

A rank-one, rigid, simple, prime map

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Abstract. We construct a rank-1 map T similar to Chacon's map [C], [JRS], but with 2^{n+1} n -blocks in an $(n+1)$ -block and a single spacer in the middle. Hence T is rigid and the centralizer of T , $C(T)$, is uncountable. We show T is simple [V], [JR], and hence any factor of T is the algebra of invariant sets of some weakly compact subgroup of $C(T)$. We show $C(T)$ has no such subgroups and hence T is prime. Lastly we show that T is graphic in the sense of [AM].

0. Introduction

The notion of a 2-fold simple measure-preserving map T of a probability space (X, μ) was introduced by Veech [V] under a different name (property S). (See [JR] for a clarification of the history of the terminology). T is 2-fold simple if any ergodic $T \times T$ -invariant measure λ on $X \times X$ which has marginals μ is either $\mu \times \mu$ or a measure $\mu_S = (\text{id} \times S)\mu_\Delta$ where S is a measure-preserving map commuting with T and μ_Δ is the diagonal measure over μ . We denote by $C(T)$ the centralizer of T , that is, all measure preserving maps commuting with T .

Veech showed that if T is simple and \mathcal{F} is a non-trivial T -invariant sub- σ -algebra then there is a weakly compact subgroup K of $C(T)$ such that, up to null sets, $\mathcal{F} = \{A: SA = A \quad \forall S \in C(T)\}$. The main application which he had in mind was that if $C(T)$ has no non-trivial compact subgroups then T is prime – the only invariant σ -algebras are, up to null sets, the trivial one and the full σ -algebra.

In [JR] the notion of simplicity was extended to higher orders (and to more general group actions). T is said to be k -fold simple if every $T \times \cdots \times T$ -invariant ergodic measure on X^k is a product of off-diagonal measures, where by an off-diagonal measure on X^k , $k' \subset \{1, \dots, k\}$ we mean a measure $(\bigotimes_{i \in k'} S_i)\mu_\Delta$ where $S_i \in C(T)$ and μ_Δ is diagonal measure on $X^{k'}$. We say T is simple if it is k -fold simple of all orders. In [JR] Veech's result together with results of [JR] was applied to obtain new examples of prime maps, for example the time-1 map of any weakly-mixing flow with minimal self-joinings (see [JR] for the definition and [JP] for an example of such).

The purpose of this paper is to construct a quite different sort of simple prime map. In fact our example is rigid which means that there is a sequence $n_i \rightarrow \infty$ such that $T^{n_i} \rightarrow \text{id}$ weakly. As is well-known (see [KSS]) this forces the centralizer to be uncountable. As opposed to the time-1 map of a flow our example has a non-locally

compact centralizer. In the course of establishing 2-fold simplicity we obtain a fairly explicit description of the centralizer.

As shown in [JR], simplicity has important consequences which go beyond those of mere 2-fold simplicity. We mention just one example; any two simple prime maps are either disjoint or isomorphic. We establish simplicity (of all orders) for our example. In addition the description of the centralizer can be used to show that T^m has only the obvious roots (e.g. T has no roots at all) whence T^m and T^n are non-isomorphic for $|m| > |n| > 0$. (It is obvious from the construction that T and T^{-1} are isomorphic). The general theory then implies that T^m and T^n are disjoint, and much more, in fact.

Finally we consider the topological behaviour of (X, T) , which is much simpler. (X, T) is POD in the sense of [FKS] hence it is prime and its topological centralizer is just its powers.

1. Construction and simplicity of T

The map T will be described as the left shift on a certain closed subset of $\{0, 1\}^{\mathbb{Z}}$ and will be minimal and uniquely ergodic. We begin by listing a sequence of block names, each a finite string of 0's and 1's.

$$\begin{aligned} B_1 &= 010 \\ B_2 &= B_1B_11B_1B_1 \\ &\vdots \\ B_{n+1} &= B_n^{2^n}1B_n^{2^n}. \end{aligned}$$

As always multiplication of such strings is concatenation. Letting h_n be the length of B_n ,

$$h_{n+1} = 2^{n+1}h_n + 1 > 2^{n^2/2}.$$

We represent points in $\{0, 1\}^{\mathbb{Z}}$ by $\bar{x} = x_{-n} \cdots x_0 \cdots x_n \cdots$. We define $X = \{\bar{x}; \text{ for all } i < j, x_i x_{i+1} \cdots x_j \text{ occurs as a string of consecutive 0's and 1's in some } B_n\}$. This is clearly a closed, shift-invariant subset of $\{0, 1\}^{\mathbb{Z}}$.

For any finite string a_0, a_1, \dots, a_{j-1} of 0's and 1's, we will call $C = \{\bar{x}; x_k = a_{k-i}, i \leq k \leq j\}$ a cylinder set. The set $(i, i+1, \dots, j)$ we will call the indices of C , and a_0, \dots, a_{j-1} the name of C .

A few facts about the n -blocks follow easily.

LEMMA 1.1. (i) *In any B_n , two consecutive 1's never occur and three consecutive 0's never occur.*

(ii) *If a_0, \dots, a_{l-1} is a sequence of consecutive 0's and 1's in some B_n , then*

$$\frac{\#\{i \in (0, \dots, l-2); a_i \neq a_{i+1}\}}{l-1} \geq \frac{2}{3} - \frac{2}{l-1}.$$

(iii) *In B_n^2 the only strings of h_n consecutive symbols that are identically B_n are the two given copies.*

We say a string of 0's and 1's $a_0 a_1 \cdots a_{l-1}$ has a k -block cover if there are integers

$k_1, k_2 < 2h_k$ and

$$a_{k_1} \cdots a_{l-k_2-1} = \left(\prod_{i=1}^{s-1} B_k^i 1 \right) B_k^j,$$

where all $j_i \geq 2$. That is to say except for short ($< 2h_k$) pieces at either end, \bar{a} can be written as a concatenation of strings of k -blocks, each at least a B_k^2 , separated by single 1's.

LEMMA 1.2. (iv) Any $a_0 \cdots a_{l-1}$ has at most one k -block cover and if it does, then any substring of more than $4h_k$ consecutive symbols also does.

(v) For any $n > k$, B_n can be written uniquely as a concatenation of strings of the form $B_k^{2^k}$ or $B_k^{2^{k+1}}$, separated by single 1's, the first and last such of the form $B_k^{2^k}$. Hence any substring of at least $4h_k$ symbols in B_n has a unique k -block cover.

(vi) For any $\bar{x} \in X$, $j - i \geq 4h_k$, $x_i \cdots x_j$ has a unique k -block cover, and hence the entire name \bar{x} has a unique k -block cover where the k -blocks occur in strings of the form $B_k^{2^k}$ or $B_k^{2^{k+1}}$.

If in this unique k -block cover, $x_0 = 1$ corresponds to a 1 between strings of k -blocks, then $\bar{x} \in S_k$. This defines the set S_k as a union of cylinder sets on indices $-2h_k$ to $2h_k$. S_k is open and closed, and $S_{k+1} \subset S_k$.

For $\bar{x} \in X$, the occurrences on the T orbit of \bar{x} of points in S_k are separated by exactly h_{k+1} , or $(h_{k+1} + 1)/2$ points. Notice $T^i(\bar{x}) \in S_k$ exactly when index i in the k -block cover for \bar{x} lies between two k -block strings. Between two consecutive such occurrences, the name of \bar{x} is either $B_k^{2^k}$ or $B_k^{2^{k+1}}$. Thus the set of values j with $T^j(\bar{x}) \in S_k$ determines $\bar{x} \in X$.

LEMMA 1.3. (X, T) is minimal.

Proof. Any string a_0, \dots, a_{l-1} that is the name of some cylinder C , $C \cap X \neq \emptyset$, occurs in some B_k . For any $\bar{x} \in X$ there must exist a $-h_k - 1 \leq j \leq h_k$ with $T^j(\bar{x}) \in C$. \square

For any 0, 1-name a_0, \dots, a_{l-1} let $d_n(\bar{a})$ be the density in $\cdots B_n B_n \cdots = B_n^z$ of occurrences of \bar{a} .

LEMMA 1.4. For $\bar{a} = a_0 \cdots a_{l-1}$, $|d_{k+1}(\bar{a}) - d_k(\bar{a})| < 2l/h_{k+1}$.

Proof.

$$B_{k+1}^z = \cdots 1 B_k^{2^{k+1}} 1 B_k^{2^{k+1}} 1 B_k^{2^{k+1}} \cdots$$

The difference can only arise within l indices of a 1. \square

COROLLARY 1.5. For any $\bar{a} = a_0 \cdots a_{l-1}$, $\lim_{k \rightarrow \infty} d_k(\bar{a}) = d(\bar{a})$ exists and in fact

$$|d(\bar{a}) - d_k(\bar{a})| \leq \sum_{n=k+1}^{\infty} \frac{2l}{h_n} < \frac{l}{2^{k-1}}. \quad \square$$

For a cylinder set C , we define $\mu(C) = d(\bar{a})$, \bar{a} the name of C . This finitely additive measure extends to a shift invariant Borel measure on (X, T, B) .

COROLLARY 1.6. (X, μ, B, T) is uniquely ergodic.

COROLLARY 1.7. $\mu(S_k) < 1/2^k h_k$.

Proof. Occurrences of S_k on the orbit of any $\bar{x} \in X$ are separated by at least $(h_{k+1} + 1)/2 > 2^k h_k$. □

COROLLARY 1.8. For $\bar{x} \in X$,

$$\lim_{N \rightarrow \infty} \left(\frac{\#\{i \leq N; x_{i+h_k} \neq x_i\}}{2N+1} \right) \leq \frac{1}{2^k}.$$

Proof. If $x_{i+h_k} \neq x_i$, then for some $j \in (i, \dots, i + h_k - 1)$, $T^j(\bar{x}) \in S_k$. Hence the above \lim is

$$\leq \mu \left(\bigcup_{i=0}^{h_k-1} T^{-i}(S_k) \right) \leq h_k \mu(S_k) \leq \frac{1}{2^k}.$$

That it is actually a limit follows from unique ergodicity. □

COROLLARY 1.9. For any cylinder set C , $\mu(T^{h_k}(C)\Delta C) \xrightarrow{k} 0$, and T is rigid.

COROLLARY 1.10. The weak closure of the powers of T is uncountable and hence $C(T)$ is uncountable [KSS].

In fact $C(T) = \text{wkcl}(\{T^i\})$ can be concluded on three different levels. First we will see it directly for this example from lemma 1.25. Second, for a strongly rigid rank-1 T such as this the result follows from [CS]. Third, King [K] has shown this holds for any rank-1 map.

Let $n(k, \bar{x}) = \text{largest } n \leq 0 \text{ with } T^n(\bar{x}) \in S_k$, the return time under T^{-1} to S_k . Now define

$$\begin{aligned} A_k &= \{\bar{x}; x_{n(k,\bar{x})-h_{k+1}} \cdots x_0 \cdots x_{n(k,\bar{x})+2h_{k+1}} \\ &= 1 B_k^{2^{k+1}} 1 B_k^{2^{k+1}} 1 B_k^{2^{k+1}} 1 \end{aligned}$$

and

$$-h_k > n(k, \bar{x}) > -(2^{k+1} - 1)h_k\}.$$

Thus if $\bar{x} \in A_k$, in its k -block cover index 0 lies in a string of $2^{k+1} B_k$'s, at least a full B_k from either end, and the strings of B_k 's to either side of this one also are $B_k^{2^{k+1}}$.

LEMMA 1.11.

$$\mu(A_k) > 1 - (3h_{k+1} + 1)\mu(S_{k+1}) - 2h_k\mu(S_k) > 1 - \frac{1}{2^{k-3}}.$$

Proof. If

$$-\frac{3h_{k+1} + 1}{2} < n(k+1, \bar{x}) < \frac{3h_{k+1} + 1}{2} - h_{k-2}$$

and

$$-h_k > n(k, \bar{x}) > -(2^k - 1)h_k$$

then $\bar{x} \in A_k$. □

Let $E_k = \bigcap_{j=k}^{\infty} A_j$, those $\bar{x} \in A_j$ for all $j \geq k$. Thus $\bar{x} \in E_k$ exactly if it is at least h_k from a point in S_k on its orbit and at least $(3h_{k'} + 1)/2$ from a point in $S_{k'}$ for $k' > k$.

COROLLARY 1.12. (i) $\lim_{k \rightarrow \infty} \mu(E_k) = 1$, i.e. for μ -a.e. \bar{x} , once k is large enough, $\bar{x} \in E_k$.

(ii) For μ -a.e. \bar{x} , $\lim_{k \rightarrow \infty} n(k, \bar{x}) = -\infty$, $\lim_{k \rightarrow \infty} n(k, \bar{x}) + h_{k+1} = \infty$.

Our next step is to show that T is 2-fold simple, i.e. any $T \times T$ invariant ergodic measure on $X \times X$ is either $\mu \times \mu$ or is supported on the graph of some $S \in C(T)$. To proceed, assume λ is such a $T \times T$ invariant ergodic Borel measure on $X \times X$, which must have marginals μ by corollary 1.6.

For λ -a.e. (\bar{x}, \bar{y}) , once k is large enough both $\bar{x}, \bar{y} \in E_k$ and thus among the points

$$T^{n(k,\bar{x})}(\bar{y}), \dots, T^{n(k,\bar{x})+h_{k+1}-1}(\bar{y})$$

there is exactly one point $T^{n(k,\bar{x})+l}(\bar{y}) \in S_k$,

$$l = \begin{cases} n(k, \bar{y}) - n(k, \bar{x}) & \text{if this is } \geq 0 \\ n(k, \bar{y}) - n(k, \bar{x}) + h_{k+1} & \text{otherwise.} \end{cases}$$

This point l splits the indices $n(k, \bar{x}), \dots, n(k, \bar{x}) + h_{k+1} - 1$ into two pieces

$$I = I_k(\bar{x}, \bar{y}) = \{i; n(k, \bar{x}) \leq i < l\}$$

$$J = J_k(\bar{x}, \bar{y}) = \{i; l \leq i < n(k, \bar{x}) + h_{k+1}\}.$$

We define

$$U_k(\bar{x}, \bar{y}) = \bigcup_{i \in I} (T \times T)^i(\bar{x}, \bar{y}) \cap (E_k \times X) \cap (X \times E_k)$$

$$V_k(\bar{x}, \bar{y}) = \bigcup_{i \in J} (T \times T)^i(\bar{x}, \bar{y}) \cap (E_k \times X) \cap (X \times E_k),$$

i.e. the corresponding pieces of orbit as long as both coordinates are in E_k . Notice that if $(\bar{x}', \bar{y}') \in U_k(\bar{x}, \bar{y}) \cup V_k(\bar{x}, \bar{y})$ then

$$U_k(\bar{x}, \bar{y}) = U_k(\bar{x}', \bar{y}')$$

and

$$V_k(\bar{x}, \bar{y}) = V_k(\bar{x}', \bar{y}'),$$

and both of these are strings of consecutive points on orbits.

Define

$$W_k = \{(\bar{x}, \bar{y}); (\bar{x}, \bar{y}) \text{ is in the smaller of } U_k(\bar{x}, \bar{y}), V_k(\bar{x}, \bar{y})\},$$

(one must in fact be smaller).

LEMMA 1.13. $\overline{\lim} W_k$ is λ -a.s. $T \times T$ invariant, hence

$$\lambda(\overline{\lim} W_k) = 0 \text{ or } 1.$$

Proof. Split W_k into two sets

$$W'_k = \{(\bar{x}, \bar{y}) \in W_k; \text{ the block } U_k(\bar{x}, \bar{y}),$$

$$\text{ or } V_k(\bar{x}, \bar{y}) \text{ containing } (\bar{x}, \bar{y}) \text{ is shorter than } 2h_k\}$$

$$W''_k = W_k \setminus W'_k.$$

Now if $(\bar{x}, \bar{y}) \in W'_k$, then one of \bar{x}, \bar{y} is in the set

$$\bigcup_{i=-3h_k}^{3h_k} T^{-1}(S_k),$$

a set of measure $< 6/2^{k-1}$ so $\lambda(\overline{\lim} W'_k) = 0$.

Points $(\bar{x}, \bar{y}) \in W_k''$ come in segments on a $T \times T$ orbit at least h_k long, hence the same is true of $\bigcup_{j=k}^\infty W_j''$. Thus

$$\lambda \left(\bigcup_{j=k}^\infty W_j'' \right) \cap (T \times T) \left(\bigcup_{j=k}^\infty W_j'' \right) \geq \left(1 - \frac{2}{h_k} \right) \lambda \left(\bigcup_{j=k}^\infty W_j'' \right),$$

and

$$\lambda(\overline{\lim} W_j'' \cap (T \times T)(\overline{\lim} W_j'')) = (\overline{\lim} W_j'')$$

and $\overline{\lim} W_j''$ is λ -a.s. $T \times T$ invariant. □

That T is 2-fold simple is a result of the following dichotomy.

THEOREM 1. (a) *If $\lambda(\overline{\lim} W_j) = 1$ then $\lambda = \mu \times \mu$.*

(b) *If $\lambda(\overline{\lim} W_j) = 0$ then λ is supported on the graph of some $S \in$ weak closure $\{T^i\}$.*

We prove this in steps. The basic tool to prove (a), and furthermore that $C(T)$ contains no compact subgroups, and the simplicity of all orders for T , is the construction of a certain sequence of maps.

$$\hat{\phi}_k : \hat{D}_k \rightarrow \hat{R}_k, \quad \hat{D}_k, \hat{R}_k \subset X \times X,$$

so that for any particular $(\bar{x}, \bar{y}) \in \hat{D}_k$, there is a $t > k$ with $(\bar{x}, \bar{y}) \in U_t(\bar{x}, \bar{y})$ and

$$\hat{\phi}_k(\bar{x}, \bar{y}) = (T \times T)^{jh_k}(\bar{x}, \bar{y}) \in V_t(\bar{x}, \bar{y}).$$

Notice that for such a $\hat{\phi}_k$, and $(\bar{x}, \bar{y}) \in \hat{D}_k$, if C, D are cylinder sets whose indices lie in $(-h_k + 1, \dots, h_k - 1)$ then $(\bar{x}, \bar{y}) \in C \times D$ iff

$$\hat{\phi}_k(\bar{x}, \bar{y}) \in C \times T^{-1}(D).$$

This leads to the following fact.

LEMMA 1.14. *Suppose there exist λ -preserving 1-1 Borel maps $\hat{\phi}_k : \hat{D}_k \rightarrow \hat{R}_k$ so that for $(\bar{x}, \bar{y}) \in \hat{D}_k$ there is a $t > k$ with $(\bar{x}, \bar{y}) \in U_t(\bar{x}, \bar{y})$ and*

$$\hat{\phi}_k(\bar{x}, \bar{y}) = (T \times T)^{jh_k}(\bar{x}, \bar{y}) \in V_t(\bar{x}, \bar{y}).$$

Furthermore suppose

$$\lim \lambda(\hat{D}_k \cup \hat{R}_k) > 0.$$

It follows that λ is $\text{id} \times T^{-1}$ invariant and hence is $\mu \times \mu$.

Proof. If C, D are cylinder sets whose indices lie in $(-(h_k - 1)/2, \dots, (h_k - 1)/2)$ and $N_k = (h_k - 1)/2$, then

$$\frac{1}{N_k} \sum_{i=0}^{h_k-1} \chi_{C \times D}((T \times T)^i(\bar{x}, \bar{y})) = \frac{1}{N_k} \sum_{i=0}^{h_k-1} \chi_{C \times T^{-1}(D)}((T \times T)^i \hat{\phi}_k(\bar{x}, \bar{y})).$$

As $\hat{\phi}_k$ is λ -preserving and 1-1, and $\lim \lambda(\hat{D}_k \cup \hat{R}_k) > 0$, fixing C, D , we can select points $(\bar{x}_k, \bar{y}_k) \in \hat{D}_k$ so that

$$\begin{aligned} \frac{1}{N_k} \sum_{i=0}^{h_k-1} \chi_{C \times D}((T \times T)^i(\bar{x}_k, \bar{y}_k)) &\xrightarrow{k} \lambda(C \times D) \\ \frac{1}{N_k} \sum_{i=0}^{h_k-1} \chi_{C \times T^{-1}(D)}((T \times T)^i \hat{\phi}_k(\bar{x}_k, \bar{y}_k)) &\xrightarrow{k} \lambda(C \times T^{-1}(D)) \end{aligned}$$

by the Birkhoff ergodic theorem. Hence λ is $\text{id} \times T^{-1}$ -invariant on all cylinder sets,

hence all Borel sets. Thus λ is $T^i \times T^j$ -invariant for all i, j . But as T is uniquely ergodic, so is this Z^2 action on $X \times X$ with unique invariant measure $\mu \times \mu$. \square

Under the assumption $\lambda(\overline{\lim} W_k) = 1$ we proceed to construct the $\hat{\phi}_k$. First we define maps ϕ_k as follows.

If $(\bar{x}, \bar{y}) \in U_k(\bar{x}, \bar{y})$, let,

$$\phi_k(\bar{x}, \bar{y}) = (T \times T)^{jh_k}(\bar{x}, \bar{y}),$$

where

$$j = \left\lceil \frac{\min(\# I_k(\bar{x}, \bar{y}), \# J_k(\bar{x}, \bar{y}))}{h_k} \right\rceil + 1,$$

if this puts $\phi_k(\bar{x}, \bar{y})$ in $V_k(\bar{x}, \bar{y})$. (The square brackets here denote the greatest integer function.)

The value j above is determined solely by where the cut between the indices in I and J occurs. Furthermore, if $(\bar{x}, \bar{y}) \in W_k$, then either $(\bar{x}, \bar{y}) \in \text{Dom}(\phi_k) = D_k$ or $(\bar{x}, \bar{y}) \in \text{Range}(\phi_k) = R_k$ or \bar{y} is within $3h_k$ of an occurrence of S_k on its T orbit.

LEMMA 1.15. (i) The maps ϕ_k are λ -preserving and 1-1.

(ii) $D_k \cap R_k = \emptyset$.

(iii) If $(\bar{x}, \bar{y}) \in D_k \cap D_{k'}$, $k' > k$ then $\phi_k(\bar{x}, \bar{y}) \in D_{k'}$.

(iv) If $(\bar{x}, \bar{y}) \in R_k \cap R_{k'}$, $k' > k$, then $\phi_k^{-1}(\bar{x}, \bar{y}) \in R_{k'}$.

(v) $W_k \subset D_k \cup R_k \cup X \times \sum_{i=-3h_k}^{3h_k} T^i(S_k)$

Proof. That ϕ_k is 1-1, λ -preserving and $D_k \cap R_k = \emptyset$ are easily seen.

To obtain (iii), suppose $(\bar{x}, \bar{y}) \in D_k \cap D_{k'}$. We know then that $\phi_k(\bar{x}, \bar{y}) = (\bar{x}', \bar{y}')$ has $\bar{x}', \bar{y}' \in E_k \subset E_{k'-1}$ and hence that $(\bar{x}', \bar{y}') \in U_{k'}(\bar{x}, \bar{y})$.

Now $(\bar{x}', \bar{y}') \in D_{k'}$ as long as $(\bar{x}'', \bar{y}'') = \phi_{k'}(\bar{x}', \bar{y}') \in V_{k'}(\bar{x}, \bar{y})$. We know (\bar{x}'', \bar{y}'') is in the block given by indices J , what we must check is that $\bar{x}'', \bar{y}'' \in E_{k'}$.

We get $\bar{x}'' \in E_{k'}$, as it is still a full $B_{k'}$ from the next occurrences of $S_{k'}$ on the orbit of \bar{x} , lying as it does between the same occurrences of $S_{k'}$ on the orbit as \bar{x} . The k -block containing \bar{y}' is not the last in a string in the k' -block cover of \bar{y} , as $\bar{y}' \in E_{k'-1}$. Hence the same is true of \bar{y}'' and $\bar{y}'' \in E_{k'}$. (iv) is analogous to (iii) and (v) we have already seen. \square

We use the maps ϕ_k to construct $\hat{\phi}_k$ as follows. Select K so large that

$$\lambda \left(\bigcup_{j=k+1}^K (D_j \cup R_j) \right) > \lambda(\overline{\lim} (D_j \cup R_j)) - 1/k.$$

For $(\bar{x}, \bar{y}) \in \bigcup_{j=k+1}^K D_j \cup R_j$, select the largest $k < j \leq K$ with $(\bar{x}, \bar{y}) \in D_j \cup R_j$. If $(\bar{x}, \bar{y}) \in D_j$ then define

$$\hat{\phi}_k(\bar{x}, \bar{y}) = \phi_j(\bar{x}, \bar{y}).$$

Let \hat{D}_k be the domain of definition of $\hat{\phi}_k$, and \hat{R}_k its range.

LEMMA 1.16. $\hat{D}_k \cup \hat{R}_k = \bigcup_{j=k+1}^K D_j \cup R_j$ and $\hat{\phi}_k$ is 1-1, λ -preserving.

Proof. If j is the largest value with $(\bar{x}, \bar{y}) \in D_j \cup R_j$ and $(\bar{x}, \bar{y}) \in R_j$, then j is the largest value with $\phi_j^{-1}(\bar{x}, \bar{y}) \in D_j$ by (iv) of lemma 1.15. Thus

$$\hat{D}_k \cup \hat{R}_k = \bigcup_{j=k+1}^K D_j \cup R_j.$$

If $\hat{\phi}_k(\bar{x}, \bar{y}) = \phi_j(\bar{x}, \bar{y}) = \phi_{j'}(\bar{x}', \bar{y}') = \hat{\phi}_k(\bar{x}', \bar{y}')$ where $j' > j$ then by (iv) of lemma 1.5, $(\bar{x}, \bar{y}) \in D_j$ conflicting with the choice of j . $\hat{\phi}_k$ is λ -preserving as it is equal to ϕ_j on disjoint pieces of \hat{D}_k . □

COROLLARY 1.17 ((a) of theorem 1). *If $\lambda(\overline{\lim} W_k) = 1$, then $\lambda = \mu \times \mu$.*

Proof. $\lambda(\hat{D}_k \cup \hat{R}_k) > \lambda(\overline{\lim} D_j \cup R_j) - 1/k$ so $\lim \lambda(\hat{D}_k \cup \hat{R}_k) \geq \lambda(\overline{\lim} D_j \cup R_j) \geq \lambda(\overline{\lim} W_k) = 1$ and the result follows from lemma 1.14. □

We can now assume $\lambda(\overline{\lim} W_k) = 0$, i.e. for λ -a.e. \bar{x}, \bar{y} , once k is sufficiently large, (\bar{x}, \bar{y}) belongs to the larger of $U_k(\bar{x}, \bar{y})$ and $V_k(\bar{x}, \bar{y})$. Set

$$G_k = \left(\bigcup_{j=k}^{\infty} W_k^c \right) \cap (E_k \times X) \cap (X \times E_k),$$

and we now can assume $\lim_{k \rightarrow \infty} \lambda(G_k) = 1$. Define

$$f_k(\bar{x}, \bar{y}) = n(k, \bar{x}) - n(k, \bar{y}),$$

the shift of the two string $1 B_k^{2k+1}$ containing \bar{x} and \bar{y} relative to one another, when both are in such $(f_k$ though is defined on all $X \times X)$.

The function f_k maps the measure λ to a measure on $(-h_{k+1}, \dots, h_{k+1})$. Our investigation from here out will consist of understanding what these measures $f_k^*(\lambda)$ look like.

LEMMA 1.18. *If $(\bar{x}, \bar{y}), (\bar{x}', \bar{y}') \in G_k$ and $f_{k+1}(\bar{x}, \bar{y}) = f_{k+1}(\bar{x}', \bar{y}')$ then*

$$f_k(\bar{x}, \bar{y}) = f_k(\bar{x}', \bar{y}').$$

Proof. As $f_{k+1}(\bar{x}, \bar{y}) = f_{k+1}(\bar{x}', \bar{y}')$, the blocks $1 B_{k+1}^{2k+2}$ containing them are shifted the same amounts relative to one another. Hence

$$\begin{aligned} \# I_k(\bar{x}, \bar{y}) &= f_{k+1}(\bar{x}, \bar{y}) \bmod (h_{k+1}), \\ \# J_k(\bar{x}, \bar{y}) &= h_{k+1} - \# I_k(\bar{x}, \bar{y}), \end{aligned}$$

and as $(x, y) \in G_k$, we know

$$f_k(\bar{x}, \bar{y}) = \begin{cases} \# I_k(\bar{x}, \bar{y}) & \text{if } (\bar{x}, \bar{y}) \in U_k(\bar{x}, \bar{y}) \\ -\# J_k(\bar{x}, \bar{y}) & \text{if } (\bar{x}, \bar{y}) \in V_k(\bar{x}, \bar{y}). \end{cases}$$

Note: the larger of I, J is the same as the larger of U, V . □

COROLLARY 1.19. *If $(\bar{x}, \bar{y}), (\bar{x}', \bar{y}') \in G_k$ and for some $k' > k$, $f_{k'}(\bar{x}, \bar{y}) = f_{k'}(\bar{x}', \bar{y}')$ then $f_k(\bar{x}, \bar{y}) = f_k(\bar{x}', \bar{y}')$.*

COROLLARY 1.20. *f_k is constant λ -a.e. on G_k .*

Proof. If (\bar{x}, \bar{y}) and (\bar{x}', \bar{y}') are both on the same $T \times T$ orbit and in some G_k , then once k' is sufficiently large,

$$T^{n(k', \bar{x})}(\bar{x}) = T^{n(k', \bar{x}')}(\bar{x}') \quad \text{and} \quad T^{n(k', \bar{y})}(\bar{y}) = T^{n(k', \bar{y}')}(\bar{y}').$$

Thus

$$f_{k'}(\bar{x}, \bar{y}) = f_{k'}(\bar{x}', \bar{y}')$$

and f_k is invariant under the induced map $(T \times T)_{G_k}$ which is ergodic for λ/G_k . □

We will now, w.l.o.g, assume f_k is strictly constant on G_k , $f_k(G_k) = n_k$. Thus we have an increasing sequence of sets G_k with $f_k; G_k \rightarrow n_k$. Recalling the measures $f_k^*(\lambda)$ on $(-h_{k+1}, \dots, h_{k+1})$, this says they become ever more concentrated on a single point. More precisely if we define

$$f; X \times X \rightarrow \prod_{k=1}^{\infty} (-h_{k+1}, \dots, h_{k+1}) = Y$$

by $F(\bar{x}, \bar{y}) = (f_1(\bar{x}, \bar{y}), f_2(\bar{x}, \bar{y}), \dots)$ then λ projects to a measure $f^*(\lambda)$ on Y . For any infinite sequence n_1, n_2, \dots , set

$$S_k(n_1, n_2 \dots) = \{\bar{y} \in Y; y_i = n_i, i \geq k\}.$$

We call a Borel measure λ on Y an asymptotic point mass on $n_1, n_2 \dots$ if

$$\lim_{k \rightarrow \infty} \lambda(S_k(n_1, n_2 \dots)) = 1.$$

COROLLARY 1.21. *If $\lambda(\overline{\text{lim}} W_k) = 0$ then $f^*(\lambda)$ is an asymptotic point mass on the sequence $f_1(G_1), f_2(G_2) \dots$.*

Denote this sequence by $n_k = f_k(G_k)$.

COROLLARY 1.22. $\lim_{k \rightarrow \infty} n_k/h_{k+1} = 0$.

Proof. For $(\bar{x}, \bar{y}) \in G_k$, first suppose $n_k > 0$. Thus $(\bar{x}, \bar{y}) \in U_k(\bar{x}, \bar{y})$ and $f_k(V_k(\bar{x}, \bar{y})) = h_{k+1} - n_k$. Hence

$$\begin{aligned} \frac{f_k^*(\lambda)(n_k^c)}{f_k^*(\lambda)(n_k)} &\geq \frac{\lambda(\{(T \times T)^j(\bar{x}, \bar{y}); (\bar{x}, \bar{y}) \in G_k, i \in J_k(\bar{x}, \bar{y})\})}{\lambda(\{(T \times T)^i(\bar{x}, \bar{y}); (\bar{x}, \bar{y}) \in G_k, i \in I_k(\bar{x}, \bar{y})\})} \\ &= \frac{\# J_k(\bar{x}, \bar{y})}{\# I_k(\bar{x}, \bar{y})} = \frac{f_k(U_k(\bar{x}, \bar{y}))}{f_k(V_k(\bar{x}, \bar{y}))} = \frac{n_k}{h_{k+1} - n_k} \geq 0. \end{aligned}$$

If $n_k < 0$, $(\bar{x}, \bar{y}) \in V_k(\bar{x}, \bar{y})$ and similar reasoning gives

$$\frac{f_k^*(\lambda)(n_k^c)}{f_k^*(\lambda)(n_k)} = \frac{-n_k}{h_{k+1} + n_k} \geq 0 \quad (n_k^c = \text{complement of } \{n_k\}).$$

As $(f_k^*(\lambda)(n_k^c))/(f_k^*(\lambda)(n_k)) \xrightarrow{k} 0$ the result follows. □

Given that $f^*(\lambda)$ is an asymptotic point mass, we now attempt to define a Borel map $S; X \rightarrow X$ on whose graph λ is supported.

For $\bar{x} \in X$ define a cylinder set $N_k(\bar{x})$ whose indices are

$$(\max(n(k, \bar{x}), n(k, \bar{x}) - n_k), \dots, \min(n(k, \bar{x}) + h_{k+1}, n(k, \bar{x}) - n_k + h_{k+1}))$$

and whose name is the 0, 1-name of $T^{-n_k}(\bar{x})$ across these indices.

If it is the case that for some k_0 and all $k \geq k_0$, the indices of $N_{k+1}(\bar{x})$ contain those of $N_k(\bar{x})$ and increase in k to all of \mathbb{Z} , and furthermore

$$N_{k+1}(\bar{x}) \subset N_k(\bar{x}),$$

i.e. the names agree where they overlap, then $\bigcap_k N_k(\bar{x}) = S(\bar{x})$ is a single point in X .

It is an easy fact that the set of $\bar{x} \in X$ where S is defined is a T -invariant Borel set, S is a Borel map and $TS(\bar{x}) = ST(\bar{x})$.

Furthermore we can define a Borel map \hat{S} similarly but by translating by n_k , instead of $-n_k$.

It follows that $\bar{y} = S(\bar{x})$ iff $\bar{x} = \hat{S}(\bar{y})$, and so where defined S is 1-1 and $\hat{S} = S^{-1}$.

LEMMA 1.23. *If for some k_0 , $(\bar{x}, \bar{y}) \in G_{k_0}$ then $\bar{y} = S(\bar{x})$, $\bar{x} = S^{-1}(\bar{y})$.*

Proof. As $(\bar{x}, \bar{y}) \in G_{k_0}$, for all $k \geq k_0$

$$f_k(\bar{x}, \bar{y}) = n(k, \bar{x}) - n(k, \bar{y}) = n_k.$$

Thus the name of $N_k(\bar{x})$ is

$$y_{\max(n(k,x), n(k,y))} \cdots y_{\min(n(k,x)+h_{k+1}, n(k,y)+h_{k+1})}.$$

These blocks in the name of \bar{y} increase to all of \bar{y} . □

COROLLARY 1.24. *S and S^{-1} are defined μ -a.e. and $\lambda(\text{graph}(S)) = 1$.*

LEMMA 1.25. *For any cylinder set C ,*

$$\lim_{k \rightarrow \infty} \mu(S^{-1}(C)\Delta T^{n_k}(C)) = 0.$$

Proof. Suppose the indices of C are in $(-N, N)$. Given $\varepsilon > 0$, once k is large enough,

$$\lambda(\{(\bar{x}, S(\bar{x})) \in G_k; \max(n(k, \bar{x}), n(k, S(\bar{x}))) < -N, \min(n(k, \bar{x}) + h_{k+1}, n(k, S(\bar{x}) + h_{k+1})) > N\}) > 1 - \varepsilon.$$

For such an x , $T^{-n_k}(x) \in C$ iff $S(x) \in C$. Thus

$$\mu(S^{-1}(C)\Delta T^{n_k}(C)) < \varepsilon. \quad \square$$

COROLLARY 1.26. ((b) of theorem 1). *S is μ -preserving and in the weak closure of $\{T^i\}$, and for any Borel set A , $X \times X$*

$$\lambda(A) = \int_x \chi_A(\bar{x}, S(\bar{x})) d\mu.$$

Proof. The first two remarks follow from lemma 1.25. Letting $\lambda = \int_x \lambda_x d\mu(x)$ be the disintegration of λ over its first coordinate, as $\lambda(\text{graph}(S)) = 1$, $\lambda_x(S(\bar{x})) = 1$ for μ -a.e. x . Thus

$$\lambda(A) = \int_x \lambda_x(A)_k d\mu = \int_x \chi_A(\bar{x}, S(\bar{x})) d\mu. \quad \square$$

Notice that for free we obtain that $C(T) = \text{weak closure } \{T^i\}$ as certainly any $S \in C(T)$ gives rise to a λ_S , supported on $\text{graph}(S)$, and as such a λ_S is unique to S , puts $S \in \text{weak closure } \{T^i\}$. This, of course, is King's theorem [K] for our particular example.

2. Primality of T

Veech's theorem, which we mentioned in the introduction, now gives the following corollary to the 2-fold simplicity of T .

COROLLARY 2.1. *If $G \subset B$ is a T -invariant factor algebra, then there is a weakly compact subgroup $H \subseteq (CT)$ so that $A \in G$ iff $\mu(S(A)\Delta A) = 0$ for all $S \in H$.*

Proof. See [V].

Our next task is to show that $C(T)$ has no non-trivial compact subgroups, and hence T is prime.

To begin this discussion, consider the set Λ of $T \times T$ invariant Borel probability measures on $X \times X$. This is a convex, weakly compact subspace of the vector space of all Borel measures, and has as its extreme points the ergodic joinings. The projection f^* to the probability measures on Y is a homeomorphism in the weak topology to its range, (that $f^*(\lambda)$ determines λ is an easy exercise). We can identify those $\lambda \in \Lambda$ which are supported on the graph of some $S \in C(T)$.

LEMMA 2.2. $\lambda \in \Lambda$ is supported on the graph of some $S \in C(T)$ iff $f^*(\lambda)$ is an asymptotic point mass.

Proof. If λ is so supported, the result is just theorem 1 and corollary 1.21. Suppose λ is an asymptotic point mass on $\{n_1, n_2, \dots\}$. Now λ has an ergodic decomposition

$$\lambda = \alpha(\mu \times \mu) + (1 - \alpha) \int_{C(T)} \lambda_S d\tau(S),$$

where λ_S is supported on graph (S) and τ is a Borel probability measure on $C(T)$.

$$f^*(\lambda) = \alpha f^*(\mu \times \mu) + (1 - \alpha) \int_{C(T)} f^*(\lambda_S) d\tau(s).$$

As $f^*(\lambda)$ is an asymptotic point mass on some $\{n_i\}$, $\alpha = 0$ and for τ -a.e. S , $T^{n_i} \rightarrow S$ weakly. Thus λ is a point mass on some $S \in C(T)$ and $\lambda = \lambda_S$. □

To show that $C(T)$ has no compact subgroups all we need do is show that for any $S \neq \text{id}$ in $C(T)$, weak closure $(\{f^*(\lambda_{S^i})\})$ contains measures which are not asymptotic point masses. As this weak closure is compact, all we need is to find a sequence of powers t_i so that $f^*(\lambda_{S^{t_i}})$ cannot have an asymptotic point mass as an accumulation point. This argument uses the same approach seen in lemmas 1.15, 1.16, building maps ϕ_k and then $\hat{\phi}_k$.

LEMMA 2.3. For $S \in C(T)$ and $k > 0$ suppose there is a μ -preserving 1-1 Borel map $\hat{\phi}_k; \hat{D}_k \rightarrow \hat{R}_k$, both Borel subsets of X , so that for any $\bar{x} \in \hat{D}_k$ there is a $t > k$ with $(\bar{x}, S(\bar{x})) \in U_t(\bar{x}, S(\bar{x}))$ and $\hat{\phi}_k(\bar{x}) = T^{jt}(\bar{x})$ has $(\hat{\phi}_k(\bar{x}), S(\hat{\phi}_k(\bar{x}))) \in V_t(\bar{x}, S(\bar{x}))$. It follows that for all $j < k$, if $\bar{x}, S(\bar{x}) \in E_j$ then

$$f_j(\hat{\phi}_k(\bar{x}), S(\hat{\phi}_k(\bar{x}))) = f_j(x, S(\bar{x})) - 1.$$

Proof. If $\bar{x}, S(\bar{x}) \in E_j$ then $n(j, \bar{x}), n(j, S(\bar{x})) > 1$ and

$$n(j, S(\hat{\phi}_k(\bar{x}))) = n(j, \bar{x}),$$

$$n(j, S(\hat{\phi}_k(\bar{x}))) = n(j, S(\bar{x})) + 1. \quad \square$$

COROLLARY 2.4. Suppose that for some sequence $S_k \in C(T)$ there is, for each S_k , a μ -preserving 1-1 Borel map $\hat{\phi}_k; \hat{D}_k \rightarrow \hat{R}_k$ as described in lemma 2.3. Furthermore suppose $\lim \mu(\hat{D}_k) = \alpha > 0$. The S_k then can have no accumulation points in $C(T)$.

Proof. For any j , if $k > j$ and $\bar{x} \in \hat{D}_k$,

$$f_j(\hat{\phi}_k(\bar{x}), S_k(\hat{\phi}_k(\bar{x}))) = f_j(\bar{x}, S_k(\bar{x})) - 1,$$

the largest mass $f_j^*(\lambda_{S_k})$ can give to any single point is

$$1 - \left(\frac{\mu(\hat{D}_k \cup \hat{R}_k)}{2} - \mu(E_j^c) \right).$$

Thus for any accumulation point λ of the λ_{S_k} , the largest mass $f_j^*(\lambda_{S_k})$ can give to any single point is

$$1 - \frac{\mu(\hat{D}_k \cup \hat{R}_k)}{2} - \mu(E_j^c).$$

Thus for any accumulation point λ of the λ_{S_k} , the largest mass $f_j^*(\lambda)$ can give to any point is $1 - ((\alpha/2) - \mu(E_j^c))$. As $\mu(E_j^c) \xrightarrow{j} 0$, λ cannot be an asymptotic point mass. □

Now all that remains is to find a sequence t_k so that $S_k = S^{t_k}$ satisfies this corollary.

Fix $S \neq \text{id}$. Thus $f^*(\lambda_S)$ is an asymptotic point mass on $\{n_1, n_2, \dots\}$, w.l.o.g. $n_i \neq 0$ and $\lim_{i \rightarrow \infty} n_i/h_{i+1} = 0$ by corollary 1.22. Fix the value k and select $k' > k$ so that:

- (a) $1/20 > n_{k'}/h_{k'+1} > n_{k''}/h_{k''+1}$ for all $k'' > k'$;
- (b) $f^*(\lambda_S)(\{\{j_i\} \in Y; j_i = n_i, i \geq k'\}) > 0.99$; and
- (c) $\mu(E_{k'}^c) < 0.01$, and $\mu(S_{k'}) < 0.01/h_k$.

Select the integer t_k so that

$$\frac{1}{4} < \frac{|t_k n_{k'}|}{h_{k'+1}} < \frac{1}{3}.$$

Define

$$0_k = \{ \bar{x}; \bar{x}, S^{t_k}(\bar{x}) \in E_{k'}, f_j(\bar{x}, S(\bar{x})) = n_j, j \geq k' \text{ and} \\ 0 \notin (\max(n(k', \bar{x}), n(k', \bar{x}) + t_k n_{k'}, \dots, \\ \min(n(k', \bar{x}) + h_{k'+1}, n(k', \bar{x}) + h_{k'+1} + t_k n_{k'})) \}.$$

Thus, if $n_{k'} > 0$, then $\bar{x} \in 0_k$ if it is within $t_k n_{k'}$ of the right end of its block $1 B_{k'}^{2^{k'}+1} 1$, and if $n_{k'} < 0$, it is within this amount of the left end. As $|t_k n_{k'}| > h_{k'+1}/4$, $\mu(0_k) > 0.22$.

We will define $\hat{\phi}_k$ so that

$$\hat{D}_k \cup \hat{R}_k \supseteq 0_k.$$

For a.e. $\bar{x} \in 0_k$, once $j > k'$ is large enough, we will have all of $\bar{x}, S(\bar{x}), \dots, S^{t_k}(\bar{x}) \in E_j$ and $f_j(S^i(\bar{x}), S^{i+1}(\bar{x})) = n_j$ for $i = 0, \dots, t_k - 1$. Let $j_0 + 1$ be the smallest such $j \geq k'$. Figure 1 illustrates how the t_k different blocks $1 B_{j_0+1}^{2^{j_0+1}} 1$ look.

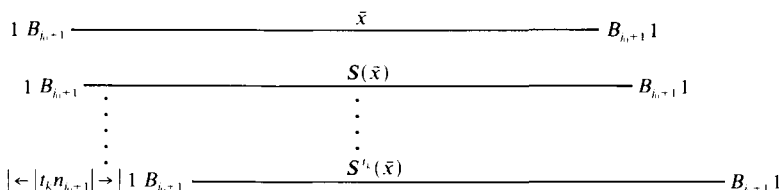


FIGURE 1

As $\bar{x}, \dots, S^{t_k}(\bar{x}) \in E_{j_0+1}$, all have 0, 1-name on indices

$$(n(j_0, S^i(\bar{x})) \cdots n(j_0, S^i(\bar{x}) + h_{j_0+1}))$$

equal to $1 B_{j_0}^{2^{j_0+1}}$. The 0, 1-name for $S^{i+1}(\bar{x})$ has its long periodic mid section in figure 1 shifted relative to that in $S^i(\bar{x})$ by $n_{j_0+1} \bmod (h_{j_0+1})$. As $f_{j_0}(\bar{x}, S(\bar{x})) = n_{j_0}$,

$$n_{j_0+1} \bmod (h_{j_0+1}) = \begin{cases} n_{j_0} & \text{if } n_{j_0} > 0 \\ n_{j_0} + h_{j_0+1} & \text{otherwise.} \end{cases}$$

Thus the ‘1’ indicating the occurrence of S_{j_0} in the 0, 1-name of $S^i(\bar{x})$ on indices $(n(j_0, \bar{x}), \dots, n(j_0, \bar{x}) + h_{j_0+1})$ is successively shifted by n_{j_0} as i increases.

If these 1’s, as i increases, do not cross from one side of \bar{x} to the other, then first, $j_0 > k'$, as this is not true of k' , and second as both $\bar{x}, S^{t_k}(\bar{x}) \in E_{j_0}$, so are all $S^i(\bar{x})$ and $f_{j_0}(S^i(\bar{x}), S^{i+1}(\bar{x})) = n_{j_0}$ for $i = 0, \dots, t_k - 1$, contradicting the minimality of j_0 . Thus

$0 \notin (\max(n(j_0, \bar{x}), n(j_0, \bar{x}) + t_k n_{j_0}), \dots, \min(n(j_0, \bar{x}) + h_{j_0+1}, n(j_0, \bar{x}) + h_{j_0+1} + t_k n_{j_0}))$, and as

$$\frac{|t_k n_{j_0}|}{h_{j_0+1}} < \frac{|t_k n_{k'}|}{j_{k'+1}} < \frac{1}{3}$$

$(\bar{x}, S^{t_k}(\bar{x}))$ belongs to the smaller of $U_{j_0}(\bar{x}, S^{t_k}(\bar{x}))$, $V_{j_0}(\bar{x}, S^{t_k}(\bar{x}))$, (both $\bar{x}, S^{t_k}(\bar{x})$ are in E_{j_0}).

As defined, $U_{j_0}(\bar{x}, S^{t_k}(\bar{x}))$ and $V_{j_0}(\bar{x}, S^{t_k}(\bar{x}))$ are subsets of $X \times X$, but as they sit on graph (S) , their projection onto the first coordinate is 1-1, so just as preceding lemma 1.15, we can define a map ϕ_{j_0} taking as much of $U_{j_0}(\bar{x}, S^{t_k}(\bar{x}))$ to $V_{j_0}(\bar{x}, S^{t_k}(\bar{x}))$ as possible by a shift by a multiple of h_{j_0} . These maps satisfy all of lemma 1.15, except (v) is replaced by

$$0_k \subset \bigcup_{j_0=k'}^{\infty} \left(D_{j_0} \cup R_{j_0} \cup \bigcup_{i=-3h_{j_0}}^{3h_{j_0}} T^i(S^{-t_k}(S_{j_0})) \right).$$

Thus

$$\mu \left(\bigcup_{j_0=k'}^{\infty} D_{j_0} \cup R_{j_0} \right) > 0.15.$$

Construct $\hat{\phi}_k$ exactly as preceding lemma 1.16 except truncate so that

$$\mu \left(\bigcup_{j_0=k'}^{\kappa} D_{j_0} \cup R_{j_0} \right) > 0.1.$$

The conclusions of lemma 1.16 still hold for $\hat{\phi}_k$, i.e. $\hat{\phi}_k$ is 1-1, μ -preserving and $\lim \mu(D_k) \geq 0.1$.

LEMMA 2.5. *Given any $S \neq \text{id}$ in $C(T)$ there exists a sequence t_k so that S^{t_k} can have no weak accumulation points in $C(T)$.*

Proof. We have demonstrated the hypotheses of corollary 2.4. □

THEOREM 2. *T is prime.*

COROLLARY 2.6. *T is weakly mixing.*

Proof. It is, of course, a joke to use this much machinery to prove so obvious a fact. We have not assumed it, of course, never actually claiming $\mu \times \mu$ was an ergodic measure for $T \times T$.

The proof is, of course, that any non-weakly mixing map has non-trivial factor algebras, unless it is a finite rotation of prime order which T is not. \square

3. Higher order simplicity and some consequences

Our next step is to show that T is simple of all orders. Knowing T is 2-fold simple, all we need show is that any m -fold ergodic joining λ on X^m that is pairwise independent on coordinate algebras must be μ^m (in the vocabulary of [JR], T is pairwise independently determined). If we can show that any such joining is $\text{id}^{m-1} \times T^{-1}$ invariant we will be done, as permuting coordinates implies any such λ is $T^j \times T^{j_2} \times \dots \times T^{j_m}$ invariant. This \mathbb{Z}^m action on X^m is uniquely ergodic as T is. The proof that such a pairwise independent λ is $\text{id}^{m-1} \times T^{-1}$ -invariant again follows the format of lemma 1.14, by constructing maps $\hat{\phi}_k$.

Suppose λ is an ergodic m -fold joining on X^m . For $w = (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^m) \in X^m$, with all $\bar{x}^i \in E_k$, the interval of indices $n(k, \bar{x}^1), \dots, n(k, \bar{x}^1) + h_{k+1} - 1$ contains indices $l_2, l_3, \dots, l_{m-1}, l_m$ with $T^{l_i}(\bar{x}^i) \in S_k$.

Letting $l_0 = n(k, \bar{x}^1)$ and $l_i = n(k, \bar{x}^1) + h_{k+1}$, the index l_m will lie between two nearest such

$$l_a \leq l_m < l_b, \quad a, b \in \{0, \dots, m-1\}.$$

LEMMA 3.1.

$$\lambda \left(\left\{ w; \bar{x}^i \in E_k, l_a \leq 0 < l_b, l_m - l_a > \frac{h_{k+1}}{10m}, l_b - l_m > \frac{h_{k+1}}{10m} \right\} \right) > \left(\frac{4}{100m} - \frac{1}{2^{k-1}} \right) \mu(E_k) - (m-1)\mu(E_k^c).$$

Proof. If all $\bar{x}^i \in E_k$, \bar{x}^m is within a fraction $1/10m$ of the end of its block $1 B_k^{2^{k+1}}$, but for all $i \neq m$, \bar{x}^i is further than a fraction $2/10m$ from the ends of its block $1 B_k^{2^{k+1}}$, then w is in the set in question.

By pairwise independence, for $i \neq m$, $\lambda(\{w; \bar{x}^m, \bar{x}^i \in E_k, \bar{x}^m$ is within a fraction $1/10m$ of the end of its block and \bar{x}^i is within a fraction $2/10m$ of the end of its block}) is less than or equal to:

$$\frac{\left(\frac{2h_{k+1}}{10m} - 3h_k - 1 \right)}{h_{k+1}} \frac{\left(\frac{4h_{k+1}}{10m} - 3h_k - 1 \right)}{h_{k+1}} \mu(E_k)^2 \leq \left(\frac{4}{10m} - \frac{1}{2^{k-1}} \right)^2 \mu(E_k)^2.$$

Thus λ (the set in question) is greater than or equal to:

$$\begin{aligned} & \left(\frac{2}{10m} - \frac{1}{2^{k-1}} \right) \mu(E_k) - (m-1)(\mu(E_k^c)) - (m-1) \left(\frac{4}{10m} - \frac{1}{2^{k-1}} \right)^2 \mu(E_k)^2 \\ & \geq \left(\frac{4}{100m} - \frac{1}{2^{k-1}} \right) \mu(E_k) - (m-1)\mu(E_k^c). \end{aligned} \quad \square$$

Setting $0_k = \{w; \bar{x}^i \in E_k, l_a \leq 0 < l_b, l_m - l_a > h_{k+1}/10m, l_b - l_m > h_{k+1}/10m\}$, we conclude $\lim \lambda(0_k) \geq 0.04/m$.

Letting $T \times T \times \dots \times T = \hat{T}$, for $w \in 0_k$, the index l_m splits the indices $l_a, l_a + 1, \dots, l_b - 1$ into two sets

$$I_k(w) = \{i; l_a \leq i < l_m\}$$

$$J_k(w) = \{i; l_m \leq i < l_b\}.$$

Except for those $i \in I_k(w) \cup J_k(w)$ within $(3h_k + 1)/2$ of l_a, l_m or l_b , all $\hat{T}^i(w)$ are also in 0_k .

Let $U_k(w) = \{T^i(w) \in 0_k; i \in I_k(w)\}$ and $V_k(w) = \{T^i(w) \in 0_k; i \in J_k(w)\}$. Let $W_k = \{w \in 0_k; w \text{ is in the smaller of } U_k(w), V_k(w)\}$.

LEMMA 3.2. $\overline{\lim} W_k$ is λ -a.s. \hat{T} -invariant and as

$$\lambda(W_k) \geq \lambda(0_k) \cdot (((h_{k+1}/10m) - 3h_k - 1)/h_{k+1}),$$

$\overline{\lim} W_k = 1$.

Proof. The proof that W_k is λ -a.s. \hat{T} -invariant is analogous to lemma 1.13, only now we have m coordinates instead of just two. As $\min(\# U_k(w), \# V_k(w)) \geq (h_{k+1}/10m) - 3h_k - 1$, and $\# I_k(w) + \# J_k(w) = h_{k+1}$,

$$\lambda(W_k) \geq \lambda(0_k) \left(\frac{h_{k+1}}{10m} - 3h_k - 1 \right) / h_{k+1}$$

$$> \lambda(0_k) \left(\frac{1}{10m} - \frac{1}{2^{k-1}} \right).$$

Thus $\overline{\lim} \lambda(W_k) > (1/10m) \lim \lambda(0_k) > 0$. □

As in our previous two constructions, we now define maps $\phi_k; D_k \rightarrow R_k$ mapping as much of $U_k(w)$ to $V_k(w)$ as possible by a shift by a multiple of h_k . The conclusions of lemma 1.15 will again hold, except that (v) is replaced by

$$W_k \subset D_k \cup R_k \cup \{w; \text{some } \bar{x}^i, \text{ is within } 3h_k \text{ of } S_k \text{ on its } T \text{ orbit}\}.$$

This is still enough to imply

$$\lambda(\overline{\lim} D_k \cup R_k) = 1.$$

As before lemma 1.16, we construct $\hat{\phi}_k; \hat{D}_k \rightarrow \hat{R}_k$ from the ϕ_k , and the conclusions of lemma 1.16 still hold. An m -fold version of lemma 1.14 now leads to the following conclusion.

LEMMA 3.3. *If λ is an ergodic m -fold joining of T with itself, whose coordinate algebras are pairwise independent, then λ is $\text{id}^{m-1} \times T^{-1}$ -invariant.*

THEOREM 3. *T is simple of all orders.*

Applying theorem 4.1 of [JR] we now know that any joining of T to an arbitrary S arises as the relatively independent joining of the k -fold product of T and S over the factor algebra $T^{k \odot}$ of symmetric sets, which S must have as a factor. Furthermore, by corollary 4.6 of [JR] if we were to construct two ‘different’ versions of T , both simple and prime, then they must either be disjoint or isomorphic. If by ‘different’ all we meant was that the position of the single spacer among the 2^{k+1} k -blocks in a $(k+1)$ -block varied from one construction to the other, then in fact the maps would be isomorphic. We leave it to the reader’s cleverness to see why. All our arguments really only depended on h_{k+1}/h_k growing fast enough. We leave open

the question whether choosing sufficiently different numbers of k -blocks in a $(k + 1)$ -block would lead to non-isomorphic maps.

We conclude this section by showing how the results at hand can be used to show that for $|m| > |n| > 0$, T^m and T^n are disjoint. By theorem 6.1 of [JR] both T^m and T^n are simple and $C(T^m) = C(T^n) = C(T)$. By corollary 4.6 of [JR], then T^m and T^n are either disjoint or isomorphic. We rule out the possibility of isomorphism by showing that T^n has no m th root. Indeed if $S^m = T^n$ then $S \in C(T^n) = C(T)$. Then, in the notation of § 2, $f^*(\lambda_s)$ is an asymptotic point mass on a sequence $\{k_i\}$. The fact that $k_i/h_{i+1} \rightarrow 0$ makes it clear that $f^*(\lambda_{S^m})$ is an asymptotic point mass on $\{mk_i\}$. But $f^*(\lambda_{T^n})$ is evidently an asymptotic point mass on the constant sequence $\{n\}$ so we have $n = mk_i$ for large i which is impossible since $|m| > |n|$. This establishes the disjointness of T^m and T^n .

Now T and T^{-1} are evidently isomorphic via an involution ϕ . If ψ is any other isomorphism then $\psi^{-1}\phi \in C(T)$ so $\psi = \phi S$ for some $S \in C(T)$. Note that $\phi S \phi^{-1} = S^{-1}$ since S is a weak limit of powers of T , so $\phi S \phi S = \text{id}$, that is ψ is also an involution. We now have the following description of invariant measures for $\bigotimes_{i \in k} T^{n_i}$ on X^k , $n_i \neq 0$, k a finite set.

COROLLARY 3.4. *Suppose $n_1, \dots, n_k \neq 0$ and λ is a $\bigotimes_{i \in k} T^{n_i}$ -invariant probability measure on X^k . Then λ is a product of measures of the form $\bigotimes_{i \in k} S_i \mu_\Delta$ on $X^{k'}$, $k' < k$, where μ_Δ is diagonal measure on X^k and each S_i is either in $C(T)$ or of the form $\phi S'_i$, $S'_i \in C(T)$. Moreover n_i is constant on k' and $S_i \in C(T)$ either precisely for those i for which $n_i > 0$ or precisely for those i for which $n_i < 0$.*

Proof. This follows from the fact that all T^m , $m \neq 0$, are simple and prime, together with proposition 5.3 of [JR] and the above remarks. (See also the proof of proposition 6.7 of [JR] and the remarks following that proof.) □

4. Topological behaviour

In this section we will show that (X, T) is POD in the sense of [FKS]. In particular the set of continuous maps commuting with T is just $\{T^i\}$, so although there are non-trivial weakly convergent sequences of powers of T there are no non-trivial uniformly convergent sequences. Another consequence of the POD property is that (X, T) is topologically prime.

We start by distinguishing two special points in X , \bar{x}^0 whose 0, 1-name on indices $-h_k - 1$ to h_k is $B_k B_k$, and \bar{x}^1 whose name on indices $-h_k - 1$ to $h_k + 1$ is $B_k 1 B_k$. Notice these two points have identical names on negative indices, but have names differing by a shift by 1 on positive indices.

LEMMA 4.1. *If \bar{x} and \bar{y} do not lie one in the orbit of \bar{x}^0 and the other in the orbit of \bar{x}^1 and $n(k, \bar{x}) = n(k, \bar{y})$ for all k then $x_i = y_i$ for all i , i.e. $\bar{x} = \bar{y}$.*

Proof. Now $n(k, \bar{x}) = n(k, \bar{y})$ for all k , implies $x_i = y_i$ if

$$n(k, \bar{x}) \leq i \leq n(k, \bar{x}) + \frac{h_{k+1} - 1}{2}.$$

If $n(k, \bar{x}) \xrightarrow{k \rightarrow -\infty} -\infty$ and $n(k, \bar{x}) + (h_{k+1} - 1)/2 \xrightarrow{k \rightarrow \infty} \infty$ we are done. As $n(k, \bar{x})$ is non-

increasing and $n(k, \bar{x}) + (h_{k+1} - 1)/2$ is non-decreasing,

(a) if $\lim_{k \rightarrow \infty} n(k, \bar{x}) = n > -\infty$, then $T^n(\bar{x}) = T^n(\bar{y}) = \bar{x}^1$; and

(b) if $\lim_{k \rightarrow \infty} n(k, \bar{x}) + (h_{k+1} - 1)/2 = n < \infty$, then $T^{n+1}(\bar{x}) = \bar{x}^1$ or \bar{x}^0 and $T^{n+1}(\bar{y}) = \bar{x}^1$ or \bar{x}^0 . As both must be on the same orbit of \bar{x}^1 and \bar{x}^0 , $\bar{x} = \bar{y}$. \square

For $\bar{x} \in X$, let $j_i(k, \bar{x})$ be the values $j < 0$ in decreasing order in i with $T^j(\bar{x}) \in S_k$.

LEMMA 4.2. *If \bar{x} and \bar{y} do not one lie on the orbit of \bar{x}^0 and the other \bar{x}^1 , and for some $k > 0$, $j_0 \in \mathbb{Z}$, $j_i(k, \bar{x}) - j_i(k, \bar{y}) = j_0$ for all i , then $\bar{y} = T^{-j_0}(\bar{x})$.*

Proof. The values $j_i(k, \bar{x})$ determine the values $n(k', \bar{x})$ for $k' \geq k$, and hence $k' < k$. For $k' \geq k$, $n(k', \bar{y}) = n(k', \bar{x}) - j_0$ so

$$n(k', T^{j_0}(\bar{y})) = n(k', \bar{x}).$$

But then $n(k', T^{j_0}(\bar{y})) - n(k', \bar{x})$ for all k' and $T^{j_0}(\bar{y}) = \bar{x}$. \square

Fix a pair of points $(\bar{x}, \bar{y}) \in X \times X$, on distinct orbits, at most one of which is on the orbit of \bar{x}^0 or \bar{x}^1 . If it is the case that one is on the orbit of \bar{x}^0 or \bar{x}^1 , be sure it is a right shift of \bar{x}^0 or \bar{x}^1 by multiplying (\bar{x}, \bar{y}) by a sufficiently high negative power of $T \times T$, and using this pair instead of (\bar{x}, \bar{y}) .

We define a collection of pairs of cylinder sets $I(\bar{x}, \bar{y})$ where $\{C, D\} \in I(\bar{x}, \bar{y})$ if for some $j \leq 0$, $(T^j(\bar{x}), T^j(\bar{y})) \in C \times D$ and the indices of both C and D are contained in $(-\infty, -j)$, i.e. the name of $C \times D$ appears in the negative indices of (\bar{x}, \bar{y}) .

From the restriction placed on (\bar{x}, \bar{y}) , any finite set of negative indices lies completely inside the indices of k -blocks containing 0 in the k -block covers of \bar{x} and \bar{y} , if k is large enough.

Thus for $\{C, D\} \in I(\bar{x}, \bar{y})$, once k is large enough, the indices of $T^{-j}(C)$ and $T^{-j}(D)$ lie inside the intersection of the indices of the k -blocks containing 0 of \bar{x} and \bar{y} (see figure 2).

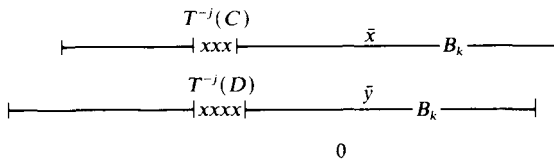


FIGURE 2

In these two k -block covers, move one block at a time to the left. If as we so proceed, every time the \bar{x} name crosses a 1 in between two B_k 's, the \bar{y} name also does, then $j_i(k, \bar{x}) = j_i(k, \bar{y}) + j_0$ for all i , and by lemma 3.2, $x = T^{j_0}(\bar{y})$, contrary to choice.

We conclude that finally one of the two names crosses an occurrence of S_k and the other does not. Focus on the first such occurrence (see figure 3).

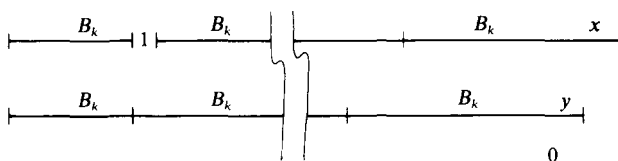


FIGURE 3

In each k -block in the name of \bar{x} we see the name of C , and in \bar{y} , that of D . In these next two k -blocks, the relative positions though have shifted by one place. Thus one of the two pairs $\{C, T(D)\}$ or $\{C, T^{-1}(D)\}$ is in $I(\bar{x}, \bar{y})$.

LEMMA 4.3. *If $\{C, D\} \in I(\bar{x}, \bar{y})$ then one of $\{C, T(D)\}$ or $\{C, T^{-1}(D)\} \in I(\bar{x}, \bar{y})$.*

LEMMA 4.4. *If $J \subset \mathbb{Z}$ is finite and C, D are such that for all $i \in J, \{C, T^i(D)\} \in I(\bar{x}, \bar{y})$ then either*

- (a) *for all $i \in J, \{C, T^{i-1}(D)\} \in I(\bar{x}, \bar{y})$; or*
- (b) *for all $i \in J, \{C, T^{i+1}(D)\} \in I(\bar{x}, \bar{y})$.*

Proof. For each $i \in J$ there is a $j(i) \leq 0$ so that

$$(x, y) \in \bigcap_{i \in J} T^{-j(i)}(C) \times T^{-j(i)+i}(D).$$

Select $N < 0$ so that all the indices of all the $T^{-j(i)}(C)$ and $T^{-j(i)+i}(D)$ are contained in $(N, N+1, \dots, 0)$. Let C' be the cylinder set with name $x_n x_{n+1} \dots x_0$ on indices n to 0 and similarly D' have name $y_n \dots y_0$ on indices n to 0 . Now $\{C', D'\} \in I(\bar{x}, \bar{y})$ and $C' \times D' \subset \bigcap_{i \in J} T^{-j(i)}(C) \times T^{-j(i)+i}(D)$. By lemma 4.3 either:

- (a) $\{C', T^{-1}(D')\} \in I(\bar{x}, \bar{y})$, and so for some $j < 0$

$$(\bar{x}, \bar{y}) \in T^{-j}(C') \times T^{-j-1}(D') \subset \bigcap_{i \in J} T^{-j(i)-j}(C) \times T^{-j(i)-j+i-1}(D)$$

and for $i \in J, \{C, T^{i-1}(D)\} \in I(\bar{x}, \bar{y})$, or:

- (b) $\{C', T(D')\} \in I(\bar{x}, \bar{y})$ in which case for all $i \in J,$
 $\{C, T^{i+1}(D)\} \in I(\bar{x}, \bar{y}).$ □

COROLLARY 4.5. *For any $\{C, D\} \in I(\bar{x}, \bar{y})$, either*

- (a) $\mathbb{Z}^+ \subset \{i; \{C, T^i(D)\} \in I(\bar{x}, \bar{y})\}$; or
- (b) $\mathbb{Z}^- \subset \{i; \{C, T^i(D)\} \in I(\bar{x}, \bar{y})\}$

(or perhaps both).

Proof. By lemma 4.4, $\{i; \{C, T^i(D)\} \in I(\bar{x}, \bar{y})\}$ cannot exclude both a positive and negative value unless it is \emptyset . It cannot be empty though as T is minimal. □

THEOREM 4. *If $\bar{x}, \bar{y} \in X$ are on distinct orbits and at most one is on the orbit of \bar{x}^1 or \bar{x}^0 the $T \times T$ orbit closure of (\bar{x}, \bar{y}) is all of $X \times X$.*

Proof. Translate (\bar{x}, \bar{y}) if necessary so that $I(\bar{x}, \bar{y})$ is defined. Fix cylinder sets U, V . As T is uniquely ergodic, on any orbit there is a maximum distance N between occurrences of U or V . Assume N is larger than the number of indices of either U or V .

Let

$$C = x_{-5n} x_{-5n+1} \dots x_0$$

and

$$D = y_{-5n} y_{-5n+1} \dots y_0,$$

so $\{C, D\} \in I(\bar{x}, \bar{y})$.

There must be values

$$-4n < j(1) \leq -3n < j(2) \leq -2n < j(3) < -n$$

with $T^{-j(1)}(\bar{x}) \in U$, $T^{-j(2)}(\bar{y}) \in V$, $T^{-j(3)}(\bar{x}) \in U$. Thus

$$C \times D \subset (T^{j(1)}(U) \times T^{j(2)}(V)) \cap (T^{j(3)}(U) \times T^{j(2)}(V)).$$

Either $\{C, T^{j(1)-j(2)}(D)\}$ or $\{C, T^{j(3)-j(2)}(D)\} \in I(\bar{x}, \bar{y})$ by corollary 4.5. In either case $\{U, V\} \in I(\bar{x}, \bar{y})$ and the closure of just the past orbit of (\bar{x}, \bar{y}) is $X \times X$.

COROLLARY 4.6. *(X, T) is POD in the sense of [FKS]. In particular (X, T) is graphic in the sense of [AM].*

Proof. We already know that (X, T) is minimal and measure theoretically, hence also topologically, weakly mixing. Thus we only have that for any $\bar{x}, \bar{y} \in X$ on different orbits the orbit closure of (x, y) contains the graph of T^m for some $m \neq 0$. If at most one of \bar{x} or \bar{y} is \bar{x}^0 or \bar{x}^1 , theorem 4 gives much more. If one is \bar{x}^0 and the other is \bar{x}^1 then the orbit closure contains the graphs of T^m and T^{m+1} for some m .

More remains to be done concerning (X, T) . For example, can the results of [JK] on Chacón’s map be proved here and can (X, T) be shown to have topological minimal self-joinings in the stronger sense of [J]. □

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