows. If $e = 1$, we set $e\varepsilon_x = xx^{-1}$. If $e = g_1g_1^{-1} \cdots g_kg_k^{-1}$ then we set

$$e\varepsilon_x = xx^{-1}(x \cdot g_1)(x \cdot g_1)^{-1} \cdots (x \cdot g_k)(x \cdot g_k)^{-1}.$$

We must show that the ε_x are well-defined on E^1, i.e. we must show that, if $e \sim f$, then $e\varepsilon_x \sim f\varepsilon_x$. The following lemma provides what is needed. If $A \subseteq G$ and $x \in G$, set $x \cdot A = \{x \cdot a \mid a \in A\}$.

Lemma 4. *Let $A \subseteq G$ and $x \in G$. Let $a \in A$ and suppose that $x \cdot a$ is a maximal element of $x \cdot A$. Then either a is a maximal element of A or $x \cdot a \leqq x$.*

Proof. Suppose that a is not a maximal element of A; then there exists b in A such that $a < b$, i.e. such that $b = ac = a \cdot c$ for some $c \neq 1$. This implies that $x \cdot a < x \cdot b$ unless the process of reducing the words absorbs the whole of a, i.e. unless $x = x'a^{-1} = x' \cdot a^{-1}$ for some x'. (In such a case $x \cdot a = x'$, while $xb = x'c$, and any further reduction between x' and c will lead to a situation where $x \cdot a < x \cdot b$.) But if $x = x' \cdot a^{-1}$ then

$$x \cdot a = x' \leqq x.$$

Now suppose that $e \sim f$, $e, f \in F^1$. If $e = 1$, then $f = 1$ and so $e\varepsilon_x = f\varepsilon_x$. If $e = g_1g_1^{-1} \cdots g_kg_k^{-1}$, set $A = \text{Dom } e$. Then $\text{Car } e\varepsilon_x = \text{Max}(\{x\} \cup x \cdot A)$, the possibility that some of the $x \cdot g_i$ equal 1 not affecting this statement. From Lemma 4, if $x \cdot a$. $a \in A$, is a maximal element of $x \cdot A$, either a is a maximal element of A, i.e. $a \in \text{Car } e$, or $x \cdot a \leqq x$. Thus $\text{Car } e\varepsilon_x = \text{Max}(\{x\} \cup x \cdot \text{Car } e)$. Since $\text{Car } e = \text{Car } f$, therefore $\text{Car } e\varepsilon_x = \text{Car } f\varepsilon_x$, i.e. $e\varepsilon_x \sim f\varepsilon_x$.

We shall frequently write e^x for $e\varepsilon_x$.

Lemma 5. *For each x in G, ε_x is an endomorphism of E^1.*

Proof. Let $e, f \in F^1$. If $e = 1$, then $(ef)^x = f^x$ and $e^xf^x = xx^{-1}f^x \sim f^x$ from the definition of $\sim$. Similarly, if $f = 1, (ef)^x = e^xf^x$.

Suppose that $e \neq 1$ and $f \neq 1$. Set $A = \text{Dom } e$ and $B = \text{Dom } f$. Then $\text{Dom}(ef) = A \cup B$. Hence

$$\begin{aligned} \text{Car}(ef)^x &= \text{Max}(\{x\} \cup x \cdot (A \cup B)) \\ &= \text{Max}((\{x\} \cup x \cdot A) \cup (\{x\} \cup x \cdot B)) \\ &= \text{Car } e^xf^x. \end{aligned}$$

Thus $(ef)^x \sim e^xf^x$. This completes the proof.

The mapping $x \to \varepsilon_x, x \in G$, is almost an antihomomorphism of G into $\text{End } E^1$. The extent to which it fails to be is clarified in the next lemma.

LEMMA 6. *Let* $e \in E^1, x, y \in G$. *Then* $xx^{-1}e^{x.y} = (e^y)^x$.

PROOF. If $e = 1$, then

$$(e^y)^x = (yy^{-1})^x$$
$$= \begin{cases} xx^{-1}(x \cdot y)(x \cdot y)^{-1}, & \text{if } y \neq 1 \\ xx^{-1} & , \textit{if } y = 1 \end{cases}$$
$$\sim xx^{-1}(x \cdot y)(x \cdot y)^{-1}, \text{ in both cases,}$$
$$= xx^{-1}e^{x.y}.$$

If $e = g_1 g_1^{-1} \cdots g_k g_k^{-1}$, then

(1) $$(e^y)^x = (yy^{-1}(y \cdot g_1)(y \cdot g_1)^{-1} \cdots (y \cdot g_k)(y \cdot g_k)^{-1})^x.$$

If $y \neq 1$ and also no $y \cdot g_i$ equals 1, then

$$(e^y)^x = xx^{-1}(x \cdot y)(x \cdot y)^{-1}(x \cdot y \cdot g_1)(x \cdot y \cdot g_1)^{-1} \cdots (x \cdot y \cdot g_k)(x \cdot y \cdot g_k)^{-1}$$
$$= xx^{-1}e^{x.y},$$

as required. If $y = 1$, then

$$(e^y)^x = e^x = xx^{-1}e^x,$$

by definition of $\sim$,

$$= xx^{-1}e^{x.y},$$

as required.

Finally, suppose that $y \neq 1$, and that some $y \cdot g_i$ equal 1. Then the corresponding factors $(y \cdot g_i)(y \cdot g_i)^{-1}$ will be suppressed in the right-hand side of (1) above. Hence, in evaluating this right-hand side the corresponding expressions $(x.y.g_i)\,(x \cdot y \cdot g_i)^{-1}$ will not occur. However, since each such expression is equal to xx^{-1}, and xx^{-1} already occurs on the right-hand side, from the definition of $\sim$, we have, with no factors missing, from (1),

$$(e^y)^x = xx^{-1}(x \cdot y)(x \cdot y)^{-1}(x \cdot y \cdot g_1)(x \cdot y \cdot g_1)^{-1} \cdots (x \cdot y \cdot g_k)(x \cdot y \cdot g_k)^{-1}$$
$$= xx^{-1}e^{x.y},$$

completing the proof.

We can now define, in terms of the endomorphisms ε_x, the semi-direct product J_X^1 as the set $E^1 \times G$ on which a product is given by

$$(e, x)(f, y) = (ef^x, x \cdot y).$$

LEMMA 7. J_X^1 *is a semigroup.*

PROOF. Let $(e, x), (f, y), (g, z) \in E^1 \times G$.
Then

$$((e,x)(f,y))(g,z) = (ef^x, x\cdot y)(g,z) = (ef^x g^{x\cdot y}, (x\cdot y)\cdot z)$$
$$= (ef^x xx^{-1} g^{x\cdot y}, x\cdot(y\cdot z)),$$

by the definition of $\sim$, since $f^x \sim xx^{-1}f^x$,

$$= (ef^x(g^y)^x, x(y\cdot z)),$$

by Lemma 6,

$$= (e,x)(fg^y, y\cdot z),$$

since $\varepsilon_x \in \operatorname{End} E^1$,

$$= (e,x)((f,y)(g,z)).$$

We now find the idempotents and the regular elements of J_X.

LEMMA 8. *$E^1 \times \{1\}$ is the set of idempotents of J_X^1. It forms a subsemigroup of J_X^1 isomorphic to E^1.*

PROOF. This result is immediate from the definition of the product in J_X^1. For $(e,x)^2 = (e,x)$ if and only if $x\cdot x = x$, i,e. if and only if $x = 1$.

LEMMA 9. *Let $(e,x)\in J_X^1$. Then (e,x) is a regular element of J_X^1 if and only if $\{x\} \leqq \operatorname{Car} e$, i.e. if and only if $\operatorname{Car} xx^{-1}e = \operatorname{Car} e$.*

PROOF. Let (e,x) be regular. Then there exists (f,y) such that $(e,x)(f,y)(e,x) = (e,x)$. Hence

$$(ef^x e^{x\cdot y}, x\cdot y\cdot x) = (e,x).$$

Thus $x\cdot y\cdot x = x$, so that $x\cdot y = 1$, and

$$ef^x e^1 = e,$$

i.e.

$$exx^{-1}f^x e = e.$$

From the definition of $\sim$ it follows that $\{x\} \leqq \operatorname{Car} e$.

Conversely, suppose that $\{x\} \leqq \operatorname{Car} e$. Set $y = x^{-1}$; then

$$(e,x)(e^y,y)(e,x) = (e(e^y)^x e, x).$$

By Lemma 6, $(e^y)^x = xx^{-1}e^{x\cdot y} = xx^{-1}e^1 = xx^{-1}e$. However, since $\{x\} \leqq \operatorname{Car} e$, $\operatorname{Car} xx^{-1}e = \operatorname{Car} e$. Hence $xx^{-1}e = e$. Thus $exx^{-1}ee = e$ and

$$(e,x)(e^y,y)(e,x) = (e,x).$$

We may now proceed in two ways. The first way is that of the next lemma.

LEMMA 10. *The regular elements of J_X^1 form an inverse subsemigroup of J_X^1.*

PROOF. Let a, b be regular elements of J_X^1. Let a', b' be inverses of a, b, respectively. Then, since the idempotents of J_X^1 commute, by Lemmas 3 and 8,

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$$ab(b'a')ab = a(bb')(a'a)b = (aa'a)(bb'b) = ab,$$

and similarly,

$$b'a'(ab)b'a' = b'a'.$$

Thus ab is regular.

Since the idempotents of this regular subsemigroup form a semilattice it forms an inverse semigroup.

COROLLARY. *If (e,x) is a regular element of J_X^1 then it has a unique inverse, namely $(e^{x^{-1}}, x^{-1})$, in J_X^1.*

PROOF. That any inverse is unique follows from Lemma 10 because any inverse of a regular element is itself regular. That $(e^{x^{-1}}, x^{-1})$ is the inverse of (e,x) when it is regular was part of the proof of Lemma 9.

Let us denote the semigroup of regular elements of J_X^1, i.e., by Lemma 10, the maximal inverse subsemigroup of J_X^1, by I_X^1.

The second possible construction of I_X^1 is as a homomorphic image of J_X^1. We shall say that $(e,x) \sim (f,y)$ if and only if $x = y$ and Car $xx^{-1}e =$ Car yy^{-1} It is immediate that $(e,1) \sim (f,1)$ if and only if $e \sim f$, where the latter $\sim$ denotes our earlier equivalence on E^1. Since we have already seen (Lemma 8) that $E^1 \times \{1\}$ may be identified with E^1, our new definition of $\sim$ may be regarded as merely extending $\sim$ from E^1 to J_X^1.

LEMMA 11. *$\sim$ is a congruence on J_X^1.*

PROOF. Let $(e,x),(f,y),(g,z) \in J_X^1$ and suppose that $(e,x) \sim (f,y)$. Thus $x = y$ and Car $xx^{-1}e =$ Car $yy^{-1}f$, i.e. Car $xx^{-1}e = Car\ yy^{-1}f$.

To prove right compatibility, consider

$$(e,x)(g,z) = (eg^x, x \cdot z)$$

and

$$(f,y)(g,z) = (fg^x, x \cdot z),$$

since $x = y$. Now, by the definition of $\sim$ on F^1, $g^x \sim xx^{-1}g^x$, so by Lemma 2,

$$\begin{aligned}(x \cdot z)(x \cdot z)^{-1}eg^x &\sim (x \cdot z)(x \cdot z)^{-1}exx^{-1}g^x \\ &\sim (x \cdot z)(x \cdot z)^{-1}fxx^{-1}g^x,\end{aligned}$$

since $exx^{-1} \sim fxx^{-1}$,

$$\sim (x \cdot z)(x \cdot z)^{-1}fg^x;$$

hence $$(e,x)(g,z) \sim (f,y)(g,z).$$

To deal with left compatibility, consider

$$(g,z)(e,x) = (ge^z, z \cdot x)$$

and

$$(g,z)(f,y) = (gf^z, z\cdot x),$$

since $x = y$. Since

$$xx^{-1}e \sim xx^{-1}f,$$

therefore,

$$(xx^{-1}e)\varepsilon_z \sim (xx^{-1}f)\varepsilon_z,$$

so, by Lemma 5,

$$zz^{-1}(z\cdot x)(z\cdot x)^{-1}e^z \sim zz^{-1}(z\cdot x)(z\cdot x)^{-1}f^z.$$

Thus

$$(z\cdot x)(z\cdot x)^{-1}e^z \sim (z\cdot x)(z\cdot x)^{-1}f^z.$$

By Lemma 2, therefore

$$(z\cdot x)(z\cdot x)^{-1}ge^z \sim (z\cdot x)(z\cdot x)^{-1}gf^z.$$

Hence $\sim$ is right compatible.

LEMMA 12. *Each congruence class of $\sim$ in J_X^1 contains precisely one regular element. Hence $J_X^1/\sim\ \cong I_X^1$.*

PROOF. The final statement follows immediately from Lemma 10, for the set of regular elements of J_X^1 form the inverse semigroup I_X^1.

Let $(e,x) \in J_X^1$. Then

$$(xx^{-1}e, x) \sim (e,x)$$

and, by Lemma 9, $(xx^{-1}e, x)$ is regular. Thus each $\sim$ class contains a regular element.

Suppose that $(e,x) \sim (f,y)$ and that (e,x) and (f,y) are regular. Since $(e,x) \sim (f,y)$, therefore $x = y$ and $\mathrm{Car}(xx^{-1}e) = \mathrm{Car}(yy^{-1}f)$. Since (e,x) is regular, $\mathrm{Car}\ xx^{-1}e = \mathrm{Car}\ e$; since (f,y) is regular, $\mathrm{Car}\ (yy^{-1}f) = \mathrm{Car}\ f$. Thus $x = y$ and $\mathrm{Car}\ e = \mathrm{Car}\ f$, i.e. $e \sim f$. Thus $(e,x) = (f,y)$.

We now define I_X to be $I_X^1 \setminus \{1\}$. I_X is clearly an inverse subsemigroup of I_X^1. Define $\phi : X \to I_X$ thus

$$\phi : x \to (xx^{-1}, x), x \in X.$$

Then ϕ is one-to-one i.e. ϕ embeds X in I_X. In the next section we show that (I_X, ϕ) is a free inverse semigroup on X.

We comment now an another possible construction of I_X, mentioned earlier, starting from the set W. We define an equivalence $\sim$, say, on W thus:

$$fg \sim eh \text{ if and only if } g = h \text{ and } \mathrm{Car}\, fgg^{-1} = \mathrm{Car}\ ehh^{-1},$$

for f, $e \in F^1, g, h \in G$, $fg, eh \in W$. On $W/\sim$ we then define a product as follows:

$$((fg)\sim)((eh)\sim) = (fgg^{-1}kg\cdot h)\sim,$$

where, if $e = h_1h_1^{-1}\cdots h_lh_l^{-1}$, then

$$k = (g \cdot h_1)(g \cdot h_1)^{-1} \cdots (g \cdot h_l)(g \cdot h_l)^{-1}.$$

With this product we then take $W/\sim$ as I_X.

3. The freedom of (I_X, ϕ)

In what follows we take I_X as the subsemigroup of J_X^1 consisting of all the regular elements of J_X^1 other than 1.

We are to show that (I_X, ϕ) is a free inverse semigroup on X. In other words, we are to show that if S is an inverse semigroup and $\alpha : X \to S$ is any mapping, then there is a unique morphism $\theta : I_X \to S$, say, such that $\phi\theta = \alpha$.

To construct θ, first let us adjoin an identity element 1, an additiona one if S already has one, to S to form the inverse semigroup S^*. Define $\beta : G \to S^*$ as follows:

$$x\beta = x\alpha, x \in X,$$

$$x^{-1}\beta = (x\alpha)^{-1}, x^{-1} \in X^{-1},$$

where $(x\alpha)^{-1}$ is the inverse in S^*,

$$1\beta = 1,$$

$$(y_1 y_2 \cdots y_n)\beta = (y_1\beta)(y_2\beta)\cdots(y_n\beta), y_i \in Y,$$

where $y_1 y_2 \cdots y_n$ is reduced.

LEMMA 13. *Let $g, h \in G$ and suppose that $g \leqq h$. Then*

$$(g\beta)(g\beta)^{-1} \leqq (h\beta)(h\beta)^{-1}$$

in S^*.

PROOF. There exists g' in G such that $h = gg' = g \cdot g'$. Hence, from the definition of β,

$$h\beta = (g\beta)(g'\beta).$$

Thus

$$\begin{aligned}(g\beta)(g\beta)^{-1}(h\beta)(h\beta)^{-1} &= (g\beta)(g\beta)^{-1}(g\beta)(g'\beta)(h\beta)^{-1}\\ &= (g\beta)(g'\beta)(h\beta)^{-1}\\ &= (h\beta)(h\beta)^{-1}.\end{aligned}$$

We may now define θ as follows. Let $(e, g) \in I_X$. Then $e \neq 1$, for Car $gg^{-1}e$ = Car e, and $(e, g) \neq (1, 1)$. Hence $e = g_1 g_1^{-1} \cdots g_k g_k^{-1}$, say. Define

$$(e, g)\theta = (g_1\beta)(g_1\beta)^{-1} \cdots (g_k\beta)(g_k\beta)^{-1} g\beta$$

where, on the right of this equation, $(g_i\beta)^{-1}$, $i = 1, \cdots, k$, denotes the inverse of $g_i\beta$ in S. From Lemma 13, if $e \sim f$ (in F^1), so that $(e, g) = (f, g)$ in I_X, then the above definition gives $(e, g)\theta = (f, g)\theta$. Thus θ is well-defined.

The following lemma contains two results needed to show that θ is a morphism.

LEMMA 14. *Let T be any inverse semigroup. Let $a, b, m \in T$. Then*

(i) $$(am)(m^{-1}b)(m^{-1}b)^{-1} = (ab)(ab)^{-1}am,$$

(ii) $$(am)(m^{-1}b) = (am)(am)^{-1}ab.$$

PROOF. (i) $(am)(m^{-1}b)(m^{-1}b)^{-1} = a \cdot mm^{-1} \cdot bb^{-1} \cdot m = abb^{-1}mm^{-1}m$

$$= a \cdot a^{-1}a \cdot bb^{-1} \cdot m = abb^{-1}a^{-1}am.$$

(ii) $$(am)(m^{-1}b) = a \cdot a^{-1}a \cdot mm^{-1} \cdot b = amm^{-1}a^{-1} \cdot ab.$$

Consider now (e, g), (f, h) in I_X. Then

$$(e, g)(f, h) = (ef^g, g \cdot h).$$

Since e and f are not equal to 1, we have $e = g_1 g_1^{-1} \cdots g_k g_k^{-1}$ and $f = h_1 h_1^{-1} \cdots h_l h_l^{-1}$, say. Set $g_i\beta = s_i, i = 1, \cdots, k$, $h_j\beta = t_j$, $j = 1, 2, \cdots, l$, $g\beta = s$, and $h\beta = t$.

Suppose that $g \cdot h = g'u \cdot u^{-1}h'$, where $g = g'u$, $h = u^{-1}h'$, and $g \cdot h = g' \cdot h'hg = g'h'$, and that

$$g \cdot h_j = g^{(j)}u_j \cdot u_j^{-1}h_j',$$

where

$$g^{(j)}u_j = g, \; u_j^{-1}h_j' = h_j, \; g \cdot h_j = g^{(j)} \cdot h_j' = g^{(j)}h_j',$$

for $j = 1, \cdots, l$. Set $g^{(j)}\beta = s^{(j)}$, $u_j\beta = r_j$, $h_j'\beta = t_j'$, $g'\beta = s'$, $h'\beta = t'$, and $u\beta = r$.

Then, by definition of θ,

$$\begin{aligned}(ef^g, g \cdot h)\theta &= (g_1 g_1^{-1} \cdots g_k g_k^{-1} g g^{-1}(g \cdot h_1)(g \cdot h_1)^{-1} \cdots (g \cdot h_l)(g \cdot h_l)^{-1}, g \cdot h)\theta \\ &= s_1 s_1^{-1} \cdots s_k s_k^{-1} s s^{-1}(g \cdot h_1)\beta((g \cdot h_1)\beta)^{-1} \cdots (g \cdot h_l)\beta((g \cdot h_l)\beta)^{-1}(g \cdot h)\beta \\ &= s_1 s_1^{-1} \cdots s_k s_k^{-1} s s^{-1}(s^{(1)}t_1')(s^{(1)}t_1')^{-1} \cdots (s^{(l)}t')(s^{(l)}t')^{-1}s't'.\end{aligned}$$

On the other hand,

$$\begin{aligned}(e, g)\theta(f, h)\theta &= (s_1 s_1^{-1} \cdots s_k s_k^{-1} s)(t_1 t_1^{-1} \cdots t_l t_l^{-1} t) \\ &= s_1 s_1^{-1} \cdots s_k s_k^{-1} s s^{-1}(s t_1 t_1^{-1}) \cdots t_l t_l^{-1} t \\ &= s_1 s_1^{-1} \cdots s_k s_k^{-1} s s^{-1}(s^{(1)}t_1)(s^{(1)}t_1)^{-1}) s t_2 t_2^{-1} \cdots t_l t_l^{-1} t, \\ &= s_1 s_1^{-1} \cdots s_k s_k^{-1} s s^{-1}(s^{(1)}t_1')(s^{(1)}t_1')^{-1} \cdots (s^{(l)}t_l')(s^{(l)}t_l')^{-1} st,\end{aligned}$$

by successive applications of Lemma 14, part (i),

$$= s_1 s_1^{-1} \cdots s_k s_k^{-1} s s^{-1}(s^{(1)}t_1')(s^{(1)}t_1')^{-1} \cdots (s^{(l)}t'_l)(s^{(l)}t_l')^{-1} s s^{-1} s' t',$$

by Lemma 14, part (ii),

$$= (ef^g, g \cdot h)\vartheta,$$

completing the proof that θ is a morphism.

Now, immediately from the definition of θ,

$$\begin{aligned} x\phi\theta &= (xx^{-1}, x)\theta \\ &= (x\alpha)(x\alpha)^{-1}(x\alpha) \\ &= x\alpha, \text{ if } x \in X. \end{aligned}$$

Thus $\phi\theta = \alpha$. It remains to show that θ is uniquely determined by this condition. This will be so if $X\phi$ is a set of generators of I_X. This is so, for we easily check that, if $e = g_1 g_1^{-1} \cdots g_k g_k^{-1}$ then

(1) $$(e, g) = (g_1 g_1^{-1}, 1) \cdots (g_k g_k^{-1}, 1)(gg^{-1}, g),$$

and for any h in G,

(2) $$(hh^{-1}, 1) = (hh^{-1}, h)(h^{-1}h, h^{-1}).$$

Moreover, if $h = y_1 \cdots y_n, y_i \in Y$, then

(3) $$(hh^{-1}, h) = (y_1 y_1^{-1}, y_1)(y_2 y_2^{-1}, y_2) \cdots (y_n y_n^{-1}, y_n),$$

a product of elements of $X\phi$ and their inverses.

The final remarks of the above proof lead to a canonical form for the elements of I_X. If $(e, g) \in I_X$, we may choose e, by the definition of $\sim$ (on F^1), so that Car e = Dom e. Suppose that then $e = g_1 g_1^{-1} \cdots g_k g_k^{-1}$. Then e is uniquely expressible in this form, modulo $\sim$, up to a permutation of the factors $g_i g_i^{-1}$. Modulo such permutations, equations (1), (2) and (3), above, enable each element of I_X to be expressed as a unique product of elements of X and their inverses.

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