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Modified quantum dimensions and re-normalized link invariants

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Abstract

In this paper we give a re-normalization of the Reshetikhin–Turaev quantum invariants of links, using modified quantum dimensions. In the case of simple Lie algebras these modified quantum dimensions are proportional to the usual quantum dimensions. More interestingly, we give two examples where the usual quantum dimensions vanish but the modified quantum dimensions are non-zero and lead to non-trivial link invariants. The first of these examples is a class of invariants arising from Lie superalgebras previously defined by the first two authors. These link invariants are multivariable and generalize the multivariable Alexander polynomial. The second example is a hierarchy of link invariants arising from nilpotent representations of quantized $\mathfrak{sl}(2)$ at a root of unity. These invariants contain Kashaev's quantum dilogarithm invariants of knots.

1. Introduction

One obstruction to applications of quantum link invariants associated with a ribbon category $\mathcal{C}$ stems from the fact that certain simple (irreducible) objects of $\mathcal{C}$ may have zero quantum dimensions. If the dimension of a simple object $V \in Ob(\mathcal{C})$ is zero, then the quantum invariants of all (framed oriented) links with components labeled by V are equal to zero. A well-known topological trick allows us to derive possibly non-trivial invariants in this setting, at least in the case of knots. Namely, one presents a V-labeled knot L as the closure of a $(1, 1)$ tangle T and considers the endomorphism of V associated with T. This endomorphism is the product of the identity $\mathrm{Id}_V : V \to V$ with an element $\langle T\rangle$ of the ground ring of $\mathcal{C}$. The tangle T is determined by L uniquely up to isotopy and therefore $\langle T\rangle$ is an isotopy invariant of L. This invariant may be non-trivial even when $\dim_{\mathcal{C}}(V) = 0$. Note that the usual quantum invariant of L is equal to $\langle T\rangle \dim_{\mathcal{C}}(V)$.

For a link L with at least two components labeled by V, the situation is more involved because $\langle T\rangle$ may depend on the choice of T. In many known examples of ribbon categories, an appropriate re-normalization of $\langle T\rangle$ does not depend on the choice of T and yields a possibly non-trivial invariant of L, see [ADO92, GP08b, Kas95, KS91]. A systematic explanation of this phenomenon seems to be missing in the literature. In this paper we suggest such an explanation. It is based on a new notion of an ambidextrous object in $\mathcal{C}$. Every simple ambidextrous object $J \in Ob(\mathcal{C})$ determines a certain set $A(J)$ of (isomorphism classes of) simple objects of $\mathcal{C}$. For all

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simple objects V belonging to this set we define a modified (quantum) dimension depending on J. The modified dimension may be non-zero when $\dim_{\mathcal{C}}(V) = 0$. Using the modified dimensions we define an isotopy invariant $F'(L)$ for any link whose components are labeled with objects of $\mathcal{C}$ under the only assumption that at least one of the labels belongs to $A(J)$. Most of these results extend to closed $\mathcal{C}$-colored ribbon graphs (i.e. to $\mathcal{C}$-colored ribbon graphs with no inputs and no outputs).

We give three families of examples illustrating our constructions. In the first example, $\mathcal{C}$ is the category of finite-dimensional $U_q(\mathfrak{g})$-modules, where $U_q(\mathfrak{g})$ be the Drinfeld–Jimbo $\mathbb{C}(q)$-algebra associated to a simple complex Lie algebra. In this case we recover the standard Reshetikhin–Turaev link invariants. In the second example $\mathcal{C}$ is the category of topologically free $U_h(\mathfrak{g})$-modules of finite rank, where $\mathfrak{g}$ is a Lie superalgebra of type I and $U_h(\mathfrak{g})$ is its quantized universal enveloping $\mathbb{C}[[h]]$-superalgebra. In this case we recover the link invariants defined by the first two authors in [GP08b, GP08a]. These invariants generalize both the multivariable Alexander polynomial of links and Kashaev's link invariants. In the final example $\mathcal{C}$ is the category of finite-dimensional weight $U_q(\mathfrak{sl}(2))$-modules where q is a root of unity. We show that our construction in this case gives a generalization of the invariants defined by Akutsu, Deguchi and Ohtsuki [ADO92], using a regularization of the Markov trace and nilpotent representations of $U_q(\mathfrak{sl}(2))$ at a root of unity. In the later two examples the standard Reshetikhin–Turaev link invariant coming from $\mathcal{C}$ is generically zero.

The paper is organized as follows. In § 2 we recall the basic results on ribbon categories. In § 3 we introduce the ambidextrous objects, the modified dimensions and the invariant F' of closed $\mathcal{C}$-colored ribbon graphs. In § 4 we extend F' to arbitrary $\mathcal{C}$-colored ribbon graphs (this does not yield a functor as in the standard theory but only a quasi-functor). In § 5 we study the basic properties of F'. Section 6 is devoted to the presentation of examples.

2. Ribbon Ab-categories

We describe the concept of a ribbon Ab-category (for details see [Tur94]). A *tensor category* $\mathcal{C}$ is a category equipped with a covariant bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product, a unit object $\mathbb{I}$, an associativity constraint, and left and right unit constraints such that the triangle and pentagon axioms hold. When the associativity constraint and the left and right unit constraints are all identities we say that the category $\mathcal{C}$ is a *strict* tensor category. By Mac Lane's coherence theorem any tensor category is equivalent to a strict tensor category.

A tensor category $\mathcal{C}$ is said to be an *Ab-category* if for any pair of objects V, W of $\mathcal{C}$ the set of morphisms $\mathrm{Hom}(V, W)$ is an additive abelian group and the composition and tensor product of morphisms are bilinear.

Let $\mathcal{C}$ be a (strict) ribbon Ab-category, i.e. a (strict) tensor Ab-category with duality, a braiding and a twist. Composition of morphisms induces a commutative ring structure on $\mathrm{End}(\mathbb{I})$. This ring is called the *ground ring* of $\mathcal{C}$ and denoted by K. For any pair of objects V, W of $\mathcal{C}$ the abelian group $\mathrm{Hom}(V, W)$ becomes a left K-module where the action is defined by $kf = k \otimes f$ where $k \in K$ and $f \in \mathrm{Hom}(V, W)$. An object V of $\mathcal{C}$ is *simple* if $\mathrm{End}(V) = K\,\mathrm{Id}_V$.

We denote the braiding in $\mathcal{C}$ by $c_{V,W} : V \otimes W \to W \otimes V$ and duality morphisms in $\mathcal{C}$ by

$$b_V : \mathbb{I} \to V \otimes V^*, \quad b'_V : \mathbb{I} \to V^* \otimes V, \quad d_V : V^* \otimes V \to \mathbb{I}, \quad d'_V : V \otimes V^* \to \mathbb{I}.$$

The *trace* of any endomorphism $f \in \mathrm{End}(V)$ of an object V of $\mathcal{C}$ is defined by

$$\mathrm{tr}_{\mathcal{C}}(f) = d'_V \circ (f \otimes \mathrm{Id}^*_V) \circ b_V \in \mathrm{End}(\mathbb{I}) = K.$$

Define $\dim_{\mathcal{C}} : Ob(\mathcal{C}) \to K$ by $\dim_{\mathcal{C}}(V) = \mathrm{tr}_{\mathcal{C}}(\mathrm{Id}_V)$. We call $\dim_{\mathcal{C}}(V)$ the *dimension* of V.

3. The invariant F' of closed ribbon graphs

Let $\mathcal{C}$ be a strict ribbon Ab-category with ground ring K and the set of objects $Ob(\mathcal{C})$. We assume everywhere that K is an integral domain with field of fractions $\mathbb{F}$.

For any object V of $\mathcal{C}$ and any endomorphism f of $V \otimes V$, set

$$\mathrm{tr}_L(f) = (d_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_{V^*} \otimes f) \circ (b'_V \otimes \mathrm{Id}_V) \in \mathrm{End}(V),$$
$$\mathrm{tr}_R(f) = (\mathrm{Id}_V \otimes d'_V) \circ (f \otimes \mathrm{Id}_{V^*}) \circ (\mathrm{Id}_V \otimes b_V) \in \mathrm{End}(V).$$

An object V of $\mathcal{C}$ is called *ambidextrous* if $\mathrm{tr}_L(f) = \mathrm{tr}_R(f)$ for all $f \in \mathrm{End}(V \otimes V)$.

The following lemma gives examples of ambidextrous elements.

Lemma 1. *We have two examples of ambidextrous elements.*

(1) *If J is an object of $\mathcal{C}$ such that the braiding $c_{J,J}$ commutes with any element of $\mathrm{End}(J \otimes J)$, then J is ambidextrous.*

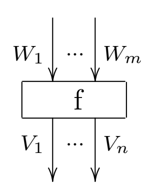
(2) *If J is a simple object of $\mathcal{C}$ such that $\dim_{\mathcal{C}}(J) \neq 0$, then J is ambidextrous.*

Proof. (1) Let $f \in \mathrm{End}(J \otimes J)$. We have $\mathrm{tr}_R(f) = \mathrm{tr}_L(c^{-1}_{J,J} \circ f \circ c_{J,J})$. However, $c_{J,J}$ commutes with $\mathrm{End}(J \otimes J)$ and so $c^{-1}_{J,J} \circ f \circ c_{J,J} = f$.

(2) Let $f \in \mathrm{End}(J \otimes J)$. We have

$$\mathrm{tr}_L(f) = \frac{\mathrm{tr}_{\mathcal{C}}(f)}{\dim_{\mathcal{C}}(J)} \ \mathrm{Id}_J = \mathrm{tr}_R(f). \qquad \square$$

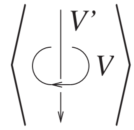
Next we recall the category of $\mathcal{C}$-colored ribbon graphs $Rib_{\mathcal{C}}$ (for more details see [Tur94, ch. I]). A morphism $f : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_m$ in the category $\mathcal{C}$ can be represented by the following box and arrows:



Such boxes are called coupons. A ribbon graph is formed from several oriented framed edges colored by objects of $\mathcal{C}$ and several coupons colored with morphisms of $\mathcal{C}$. The objects of $Rib_{\mathcal{C}}$ are sequences of pairs (V, ϵ), where $V \in Ob(\mathcal{C})$ and $\epsilon = \pm$ determines the orientation of the corresponding edge. The morphism of $Rib_{\mathcal{C}}$ are isotopy classes of $\mathcal{C}$-colored ribbon graphs and their formal linear combinations with coefficients in K. From now on we write V for $(V, +)$.

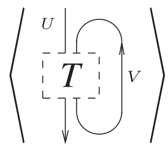
Let F be the usual ribbon functor from $Rib_{\mathcal{C}}$ to $\mathcal{C}$ (see [Tur94]). Let T_V (T^-_V) be a $\mathcal{C}$-colored $(1,1)$-ribbon graph whose open string is oriented downward (respectively upward) and colored with a simple object V of $\mathcal{C}$. Then $F(T_V) \in \mathrm{End}_{\mathcal{C}}(V) = K\,\mathrm{Id}_V$. Let $\langle T_V \rangle \in K$ be such that $F(T_V) = \langle T_V \rangle\,\mathrm{Id}_V$.

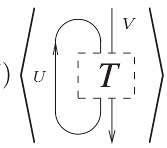
Let V and V' be objects of $\mathcal{C}$ such that V' is simple and define the following diagram.

$$S'(V,V') =$$


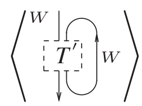
$$\in K.$$

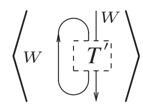
Lemma 2. *For all simple objects U, V, W of $\mathcal{C}$ such that W is ambidextrous and for any $\mathcal{C}$-colored ribbon graph T with two inputs and two outputs colored by U, V,*

$$S'(U,W)S'(W,V)$$


$$= S'(V,W)S'(W,U)$$


Proof. Recall that $\operatorname{tr}_L(f) = \operatorname{tr}_R(f)$ for all $f \in \operatorname{End}(W \otimes W)$. This implies that:

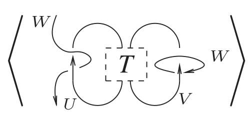


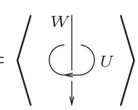
$$=$$


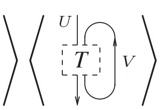
$$(1)$$

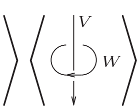
for all $\mathcal{C}$-colored ribbon graphs T' with two inputs and two outputs all colored by W.

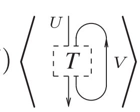
By definition we have



$$=$$


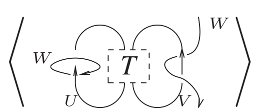


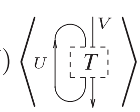


$$= S'(U,W)\, S'(W,V)$$


$$(2)$$

Similarly,



$$= S'(V,W)\, S'(W,U)$$


$$(3)$$

Then (1) implies that the left-hand sides of the above equations are equal and so the lemma follows. □

Applying this lemma to $U = V$, we obtain that if U, W are simple objects of $\mathcal{C}$ such that W is ambidextrous and $S'(U,W)S'(W,U) \neq 0$, then U is also ambidextrous.

If A is a subset of $Ob(\mathcal{C})$, then let $\mathcal{L}_A$ be the set of closed $\mathcal{C}$-colored ribbon graphs, such that at least one of the colors of the edges is in A. For a simple ambidextrous object J of $\mathcal{C}$, set

$$A(J) = \{V \in Ob(\mathcal{C}) \mid V \text{ simple and } S'(J,V) \neq 0\}.$$

Fix a non-zero $d_0 \in \mathbb{F}$. For $V \in A(J)$ define

$$\mathsf{d}_J(V) = d_0 \frac{S'(V,J)}{S'(J,V)} \in \mathbb{F}.$$

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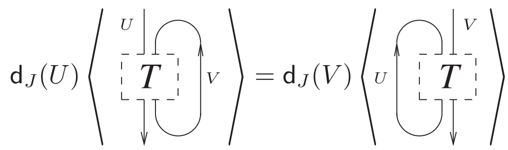
We view $\mathsf{d}_J(V)$ as a modified quantum dimension of V determined by J. For any $U, V \in A(J)$, Lemma 2 implies that



for any T. In particular, when T consists of two vertical intervals colored by $U, V \in A(J)$,

$$\mathsf{d}_J(U)\dim_{\mathcal{C}}(V) = \mathsf{d}_J(V)\dim_{\mathcal{C}}(U).$$

This shows that the functions d_J and $\dim_{\mathcal{C}}$ are proportional to each other. This is especially interesting when $\dim_{\mathcal{C}} = 0$ and $\mathsf{d}_J \neq 0$, see the examples below.

Theorem 3. *Let $L \in \mathcal{L}_{A(J)}$ and V be the color of an edge of L belonging to $A(J)$. Cutting this edge, we obtain a colored $(1,1)$-ribbon graph T_V whose closure is L. Then*

$$F'(L) = \mathsf{d}_J(V)\langle T_V \rangle \in \mathbb{F}$$

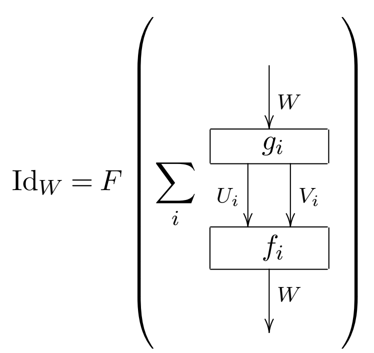
is independent of the choice of the edge to be cut and yields a well-defined invariant of L.

Proof. The theorem follows from Lemma 2 and the definition of d_J. □

We call the invariant F' the re-normalized Reshetikhin–Turaev link invariant.

Let $\overline{A} = \overline{A}(J)$ be the set of objects W of $\mathcal{C}$ such that there exists a finite family of tuples (U_i, V_i, f_i, g_i), where $U_i \in A(J)$, $V_i \in Ob(\mathcal{C})$ and $g_i : U_i \otimes V_i \to W$, $f_i : W \to U_i \otimes V_i$ satisfying $\mathrm{Id}_W = \sum g_i \circ f_i$. Note that $\overline{A}$ has the property that $W \otimes V \in \overline{A}$ for all $V \in Ob(\mathcal{C})$ and $W \in \overline{A}$.

The map $\mathcal{L}_A \to \mathbb{F}$, $L \mapsto F'(L)$ extends to a map $\mathcal{L}_{\overline{A}} \to \mathbb{F}$ as follows. Let L be a closed $\mathcal{C}$-colored ribbon graph with one edge colored by $W \in \overline{A}$. Pick a decomposition $\mathrm{Id}_W = \sum g_i \circ f_i$ as above. Then



We define $F'(L)$ applying this expansion to the edge of L colored by W and then cutting as above the edge of the resulting graph labeled by U_i. It is easy to show using Theorem 3 that this extension is independent of the decomposition of Id_W.

4. The quasi-functor F'

In this section we extend the invariant F' to $\mathcal{C}$-colored ribbon graphs with endpoints. This leads us to a notion of a quasi-functor which we briefly discuss in a more general setting. This section is essentially independent from the rest of the paper.

Let $\mathcal{E}$ be a category. Given a map $\mathrm{M} : Ob(\mathcal{E}) \times Ob(\mathcal{E}) \to \mathcal{S}ets$ we use the notation $m \in \mathrm{M}$ to mean that there exist two objects X and Y of $\mathcal{E}$ such that $m \in \mathrm{M}(X, Y)$.

Definition 4. A $\mathcal{E}$*-bimodule* is a map $\mathrm{M} : Ob(\mathcal{E}) \times Ob(\mathcal{E}) \to \mathcal{S}ets$ endowed with two operations:

$$\triangleright : \mathrm{Hom}_{\mathcal{E}}(Y, Z) \times \mathrm{M}(X, Y) \to \mathrm{M}(X, Z),$$
$$\triangleleft : \mathrm{M}(Y, Z) \times \mathrm{Hom}_{\mathcal{E}}(X, Y) \to \mathrm{M}(X, Z),$$

where X, Y and Z are any objects of $\mathcal{E}$. Given morphisms f and g of $\mathcal{E}$ and $m \in \mathrm{M}$ we require that:

(1) $(f \circ g) \triangleright m = f \triangleright (g \triangleright m)$;

(2) $m \triangleleft (f \circ g) = (m \triangleleft f) \triangleleft g$;

(3) $(f \triangleright m) \triangleleft g = f \triangleright (m \triangleleft g)$;

whenever the operations in these equalities make sense.

Remark that if one has $\mathrm{Id} \triangleright m = m \triangleleft \mathrm{Id} = m$ for all $m \in \mathrm{M}$ these axioms mean that M is a bifunctor contravariant in the first place and covariant in the second place with $\mathrm{M}(X, f)(m) = f \triangleright m$ and $\mathrm{M}(f, Z)(m) = m \triangleleft f$.

Example 5. The functor $\mathrm{Hom}_{\mathcal{E}}$ is a $\mathcal{E}$-bimodule with $\triangleleft = \triangleright = \circ$.

Example 6. Let K be an integral domain and suppose that $(\mathrm{M}, \triangleright, \triangleleft)$ is a $\mathcal{E}$-bimodule with values in K-modules. If Γ is a K-module, then we define an $\mathcal{E}$-bimodule $\mathrm{H}_{\mathcal{E}}$ by $\mathrm{H}_{\mathcal{E}}(X, Y) = \mathrm{Hom}_K(\mathrm{M}(Y, X), \Gamma)$ with operations $\triangleright$ and $\triangleleft$ defined as follows. Let $f : Y \to Z$ and $f' : Z \to X$ be morphisms of $\mathcal{E}$ and let ϕ be an element of $\mathrm{H}_{\mathcal{E}}(X, Y)$, then

$$(f \triangleright \phi)(m) = \phi(m \triangleleft f) \quad \text{and} \quad (\phi \triangleleft f')(m') = \phi(f' \triangleright m'),$$

where $m \in M(Z, X)$ and $m' \in M(Y, Z)$.

Suppose now that $\mathcal{E}$ is a tensor *Ab*-category and that M is a $\mathcal{E}$-bimodule which takes values in abelian groups. We assume that the operations $\triangleright$ and $\triangleleft$ are bilinear.

Definition 7. We call M a *monoidal* $\mathcal{E}$*-bimodule* if it is endowed with two bilinear operations:

$$\otimes\!\!\!\!\triangleright : \mathrm{Hom}_{\mathcal{E}}(X, Y) \times \mathrm{M}(X', Y') \to \mathrm{M}(X \otimes X', Y \otimes Y'),$$
$$\triangleleft\!\!\!\!\otimes : \mathrm{M}(X, Y) \times \mathrm{Hom}_{\mathcal{E}}(X', Y') \to \mathrm{M}(X \otimes X', Y \otimes Y')$$

such that for any morphisms f, g and h in $\mathcal{E}$ and any $m \in \mathrm{M}$:

(1) $(f \otimes g) \otimes\!\!\!\!\triangleright\, m = f \otimes\!\!\!\!\triangleright\, (g \otimes\!\!\!\!\triangleright\, m)$;

(2) $m \triangleleft\!\!\!\!\otimes (f \otimes g) = (m \triangleleft\!\!\!\!\otimes f) \triangleleft\!\!\!\!\otimes g$;

(3) $f \otimes\!\!\!\!\triangleright\, (m \triangleleft\!\!\!\!\otimes g) = (f \otimes\!\!\!\!\triangleright\, m) \triangleleft\!\!\!\!\otimes g$;

(4) $(f \circ g) \otimes\!\!\!\!\triangleright\, (h \triangleright m) = (f \otimes h) \triangleright (g \otimes\!\!\!\!\triangleright\, m)$;

(5) $(h \triangleright m) \triangleleft\!\!\!\!\otimes (f \circ g) = (h \otimes f) \triangleright (m \triangleleft\!\!\!\!\otimes g)$;

(6) $(f \circ g) \otimes\!\!\!\!\triangleright\, (m \triangleleft h) = (f \otimes\!\!\!\!\triangleright\, m) \triangleleft (g \otimes h)$;

(7) $(m \triangleleft h) \triangleleft\!\!\!\!\otimes (f \circ g) = (m \triangleleft\!\!\!\!\otimes f) \triangleleft (f \otimes g)$;

whenever the operations in these equalities make sense.

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Definition 7 can be illustrated with diagrams. For example, Axiom (4) is given by

$$\boxed{\begin{array}{c} f \\ \circ \\ g \end{array}} \quad \otimes\!\!\!\!\lhd \quad \boxed{\begin{array}{c} h \\ \nabla \\ m \end{array}} = \begin{array}{c} \boxed{f \otimes h} \\ \nabla \\ \boxed{g \otimes\!\!\!\!\rhd\, m} \end{array}$$

where the composition operations should be read from the top to the bottom and tensor operations from left to right.

Example 8. The functor $\mathrm{Hom}_{\mathcal{E}}$ is a monoidal $\mathcal{E}$-bimodule with $\lhd = \rhd = \circ$ and $\otimes\!\!\!\!\lhd\; = \;\otimes\!\!\!\!\rhd\; = \otimes$.

Suppose that $G : \mathcal{D} \to \mathcal{E}$ is a monoidal functor between two tensor *Ab*-categories. Let M be a monoidal $\mathcal{E}$-bimodule and M' be a monoidal $\mathcal{D}$-bimodule.

Definition 9. A *G-bilinear monoidal quasi-functor* $G' : \mathrm{M}' \to \mathrm{M}$ is a family of maps $G' : \mathrm{M}'(X, Y) \to \mathrm{M}(G(X), G(Y))$ indexed by the objects X and Y of $\mathcal{E}$, such that for every $m \in \mathrm{M}'(X, Y)$ and every morphism f of $\mathcal{D}$ one has:

(1) $G'(f \rhd m) = G(f) \rhd G'(m)$;

(2) $G'(m \lhd f) = G'(m) \lhd G(f)$;

(3) $G'(f \otimes\!\!\!\!\rhd\, m) = G(f) \otimes\!\!\!\!\rhd\, G'(m)$;

(4) $G'(m \otimes\!\!\!\!\lhd\, f) = G'(m) \otimes\!\!\!\!\lhd\, G(f)$;

whenever the operations in these equalities make sense.

Let us now go back to the situation of § 3. Define a $\mathcal{C}$-bimodule structure on $\mathrm{H}_{\mathcal{C}} = \mathrm{Hom}_K(\mathrm{Hom}_{\mathcal{C}}, \mathbb{F})$ as in Example 6. In particular, for any objects U, V of $\mathcal{C}$ we have

$$\mathrm{H}_{\mathcal{C}}(U, V) = \mathrm{Hom}_K(\mathrm{Hom}_{\mathcal{C}}(V, U), \mathbb{F})$$

and if f and g are morphisms of $\mathcal{C}$ and $\phi \in \mathrm{H}_{\mathcal{C}}$, then

$$(f \rhd \phi)(g) = \phi(g \circ f) \text{ and } (\phi \lhd f)(g) = \phi(f \circ g)$$

when these operations make sense.

To give $\mathrm{H}_{\mathcal{C}}$ a monoidal structure, let us recall the partial traces in $\mathcal{C}$: if $f \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Z, Y \otimes Z)$ set $\mathrm{tr}_R(f) = (\mathrm{Id}_X \otimes d'_Z) \circ (f \otimes \mathrm{Id}_{Z^*}) \circ (\mathrm{Id}_X \otimes b_Z) \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and if $f \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, X \otimes Z)$ set $\mathrm{tr}_L(f) = (d_X \otimes \mathrm{Id}_Z) \circ (\mathrm{Id}_{X^*} \otimes f) \circ (b'_X \otimes \mathrm{Id}_Y) \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$.

Then

$$\begin{aligned} \otimes\!\!\!\!\rhd\; &: \mathrm{Hom}_{\mathcal{C}}(U, V) \times \mathrm{H}_{\mathcal{C}}(U', V') \to \mathrm{H}_{\mathcal{C}}(U \otimes U', V \otimes V'), \\ \otimes\!\!\!\!\lhd\; &: \mathrm{H}_{\mathcal{C}}(U, V) \times \mathrm{Hom}_{\mathcal{C}}(U', V') \to \mathrm{H}_{\mathcal{C}}(U \otimes U', V \otimes V') \end{aligned}$$

are defined as follows.

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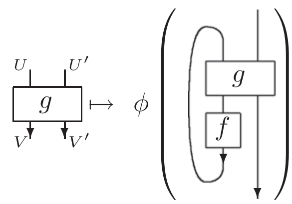
(1) If $f \in \mathrm{Hom}_{\mathcal{C}}(U, V)$ and $\phi \in \mathrm{H}_{\mathcal{C}}(U', V')$, then $f \otimes \phi \in \mathrm{H}_{\mathcal{C}}(U \otimes U', V \otimes V')$ is given by

$$(f \otimes \phi)(g) = \phi(\mathrm{tr}_L(g \circ (f \otimes \mathrm{Id}_{V'}))).$$

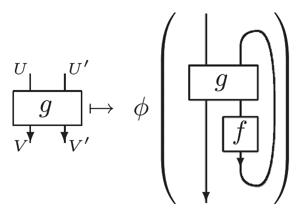
where g is any element of $\mathrm{Hom}_{\mathcal{C}}(V \otimes V', U \otimes U')$. This operation can be represented by the following diagram.



(2) If $f \in \mathrm{Hom}_{\mathcal{C}}(U', V')$ and $\phi \in \mathrm{H}_{\mathcal{C}}(U, V)$, then $\phi \otimes f \in \mathrm{H}_{\mathcal{C}}(U \otimes U', V \otimes V')$ is given by

$$(\phi \otimes f)(g) = \phi(\mathrm{tr}_R(g \circ (\mathrm{Id}_{U'} \otimes f)))$$

where g is any element of $\mathrm{Hom}_{\mathcal{C}}(V \otimes V', U \otimes U')$. Again this operation can be represented by the following diagram.



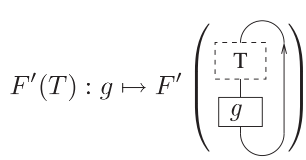
One can check that these maps make $\mathrm{H}_{\mathcal{C}}$ a monoidal $\mathcal{C}$-bilinear module.

Fix now a simple ambidextrous object J in $\mathcal{C}$ and a non-zero element d_0 of $\mathbb{F}$. Set $\overline{A} = \overline{A}(J)$. Let $Rib_{\overline{A}}$ be the monoidal $Rib_{\mathcal{C}}$-bimodule defined as follows. The operations of $Rib_{\overline{A}}$ are the composition and tensor product of $Rib_{\mathcal{C}}$, i.e. $\triangleleft = \triangleright = \circ$ and $\otimes = \otimes = \otimes$. Let V and W be objects of $Rib_{\mathcal{C}}$, then $Rib_{\overline{A}}(V, W)$ is the set of $\mathcal{C}$-colored ribbon graphs in $\mathrm{Hom}_{\mathcal{C}}(V, W)$ with at least one color in $\overline{A}$. In particular, $Rib_{\overline{A}}(\emptyset, \emptyset) = \mathcal{L}_{\overline{A}}$.

THEOREM 10. *The invariant $F' : \mathcal{L}_{\overline{A}} \to \mathbb{F}$ extends naturally to a F-bilinear monoidal quasi-functor $F' : Rib_{\overline{A}} \to \mathrm{H}_{\mathcal{C}}$ by the formula*

$$F'(T)(g) = F'(\mathrm{tr}_{Rib_{\mathcal{C}}}(T \circ c_g))$$

where $T \in Rib_{\overline{A}}(V, W)$, $g \in \mathrm{Hom}_{\mathcal{C}}(F(W), F(V))$ and where c_g is a coupon labeled by g. This expression can be represented by the following diagram.



Proof. Let T and T' be morphisms of $Rib_{\mathcal{C}}$ and $Rib_{\overline{A}}$, respectively. Suppose that $T \circ T'$ is defined. Let T'' be the trace in $Rib_{\mathcal{C}}$ of $T \circ T'$ and let C be a coupon labeled by $F(T')$. The proof follows from the fact that

$$F'(T'') = F'(\mathrm{tr}_{Rib_{\mathcal{C}}}(T \circ C)) = F'(T)(F(T')).$$

For example, we now check equality (1) of Definition 9. Let T and T' be as above. Let g be a morphism of $\mathcal{C}$ and let c_g be a coupon labeled by g. Suppose that g is a morphism such that $F'(T \triangleright T')(g)$ is defined. If c_g is a coupon labeled by g, then we have

$$\begin{aligned} F'(T \triangleright T')(g) &= F'(\mathrm{tr}_{Rib_{\mathcal{C}}}(T \circ T' \circ c_g)) = F'(T')(F(c_g \circ T)) \\ &= F'(T')(g \circ F(T)) = (F(T) \triangleright F'(T'))(g). \end{aligned}$$

Checking equalities (2)–(4) of Definition 9 are similar. □

5. Properties of d_J and F'

The following lemma shows that the function d_J satisfying the conditions of Theorem 3 is essentially unique.

Lemma 11. *Let $J \in Ob(\mathcal{C})$ be a simple ambidextrous object such that $J \in A(J)$. Suppose that $d : A(J) \to \mathbb{F}$ is a function such that the construction of Theorem 3 with d_J replaced by d yields a well-defined invariant for all $L \in \mathcal{L}_{A(J)}$. Then $d = \mathsf{d}_J$ for an appropriate choice of d_0.*

Proof. Let L be the Hopf link with components colored V and J, where V is a simple object of $\mathcal{C}$. By opening one strand of L and then the other, we obtain

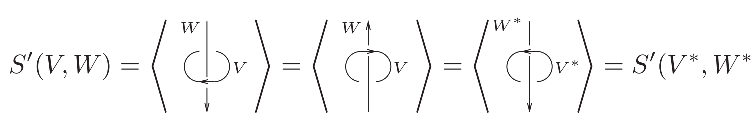
$$d(V)S'(J,V) = d(J)S'(V,J).$$

So, $d = d_0^{-1} d(J)\, \mathsf{d}_J$. □

This lemma implies, in particular, that if J_1, J_2 are two simple ambidextrous objects such that $A(J_1) = A(J_2)$, then $\mathsf{d}_{J_1} = \mathsf{d}_{J_2}$ for any choice of d_0 in the definition of d_{J_1} and an appropriate choice of d_0 in the definition of d_{J_2}.

Lemma 12. *Let V, W be simple objects in $\mathcal{C}$, then $S'(V,W) = S'(V^*, W^*)$.*

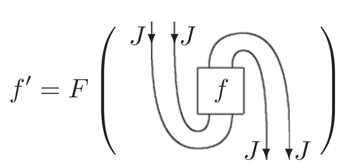
Proof. We have



where the first and the fourth equalities follow from the definition of S', the second from composing with the morphism $(\mathrm{Id}_V \otimes d_V)(b_V \otimes \mathrm{Id}_V)$ and the third from the property that $F(\uparrow_V) = F(\downarrow_{V^*})$. □

Lemma 13. *If an object J in $\mathcal{C}$ is ambidextrous, then so is J^*.*

Proof. We have to prove that $\mathrm{tr}_L(f) = \mathrm{tr}_R(f)$ for any endomorphism f of $J^* \otimes J^*$. First we define an endomorphism f' of $J \otimes J$ by the following formula.



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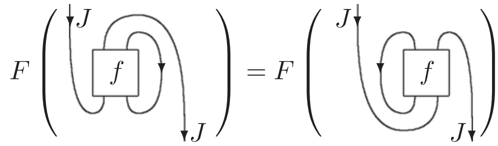
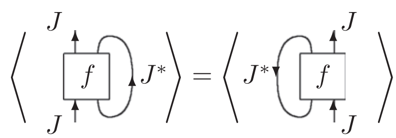
As V is ambidextrous we have $\mathrm{tr}_L(f') = \mathrm{tr}_R(f')$ and thus



In ribbon categories, modules are canonically isomorphic to their bidual and if T is a colored ribbon graph, changing both the orientation of an internal edge and its color to its dual does not affect $F(T)$. Doing this for the internal edges of the two ribbon graphs above, we deduce that



which means that $\mathrm{tr}_R(f) = \mathrm{tr}_L(f)$. $\square$

Corollary 14. *For any simple ambidextrous object $J \in Ob(\mathcal{C})$, we have $A(J^*) = \{V^* \mid V \in A(J)\}$ and $\mathsf{d}_{J^*}(V^*) = \mathsf{d}_J(V)$ for all $V \in A(J)$. If $V, V^* \in A(J)$ satisfy $S'(V, V^*) \neq 0$, then $\mathsf{d}_J(V^*) = \mathsf{d}_J(V)$.*

Proof. Lemma 12 implies that

$$A(J^*) = \{V \in Ob(\mathcal{C}) \mid V \text{ simple and } S'(J^*, V) \neq 0\} = \{V^* \mid V \in A(J)\}$$

and, for all $V \in A(J)$,

$$\mathsf{d}_J(V) = d_0 \frac{S'(V, J)}{S'(J, V)} = d_0 \frac{S'(V^*, J^*)}{S'(J^*, V^*)} = \mathsf{d}_{J^*}(V^*).$$

Consider the Hopf link H with components labeled by V and V^*. Then, by definition, $F'(H)$ is equal to both $\mathsf{d}_J(V) S'(V^*, V)$ and $\mathsf{d}_J(V^*) S'(V, V^*)$. Now Lemma 12 implies that $S'(V^*, V) = S'(V, V^*) \neq 0$ and therefore $\mathsf{d}_J(V^*) = \mathsf{d}_J(V)$. $\square$

Let I be the set of isomorphism classes of simple objects of $\mathcal{C}$. We call a subset B of I *complete* if $S'(U, V) \neq 0$ for all $U, V \in B$ and $S'(U, W) = 0$ for all $U \in B$, $W \in I - B$.

Lemma 15. *Let B be a complete subset of the set I, which contains at least one ambidextrous object. Then:*

(1) *all objects in B are ambidextrous;*

(2) *for any object $J \in B$, we have $A(J) = B$;*

(3) *let $F'_J, F'_{J'}$ be the invariants derived above from arbitrary pairs $(J \in B, d_0)$ and $(J' \in B, d'_0)$, respectively, where $d_0, d'_0 \in \mathbb{F} \setminus \{0\}$; then*

$$F'_J = \frac{\mathsf{d}_J(J')}{d'_0} F'_{J'}.$$

Proof. The first claim follows from the definition of a complete set and the remark following Lemma 2. The equality $A(J) = B$ follows from the definitions.

Lemma 11 implies that $\mathsf{d}_{J'}$ is proportional to d_J. More precisely, $\mathsf{d}_J = (d'_0)^{-1} \mathsf{d}_J(J') \mathsf{d}_{J'}$. Thus,

$$F'_J(L) = \mathsf{d}_J(V) \langle T_V \rangle = (d'_0)^{-1} \mathsf{d}_J(J') \mathsf{d}_{J'}(V) \langle T_V \rangle = (d'_0)^{-1} \mathsf{d}_J(J') F'_{J'}(L). \quad \square$$

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The completeness condition on B seems to be very strong. However, complete sets arise naturally in our examples, see §§ 6.1, 6.2 and 6.3. In such a situation Lemma 15 states that the ambidextrous object J defines a 'cluster' $A(J)$ of ambidextrous objects with the property that any element of this cluster leads to an invariant proportional to F'_J. There are two kinds of clusters depending on whether or not $\dim_{\mathcal{C}}(J)=0$. The case $\dim_{\mathcal{C}}(J)=0$ is of particular interest because then F' may be non-zero while the usual invariant F restricted to $\mathcal{L}_{\overline{A}}$ is zero, as will be clear from the next lemma.

From now on and up to the end of this section we fix a simple ambidextrous object J in $\mathcal{C}$.

Lemma 16. *For any $V \in A(J)$,*

$$\dim_{\mathcal{C}}(V) = d_0^{-1} \dim_{\mathcal{C}}(J)\, \mathsf{d}_J(V).$$

Proof. Consider the Hopf link H with components labeled by J and $V \in A$. Now $F(H)$ can be computed in two ways, namely by cutting the component labeled by J or cutting the component labeled by V. This gives

$$F(H) = S'(V,J) \dim_{\mathcal{C}}(J) = S'(J,V) \dim_{\mathcal{C}}(V).$$

Thus, $\dim_{\mathcal{C}}(V) = d_0^{-1} \dim_{\mathcal{C}}(J)\, \mathsf{d}_J(V)$. □

The next corollary follows directly from the previous lemma.

Corollary 17. *The following hold:*

1. *if $\dim_{\mathcal{C}}(J)=0$, then $\dim_{\mathcal{C}}(V)=0$ for all $V \in A(J)$ and $F(L)=0$ for all L in $\mathcal{L}_{\overline{A}}$;*
2. *if $\dim_{\mathcal{C}}(J)\neq 0$, then F' is proportional to F.*

The following two propositions show that F' has behavior similar to the functor F.

Proposition 18. *Let $U, V, W \in Ob(\mathcal{C})$ be such that $W \cong U \otimes V$. Let L be a $\mathcal{C}$-colored link, such that a component of L is colored by W. Let $L_{\parallel}$ be the link obtained from L by replacing this component of L by two parallel copies colored by U and V. If $L_{\parallel}$ is an element of $\mathcal{L}_{\overline{A(J)}}$, then $F'(L)=F'(L_{\parallel})$.*

Proof. Since $W \cong U \otimes V$ there exist morphisms $f: W \to U \otimes V$ and $g: U \otimes V \to W$ such that $f \circ g = \mathrm{Id}_{U\otimes V}$ and $g \circ f = \mathrm{Id}_W$. Use the equality $g \circ f = \mathrm{Id}_W$ to replace a portion of the W-colored component of L by two strings labeled by U and V and two coupons labeled f and g. Then by sliding one of the coupons around the component and using the equality $f \circ g = \mathrm{Id}_{U\otimes V}$ one arrives at $L_{\parallel}$. Then $F'(L)=F'(L_{\parallel})$ since F' is a well-defined $\mathcal{C}$-colored ribbon graph invariant. □

Proposition 19. *Let L be an element of $\mathcal{L}_{A(J)}$ with a circle component colored by $W \in Ob(\mathcal{C})$. If W is the only color of L in $A(J)$, then we additionally assume that $W^* \in A(J)$ and $S'(W,W^*)\neq 0$. Let L_- be obtained from L by reversing the orientation of the W-colored component and changing its color to W^*. Then $F'(L)=F'(L_-)$.*

Proof. We consider two cases: (a) if W is the only color of L in $A(J)$ and (b) otherwise. In the latter case we have $F'(L)=\mathsf{d}_J(V)\langle F(T_V)\rangle$ where $V\in A(J)$ is the label of another circle component (or an edge) of L and T_V is obtained from L by cutting this component (edge).

However, $F(T_V) = F(T'_V)$ where T'_V is obtained from T_V by reversing the orientation of the W-colored component and changing its color to W^*. In the former case we have

$$\begin{aligned} F'(L) &= \mathsf{d}_J(W)\langle T_W\rangle \\ &= \mathsf{d}_J(W)\langle T^-_{W^*}\rangle \\ &= \mathsf{d}_J(W^*)\langle \perp_{W^*}\rangle \\ &= F'(L_-) \end{aligned}$$

where $\perp_{W^*}$ is the ribbon graph $T^-_{W^*}$ rotated 180°. (We use here the second claim of Corollary 14.) □

Proposition 20. *If $L \in \mathcal{L}_{\overline{A}(J)}$, $L_+ \in \mathcal{L}_{Ob(\mathcal{C})}$, then the disjoint union $L \sqcup L_+$ belongs to $\mathcal{L}_{\overline{A}(J)}$ and $F'(L \sqcup L_+) = F'(L)F(L_+)$.*

Proof. The proof follows from the definitions. □

Remark 21. Both Propositions 18 and 19 can be extended to analogous statements for non-closed $\mathcal{C}$-colored ribbon graphs. In other words, F' behaves under cabling and reversing orientation in the same way as the standard ribbon functor F.

6. Examples

We give three classes of examples of ambidextrous objects and associated re-normalized link invariants.

6.1 Link invariants from Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra and let $U_q(\mathfrak{g})$ be the Drinfeld–Jimbo $\mathbb{C}(q)$-algebra associated to $\mathfrak{g}$ (see [Tur94, § XI.6]). (Note here that q is not a root of unity.) Let $\mathcal{C}$ be the category of finite-dimensional $U_q(\mathfrak{g})$-modules. It is well known that $\mathcal{C}$ is a ribbon Ab-category with ground ring $K = \mathbb{C}(q)$. Here $\mathbb{F} = K$ and $\mathrm{tr}_{\mathcal{C}}$ (respectively $\dim_{\mathcal{C}}$) is the quantum trace (respectively quantum dimension).

Lemma 22. *All simple objects of $\mathcal{C}$ are ambidextrous.*

Proof. This follows from Lemma 1, as $\dim_{\mathcal{C}}(J) \neq 0$ for any simple object J of $\mathcal{C}$. □

Let I be the set of isomorphism classes of simple objects of $\mathcal{C}$. One can show that $S'(V, W) \neq 0$ for any $V, W \in I$ (for a similar calculation, see [GP08b, Proposition 2.2]). Thus, $A(J) = I$ for all $J \in I$. The construction of § 3 derives from J and any non-zero $d_0 \in \mathbb{F}$ a function d_J and an invariant F'. Lemma 15 implies that F' is essentially independent of the choice of J.

Proposition 23. *For any $J \in I$ and $d_0 = \mathrm{qdim}(J)$, the J-determined quantum dimension d_J is equal to the usual quantum dimension and F' is the usual Reshetikhin–Turaev quantum group invariant arising from $\mathfrak{g}$.*

Proof. This follow from Corollary 17. □

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6.2 Link invariants from Lie superalgebras

In [GP08b, GP08a] the first two authors derived new link invariants from Lie superalgebras $\mathfrak{osp}(2|2n)$ and $\mathfrak{sl}(m|n)$ with $m \neq n$. In particular, the invariants associated with $\mathfrak{sl}(m|1)$ generalize both the multivariable Alexander polynomial of links and Kashaev's link invariants. We explain here that the construction of [GP08b, GP08a] is a special case of the construction of § 3.

In this section we work in the category of vector superspaces with even morphisms, i.e. the category whose objects are $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces and the morphisms are even linear maps. A Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(m|n)$ or $\mathfrak{g} = \mathfrak{osp}(2|2n)$ gives rise to the quantized universal enveloping $\mathbb{C}[[h]]$-superalgebra $U_h(\mathfrak{g})$. Let $\mathcal{C}$ be the category of topologically free $U_h(\mathfrak{g})$-modules of finite rank (i.e. modules of the form $V[[h]]$ where V is a finite-dimensional $\mathfrak{g}$-module). It is known that $\mathcal{C}$ is a ribbon Ab-category with ground ring $K = \mathbb{C}[[h]]$, see, for instance, [GP08b] and references therein. The object $V[[h]]$ of $\mathcal{C}$ is simple if and only if V is a simple $\mathfrak{g}$-module. The finite-dimensional simple $\mathfrak{g}$-modules are divided into two classes: typical and atypical. A simple $\mathfrak{g}$-module is typical if each time it is a submodule or a factor module of a finite-dimensional $\mathfrak{g}$-module, it splits as a direct summand. We call a $U_h(\mathfrak{g})$-module $V[[h]]$ (a)typical if V is a (a)typical $\mathfrak{g}$-module.

Let I be the set of isomorphism classes of simple objects of $\mathcal{C}$ and let B be the subset of I consisting of isomorphism classes of typical $U_h(\mathfrak{g})$-modules.

Lemma 24. *If V is an element of B, then $\dim_{\mathcal{C}}(V) = 0$. The link invariant F restricted to $\mathcal{L}_B$ is zero.*

Proof. The first statement follows from a direct calculation using the character formula of V (see [GP08b]). The second statement follows from Corollary 17. □

The following lemma is a restatement of [GP08b, Lemma 2.8].

Lemma 25. *There exists an element $J_0 \in B$ such that $J_0 \otimes J_0$ splits as a direct sum of elements of B with multiplicity one. In particular, the algebra $\mathrm{End}_{\mathcal{C}}(J_0 \otimes J_0)$ is commutative.*

Corollary 26. *The object J_0 of $\mathcal{C}$ is ambidextrous.*

Proof. Lemma 25 implies that the braiding c_{J_0,J_0} commutes with all elements of $\mathrm{End}_{\mathcal{C}}(J_0 \otimes J_0)$. Thus, the corollary follows from Lemma 1. □

In [GP08b] the first two authors have shown that $S'(U,V) \neq 0$ for all $U, V \in B$ and $S'(U,W) = 0$ for all $U \in B$, $W \in I - B$. In other words, the set B is complete.

Proposition 27. *Every element $J \in B$ is ambidextrous and $A(J) = B$. The construction in § 3 gives a function $\mathsf{d} = \mathsf{d}_J : B \to \mathbb{C}[[h]][h^{-1}]$ and an invariant $F' = F'_J$, which do not depend on J up to multiplication by a non-zero element of $\mathbb{C}[[h]][h^{-1}]$.*

Proof. Since B is complete and contains J, the lemma follows from Lemma 15. □

The link invariant introduced in [GP08b] is defined on $\mathcal{L}_B$. Its definition is similar to the definition of F' above but uses a certain function $d : B \to \mathbb{C}[[h]][h^{-1}]$ rather than d. By Lemma 11, the function d must be proportional to d and therefore the link invariant of [GP08b] is equal to the invariant F' associated with an arbitrary $J \in B$ and an appropriate d_0 (depending on J).

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Remark 28. (1) In this example there is no canonical choice for $J \in B$. However, for each J there is a suitable choice of d_0 (possibly, distinct from $d_0 = 1$) such that d_J has a nice formula (cf. [GP08b]). This justifies our choice to include the factor d_0 in the definition of F'.

(2) The extension of F' in § 3 could be useful in computing F' for links colored with non-semisimple modules.

(3) It would be interesting to extend the constructions of this section to other Lie superalgebras.

6.3 Link invariants from $U_q(\mathfrak{sl}(2))$ at roots of unity

In this section we consider the generalized multivariable Alexander invariants defined by Akutsu, Deguchi and Ohtsuki (ADO) in [ADO92], which contain Kashaev's invariants (see [Kas95, MM01]). These invariants are indexed by positive integers. In [Mur08], Murakami gives a framed version of these invariants using the universal R-matrix of $U_q(\mathfrak{sl}(2))$ and calls them the colored Alexander invariants. Here we show that these invariants are restrictions of invariants defined using ribbon categories as formulated above.

Fix a positive integer N and let $q = e^{\pi\sqrt{-1}/N}$ be a $2N$-root of unity. We use the notation $q^x = e^{\pi\sqrt{-1}x/N}$. Here we give a slightly generalized version of $U_q(\mathfrak{sl}(2))$. Let $U_q^H(\mathfrak{sl}(2))$ be the $\mathbb{C}(q)$-algebra given by generators E, F, K, K^{-1}, H and relations

$$HK = KH, \quad HK^{-1} = K^{-1}H, \quad [H, E] = 2E, \quad [H, F] = -2F,$$
$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The algebra $U_q^H(\mathfrak{sl}(2))$ is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\epsilon(E) = \epsilon(F) = \epsilon(H) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1,$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.$$

Define $\bar{U}_q(\mathfrak{sl}(2))$ as the Hopf algebra $U_q^H(\mathfrak{sl}(2))$ modulo the relations $E^N = F^N = 0$.

We say a $\bar{U}_q(\mathfrak{sl}(2))$-module V is a *weight module* if V has a weight decomposition with respect to H and if q^H acts as K. Let $\mathcal{C}$ be the tensor Ab-category of finite-dimensional weight $\bar{U}_q(\mathfrak{sl}(2))$-modules (here the ground ring is $\mathbb{C}$). We say a simple weight module is *typical* if its highest weight is in the set $(\mathbb{C} \setminus \mathbb{Z}) \cup \{-1 + kN \mid k \in \mathbb{Z}\}$, otherwise we say it is *atypical*. A typical module is N-dimensional and indexed by its highest weight λ. We denote such a module by V_λ (for a basis of this module see [Mur08]). The weights of V_λ are $\lambda - 2i$ for $0 \le i \le N - 1$, so its character formula is $\sum_{i=0}^{N-1} u^{\lambda - 2i}$ where the coefficient of u^a is the dimension of the a-weight space.

We now recall that the category $\mathcal{C}$ is a ribbon Ab-category. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we set $\{a\} = q^a - q^{-a}$ and $\{n\}! = \{n\}\{n-1\} \dots \{1\}$. In [Oht02], Ohtsuki defines an element R_t given by

$$R_t = q^{H \otimes H/2} \sum_{n=0}^{N-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n,$$

where $q^{H\otimes H/2}$ is a formal symbol. If v and v' are two weight vectors of weights of λ and λ', then $q^{H\otimes H/2}$ acts on $v\otimes v'$ by

$$q^{H\otimes H/2}\cdot(v\otimes v')=q^{\lambda\lambda'/2}v\otimes v'.$$

Thus, the action of R_t on the tensor product of two objects of $\mathcal{C}$ is well defined and induces an endomorphism on such a tensor product. Moreover, R_t gives rise to a braiding $c_{V,W}:V\otimes W\to W\otimes V$ on $\mathcal{C}$ defined by $v\otimes w\mapsto\tau(R_t(v\otimes w))$ where τ is the permutation $x\otimes y\mapsto y\otimes x$ (see [Mur08, Oht02]).

Remark 29. It is important that $\bar{U}_q(\mathfrak{sl}(2))$ contains H because the modules V_λ and $V_{\lambda+2N}$ over $U_q(\mathfrak{sl}(2))/\{E^N=F^N=0\}$ are isomorphic but the action of the R-matrix on $V_\lambda^{\otimes 2}$ and $V_{\lambda+2N}^{\otimes 2}$ are different. These modules are distinct in $\mathcal{C}$ as H acts differently.

Let V and W be objects of $\mathcal{C}$. Let $\{v_i\}$ be a basis of V and $\{v_i^*\}$ be a dual basis of V^*. Then

$$b_V:\mathbb{C}\to V\otimes V^*,\ \text{given by }1\mapsto\sum v_i\otimes v_i^*$$
$$d_V:V^*\otimes V\to\mathbb{C},\ \text{given by }f\otimes w\mapsto f(w)$$

are duality morphisms of $\mathcal{C}$.

Also, in [Oht02] Ohtsuki defines an element u given by

$$q^{-H^2/2}\sum_{n=0}^{N-1}S(t_n)s_n$$

where $R_t=q^{H\otimes H/2}\sum_{n=0}^{N-1}s_n\otimes t_n$ and $q^{-H/2}$ is a formal symbol whose action on a weight vector v_λ is given by $q^{-H^2/2}\cdot v_\lambda=q^{-\lambda^2/2}v_\lambda$. Let $\theta=uK^{N-1}=K^{N-1}u$. The twist $\theta_V:V\to V$ is defined by $v\mapsto\theta^{-1}v$ (see [Mur08, Oht02]).

If L is a link colored with objects of $\mathcal{C}$ such that one of the colors is typical, then $F(L)=0$ where as above F is the usual ribbon functor $F:Rib_{\mathcal{C}}\to\mathcal{C}$. We now show that the general construction above gives rise to a non-trivial invariant F', which contains the ADO invariants and so Kashaev's invariants.

Fix a typical $\bar{U}_q(\mathfrak{sl}(2))$-module V_{λ_0} such that $\lambda_0\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}$ and denote it by J_0.

LEMMA 30. *The tensor product $J_0\otimes J_0$ splits as a direct sum of typical $\bar{U}_q(\mathfrak{sl}(2))$-modules with no multiplicity. In particular, the algebra $\operatorname{End}_{\mathcal{C}}(J_0\otimes J_0)$ is commutative.*

Proof. First, using the character formula for J_0 one sees that all of the weights of $J_0\otimes J_0$ are not integral. Then since typical modules always split we have that $J_0\otimes J_0$ is a direct sum of typical modules. The character formula for a typical module then implies that $J_0\otimes J_0=\bigoplus_{i=0}^{N-1}V_{2\lambda_0-2i}$, which completes the proof. □

COROLLARY 31. *The element J_0 is ambidextrous.*

Proof. The corollary follows directly from Lemmas 1 and 30. □

Next we compute $\dim_{\mathcal{C}}(V_\lambda)$ and $S'(V_\lambda,V_{\lambda'})$. To do this we need the morphisms $d'_V:V\otimes V^*\to\mathbb{C}$ and $b'_V:\mathbb{C}\to V^*\otimes V$ defined by

$$d'_V=d_Vc_{V,V^*}(\theta_V\otimes\operatorname{Id}_{V^*})\quad b'_V=(\operatorname{Id}_{V^*}\otimes\theta_V)C_{V,V^*}b_V.$$

A direct computation shows that

$$d'_V(v\otimes f)=f(K^{1-N}v),\quad b'_V(1)=\sum K^{N-1}v_i\otimes v_i^*.$$

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LEMMA 32. *Let V_λ be typical $\bar{U}_q(\mathfrak{sl}(2))$-module, then* $\dim_{\mathcal{C}}(V_\lambda) = 0$.

Proof. Let $\{v_i\}$ be a basis of V_λ such that v_i is a non-zero vector of weight $\lambda - 2i$. By definition we have $\dim_{\mathcal{C}}(V_\lambda) = (d'_{V_\lambda} \circ b_{V_\lambda})(1)$ which is equal to

$$\sum_{i=0}^{N-1} v_i \otimes v_i^* = \sum_{i=0}^{N-1} v_i^*(K^{1-N} v_i) = \sum_{i=0}^{N-1} q^{(N-1)(\lambda-2i)} = q^{(N-1)\lambda} \frac{1 - q^{-2N}}{1 - q^{-2}}$$

where $q^{-2N} = 1$ and so $\dim_{\mathcal{C}}(V_\lambda) = 0$. □

LEMMA 33. *Let V_λ be a typical module and let $V_{\lambda'}$ be any simple weight module with highest weight λ'. We have*

$$S'(V_\lambda, V_{\lambda'}) = q^{(\lambda+1-N)(\lambda'+1-N)} \frac{\{N(\lambda' + 1 - N)\}}{\{\lambda' + 1 - N\}},$$

where $\{N(\lambda' + 1 - N)\}/\{\lambda' + 1 - N\}$ is a Laurent polynomial in $q^{\lambda'}$.

Proof. The proof follows from a direct computation. A detailed presentation of an analogous computation is given in [GP08b, Proposition 2.2]. □

Let I be the set of isomorphism classes of simple objects of $\mathcal{C}$ and let B be the subset of I consisting of isomorphism classes of typical $\bar{U}_q(\mathfrak{sl}(2))$-modules.

LEMMA 34. *The usual invariant F restricted to $\mathcal{L}_B$ is zero.*

Proof. The proof follows from Corollary 17 and Lemma 32. □

Let V_λ be in B, then from Lemma 33 we have $S'(V_\lambda, V) \neq 0$ for all $V \in B$ and $S'(V_\lambda, W) = 0$ for all $W \in I \setminus B$. In other words, the set B is complete.

PROPOSITION 35. *Every element of $J \in B$ is ambidextrous and $B = A(J)$. The construction in § 3 gives a function $\mathsf{d}_J : B \to \mathbb{C}$ and an invariant F'_J, which do not depend on J up to multiplication by a non-zero element of $\mathbb{C}$.*

Proof. Since B is complete and contains J, then the lemma follows from Lemma 15. □

Let $\mathsf{d} = \mathsf{d}_{J_0}$ and $F' = F'_{J_0}$ be objects defined in § 3 arising from the ambidextrous element J_0 and the constant $d_0 = 1/(\prod_{j=0}^{N-2} \{\lambda_0 + N - j\})$. We now compute d explicitly. By a direct computation, for $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ one has

$$\frac{1}{\prod_{j=0}^{N-2} \{\lambda + N - j\}} = (-1)^N q^{-N(N+1)/2} \frac{\{\lambda + 1 - N\}}{\{N(\lambda + 1 - N)\}}.$$

Therefore, from the expression of S' in Lemma 33 we have that

$$\prod_{j=0}^{N-2} \{\lambda_0 + N - j\} S'(J, V_\lambda) = \prod_{j=0}^{N-2} \{\lambda + N - j\} S'(V_\lambda, J).$$

Then, by definition, we have

$$\mathsf{d}(V_\lambda) = \frac{1}{\prod_{j=0}^{N-2} \{\lambda + N - j\}},$$

where we use the above choice of d_0.

Next we show that F' restricts to the colored Alexander invariant given by Murakami in [Mur08]. The colored Alexander invariant is a reconstruction of the link invariants defined in [ADO92]. Murakami's construction uses the universal R-matrix of $U_q(\mathfrak{sl}(2))$ and state sums. In particular, let T be a (1,1)-tangle whose ith component is colored by a parameter λ_i ($i = 1, \ldots, k$) and the first component is the open component. Murakami defines $O_T^N(\lambda_1, \ldots, \lambda_k)$ to be the element of $\mathrm{End}(V_{\lambda_1})$ obtained by assigning the matrix elements of the R-matrix for the crossings of T and particular scalars to the maximal and minimal points of T (these scalars are the same as the scalars coming from the morphisms $b_{V_{\lambda_i}}, d_{V_{\lambda_i}}, b'_{V_{\lambda_i}}, d'_{V_{\lambda_i}}$ given above). Let $\Phi_T^N(\lambda_1, \ldots, \lambda_k) = \mathsf{d}(V_{\lambda_1})O_T^N(\lambda_1, \ldots, \lambda_k)$. Then in [Mur08], Murakami showed that $\Phi_T^N(\lambda_1, \ldots, \lambda_k)$ is a framed version of the analogous invariant defined in [ADO92]. Thus, $\Phi_T^N(\lambda_1, \ldots, \lambda_k)$ is a well-defined invariant of a colored framed link L obtained by closing the tangle T; denote this invariant by Φ_L^N. Since the construction of F' uses the same R-matrix, duality, twist and scaling d as the construction of Φ_L^N we have proved the following theorem.

Theorem 36. *The invariant F' restricted to framed links colored with typical modules is equal to the colored Alexander invariant Φ_L^N.*

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