Power Coordinates and the Bicircular Quartic.

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I. *Introduction.*

* Casey, in an extensive and complete mémoire, has discussed very thoroughly the bicircular quartic.   Among the unfinished notes of †Dr Clifford the properties of the curve are established by means of a tetracyclic system of coordinates.   Such a system of coordinates has been extensively used by ‡Coolidge.   The object of this paper is to present a valid discussion of the curve in terms of a tricyclic system of "power" coordinates.

II. *The Circle Orthogonal to Three Given Circles.*

Given three fixed non-coaxal circles, whose powers with regard to a point are S_1, S_2, S_3, where $S_k = x^2 + y^2 + 2g_k x + 2f_k y + c_k$, the point is uniquely determinate when the ratios $(S_1 : S_2 : S_3)$ are given. These are the *power coordinates of the point.*

The equation of any circle of the doubly-infinite system orthogonal to the radical circle C of S_1, S_2, S_3 is $lS_1 + mS_2 + nS_3 = 0$.   The *tangential coordinates* of this circle are $(l : m : n)$.   We shall find the equation of the circle C.

The point-circles of the system, since they are orthogonal to C, lie on C.   Hence C is the locus or envelope of the point circles of the system, and all point-circles of the system must satisfy the equation

$$(lg_1 + mg_2 + ng_3)^2 + (lf_1 + mf_2 + nf_3)^2 = (lc_1 + mc_2 + nc_3)(l + m + n)$$

or $\qquad \Sigma l^2(g_1^2 + f_1^2 - c_1) + \Sigma mn(2g_2 g_3 + 2f_2 f_3 - c_2 - c_3) = 0.$

Analytically, r_1, the radius of S_1, is defined by $r_1^2 = g_1^2 + f_1^2 - c_1$, and ψ_{23}, the angle at which S_2, S_3 cut, by

$$2r_2 r_3 \cos \psi_{23} = 2g_2 g_3 + 2f_2 f_3 - c_2 - c_3.$$

* CASEY—*Trans. Royal Irish Academy.*   Dublin, 1867.

† W. K. CLIFFORD—*Mathematical Papers.*

‡ COOLIDGE—*The Circle and the Sphere.*   Oxford, 1918.

Hence the tangential equation of C is

$$\Sigma l^2 r_1^2 - 2\Sigma mnr_2 r_3 \cos \psi_{23} = 0 \quad \dots\dots\dots\dots\dots(1)$$

Reciprocally, the equation of C in power coordinates is

$$\Sigma \frac{S_1^2}{r_1^2} \sin^2 \psi_{23} + 2\Sigma \frac{S_2 S_3}{r_2 r_3} (\cos \psi_{31} \cos \psi_{12} + \cos \psi_{23}) = 0 \dots\dots(2)$$

Since S_1, S_2, S_3 in this equation represent the powers of any point on C with regard to the three circles, we have obtained a generalisation of Ptolemy's Theorem as will be verified later.

The discussion is analytical, and holds whether the circles considered are real or not. Three types of circles will arise, as follows:—

Coefficients of S real, centre real, radius real, circle referred to as *real*.

Coefficients of S real, centre real, radius imaginary, circle referred to as *imaginary*.

Coefficients of S complex, circle referred to as *complex*.

If S_1, S_2, S_3 form an orthogonal system, equations (1) and (2) simplify. S_1, S_2, S_3, S_4 $(C \equiv S_4)$ then form a system of four mutually orthogonal circles, and each one is the locus of the point-circles formed from the other three.

The equation of S_4 in power coordinates is then

$$S_1^2/r_1^2 + S_2^2/r_2^2 + S_3^2/r_3^2 = 0.$$

Every point on S_4 satisfies $S_4 = 0$ and $S_1^2/r_1^2 + S_2^2/r_2^2 + S_3^2/r_3^2 = 0$, and by symmetry similar results hold when S_1, S_2, S_3, S_4 are permuted cyclically.

Hence we have the identity

$$S_1^2/r_1^2 + S_2^2/r_2^2 + S_3^2/r_3^2 + S_4^2/r_4^2 = 0 \quad \dots\dots\dots\dots\dots(3)$$

The tangential equation of S_4 is $r_1^2 l^2 + r_2^2 m^2 + r_3^2 n^2 = 0$.

Cor.—If the circles are point-circles,

$$g_k^2 + f_k^2 - c_k = 0 \quad (K = 1, 2, 3).$$

The tangential equation of C becomes $d_{23}^2 mn + d_{31}^2 nl + d_{12}^2 lm = 0$, where $d_{hk}^2 = (g_h - g_k)^2 + f_h - f_k)^2$. Call these three base point-circles C_1, C_2, C_3. Then the circle C, passing through C_1, C_2, C_3 is $\Sigma d_{23}^4 C_1^2 - 2\Sigma d_{31}^2 d_{12}^2 C_2 C_3 = 0$, or

$$d_2 \sqrt{C_1} \pm d_{31} \sqrt{C_2} \pm d_{12} \sqrt{C_3} = 0, \quad \dots\dots\dots\dots\dots(4)$$

which is Ptolemy's Theorem.

III. *The Nature of the Orthogonal System of Base Circles.*

For real points, from (3), one at least of the four circles is non-real. Two of these circles cannot be *imaginary*. For if S_1, S_2 are so, since they are also orthogonal $(g_1 - g_2)^2 + (f_1 - f_2)^2 = r_1^2 + r_2^2$, where g_s, f_s are real, and $r_1^2 r_2^2$ both negative. This is clearly impossible.

Defining $\Sigma_k^2 \equiv S_k^2/r_k^2$, equation (3) becomes

$$\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 + \Sigma_4^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

For real points it follows from (5), that if one circle be complex and the other two real, the fourth must be conjugate complex.

It is easy to prove that the four circles cannot be two pairs of conjugate complexes. We shall show later that for a bicircular quartic with real coefficients, the complex orthogonal circles occur in pairs.

IV. *The Bicircular Quartic.*

The equation $(a, b, c, f, g, h)(S_1, S_2, S_3)^2 = 0$ represents a bicircular quartic. A comparison of coefficients shows that any bicircular quartic can be thrown into this form.

The foci are point-circles having double contact with the curve. Hence if $lS_1 + mS_2 + nS_3 = 0$ is a focus,

$$\left.\begin{aligned}&Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \text{ and}\\&r_1^2 l^2 + r_2^2 m^2 + r_3^2 n^2 - 2r_2 r_3 \cos \psi_{23} mn - 2r_3 r_1 \cos \psi_{31} nl - 2r_1 r_2 \cos \psi_{12} lm = 0\end{aligned}\right\}\ldots(6)$$

The two equations determine four sets of mutual ratios $l : m : n$, and therefore four foci lying on the circle orthogonal to S_1, S_2, S_3.

The equations in power-coordinates corresponding to (6) are

$$\left.\begin{aligned}&aS_1^2 + bS_2^2 + cS_3^2 + 2fS_2S_3 + 2gS_3S_1 + 2hS_1S_2 = 0 \text{ and}\\&\frac{\sin^2\psi_2}{r_1^2}S_1^2 + \ldots + 2\frac{(\cos\psi_{31}\cos\psi_{12} + \cos\psi_{23})}{r_2 r_3}S_2S_3 + \ldots = 0\end{aligned}\right\}\ldots\ldots\ldots(7)$$

There is a unique linear transformation

$$S_k = p_k S_1' + q_k S_2' + r_k S_3', \ (k = 1, 2, 3)$$

for which equations (7) take the canonical form

$$\left.\begin{aligned}&a'S_1'^2 + b'S_2'^2 + c'S_3'^2 = 0\\&S_1'^2/r_1'^2 + S_2'^2/r_2'^2 + S_3'^2/r_3'^2 = 0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(8)$$

S_1', S_2', S_3' belong to the same system orthogonal to the radical

circle of S_1, S_2, S_3. Further, in the transformed equations,

$$\cos \psi_{31}' \cos \psi_{12}' + \cos \psi_{23}' = 0, \text{ etc.}$$

and it is easy to show that the only solution of these equations for which corresponding forms (8) exist is given by $\cos \psi'_{hk} = 0$, $\sin \psi'_{hk} = 1$. Hence S_1', S_2', S_3' are orthogonal, and the four foci given by the transformed equations (6) lie on the radical circle S_4'.

Since S_1', S_2', S_3', S_4' form a mutually orthogonal system,

$$S_1'^2/r_1'^2 + S_2'^2/r_2'^2 + S_3'^2/r_3'^2 + S_4'^2/r_4'^2 \equiv 0,$$

and the substitution in (8) from this equation yields symmetrical results for each of S_1', S_2', S_3', S_4'. Hence the sixteen foci lie in fours on the four circles.

V. *The Uniqueness of Solution.*

Suppose the quartic to be reduced to the canonical form in two systems $a S_1^2 + b S_2^2 + c S_3^2 = 0$, $a'S_1'^2 + b'S_2'^2 + c'S_3'^2 = 0$,
where $\quad\quad S_1$, S_2, S_3, S_4; S_1', S_2', S_3', S_4'
form two mutually orthogonal sets of circles, none of which coincide. If this is so, the foci can be distributed in four concyclic sets of four in two ways.

Any circle is $l S_1 + m S_2 + n S_3 + p S_4 = 0$. Hence we can transform the second system in terms of S_1, S_2, S_3, S_4, viz.,
$l_k S_1 + m_k S_2 + n_k S_3 + p_k S_4 = (l_k + m_k + n_k + p_k)S_k', (k=1,2,3)$, giving
$\Sigma \{ a'[(l_1 S_1 + m_1 S_2 + n_1 S_3 + p_1 S_4)/(l_1 + m_1 + n_1 + p_1)]^2 \} = \Sigma a S_1^2$, etc...

From these we obtain the sets of equations

$$\Sigma[a' l_1^2/(l_1 + m_1 + n_1 + p_1)^2] = a, \ (l, m, n) \text{ and}$$
$$\Sigma[a' p_1^2/(l_1 + m_1 + n_1 + p_1)^2] = 0 ;$$
$$\Sigma[a' m_1 n_1/(l_1 + m_1 + n_1 + p_1)^2] = 0, \ (l, m, n) \text{ and}$$
$$\Sigma[a' l_1 p_1/(l_1 + m_1 + n_1 + p_1)^2] = 0, \ (l, m, n).$$

There are four linear homogeneous equations in

$a' p_1/(l_1 + m_1 + n_1 + p_1)^2$ etc....

Hence either $p_1 = p_2 = p_3 = 0$ or $\begin{Vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ p_1 & p_2 & p_3 \end{Vmatrix} = 0.$

This last condition is easily seen to be impossible. Therefore $p_1 = p_2 = p_3 = 0$, and S_1', S_2', S_3', are circles of the system $l S_1 + m S_2 + n S_3 = 0$, orthogonal to S_4. Hence $S_4 \equiv S_4'$.

This produces the identities, true for all points on the curve

$$a\,S_1^{\,2} + b\,S_2^{\,2} + c\,S_3^{\,2} \equiv a'\,S_1'^{\,2} + b'\,S_2'^{\,2} + c\,S_3'^{\,2}$$

and $S_1^{\,2}/r_1^{\,2} + S_2^{\,2}/r_2^{\,2} + S_3^{\,2}/r_3^{\,2} \equiv -S_4^{\,2}/r_4^{\,2} \equiv S_1'^{\,2}/r_1'^{\,2} + S_2'^{\,2}/r_2'^{\,2} + S_3'^{\,2}/r_3'^{\,2}$.
It is easily shown that these can only hold if

$$S_1 \equiv S_1', \; S_2 \equiv S_2', \; S_3 \equiv S_3'.$$

Hence the solution is unique.

VI. *The Foci.*

Putting $S'_k{}^2/r'_k{}^2 \equiv \Sigma_k{}^2$ as in (5) the curve is $\Sigma_1^{\,2}/\alpha + \Sigma_2^{\,2}/\beta + \Sigma_3^{\,2}/\gamma = 0$,
where $\Sigma_1^{\,2} + \Sigma_2^{\,2} + \Sigma_3^{\,2} + \Sigma_4^{\,2} \equiv 0$ and equations (8) become

$$\left. \begin{array}{l} \Sigma_1^{\,2}/\alpha + \Sigma_2^{\,2}/\beta + \Sigma_3^{\,2}/\gamma = 0 \\ \Sigma_1^{\,2} \quad + \Sigma_2^{\,2} \quad + \Sigma_3^{\,2} \quad = 0 \end{array} \right\} \quad \dots\dots\dots\dots\dots\dots (9)$$

By means of the identity, we can express the curve in terms of
any three of the four coordinates, giving four similar forms of the
equation.

The tangential forms corresponding to (9) are

$$\left. \begin{array}{l} \alpha\lambda^2 + \beta\mu^2 + \gamma\nu^2 = 0 \\ \lambda^2 + \mu^2 + \nu^2 = 0 \end{array} \right\} \text{ giving } \frac{\lambda^2}{\beta-\gamma} = \frac{\mu^2}{\gamma-\alpha} = \frac{\nu^2}{\alpha-\beta},$$

which determine four non-singular foci $\quad \lambda\Sigma_1 + \mu\Sigma_2 + \nu\Sigma_3 = 0$
lying on Σ_4.

We tabulate the results as follows :—

$$\left. \begin{array}{ll} \text{Foci} \quad \sqrt{\alpha-\beta}\,\Sigma_2 \pm \sqrt{\gamma-\alpha}\,\Sigma_3 \pm \sqrt{\beta-\gamma}\,\Sigma_4 = 0 \text{ lying on } \Sigma_1, \\ \sqrt{\beta-\gamma}\,\Sigma_3 \pm \sqrt{\gamma-\alpha}\,\Sigma_4 \pm \sqrt{\alpha-\beta}\,\Sigma_1 = 0 \qquad \Sigma_2, \\ \sqrt{\alpha-\beta}\,\Sigma_4 \pm \sqrt{\gamma-\alpha}\,\Sigma_1 \pm \sqrt{\beta-\gamma}\,\Sigma_2 = 0 \qquad \Sigma_3, \\ \sqrt{\beta-\gamma}\,\Sigma_1 \pm \sqrt{\gamma-\alpha}\,\Sigma_2 \pm \sqrt{\alpha-\beta}\,\Sigma_3 = 0 \qquad \Sigma_4, \end{array} \right\} \dots(10)$$

This determines the sixteen non-singular foci.

The coefficients in every case are $\pm\sqrt{\beta-\gamma},\, \pm\sqrt{\gamma-\alpha},\, \pm\sqrt{\alpha-\beta}$.
Hence all the confocal curves are given by the one parameter
system $\Sigma_1^{\,2}/_{\alpha+\theta} + \Sigma_2^{\,2}/_{\beta+\theta} + \Sigma_3^{\,2}/_{\gamma+\theta} = 0$.

We are concerned with the bicircular quartic of real coefficients.
Suppose the four base circles to be complex. Then the con-
jugates $\Sigma_1', \Sigma_2', \Sigma_3', \Sigma_4'$ must also form an orthogonal system, for
the equations of orthogonality still hold when the imaginary part
changes sign. Since the curve has only real coefficients it can be
written in the conjugate form $\Sigma_1'^{\,2}/_{\alpha'} + \Sigma_2'^{\,2}/_{\beta'} + \Sigma_3'^{\,2}/_{\gamma'} = 0$. From

the uniqueness of solution this must be one of the four forms (9). Hence $\Sigma_1', \Sigma_2', \Sigma_3', \Sigma_4'$ must be $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ in some order, and the complex base circles occur in pairs.

Applying the results of § III., it is clear that for a bicircular quartic of real coefficients two types only are possible, as follows:—

The base circles are (a) three real, one imaginary; (b) two real, two conjugate complex.

In case (a), equations (10) show that there are four real foci only, all lying on one of the real circles.

In case (b) let Σ_2, Σ_3 be conjugate complexes, and Σ_1, Σ_4 real.

Then in the equation of the quartic β, γ are conjugate complexes and α is real.

Write $\alpha = a, \beta = b + \iota c, \gamma = b - \iota c, \Sigma_2 = Re^{\iota\phi}, \Sigma_3 = Re^{-\iota\phi}, R, \phi,$ being functions of x, y.

The foci are easily found to be $\Sigma_1 \pm r^{\frac{1}{2}}e^{\frac{\iota\phi}{2}} Re^{\iota\phi} \pm r^{\frac{1}{2}}e^{-\frac{\iota}{2}}Re^{-\iota\phi}$,

$$\text{where} \quad re^{\iota\phi} = -\tfrac{1}{2} + \frac{\iota(a-b)}{2c}.$$

The foci are real only for both signs positive or both negative. Hence two real and two unreal foci lie on Σ_4. (10) shows that for the foci on Σ_1, the coefficients of Σ_2, Σ_3 are interchanged. This will produce real foci for the case only of different signs. Hence two real and two unreal foci lie on Σ_1. (10) shows that the foci lying on Σ_2, Σ_3 are unreal.

VII. *Fundamental Point Circles.*

If we take three foci as base circles, three sets of mutual ratios defining the foci are $(1:0:0)$, $(0:1:0)$, $(0:0:1)$. The tangential equation (6) gives

$$F\mu\nu + G\nu\lambda + H\lambda\mu = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(11)$$

and hence $A = B = C = 0$.

The equation of the quartic becomes

$$(-F^2, -G^2, -H^2, GH, HF, FG)(C_1, C_2, C_3)^2 = 0$$

or $\pm \sqrt{FC_1} \pm \sqrt{GC_2} \pm \sqrt{HC_3} = 0$, i.e. $\sqrt{F}.r_1 \pm \sqrt{G}.r_2 \pm \sqrt{H}.r_3 = 0$, where r_1, r_2, r_3 are the distances of any point on the quartic from any three foci. From (4) the equation of the orthogonal circle passing through C_1, C_2, C_3 is

$$(d_{23}^2, d_{31}^2, d_{12}^2, -d_{31}d_{12}, -d_{12}d_{23}, -d_{23}d_{31})(C_1, C_2, C_3)^2 = 0.$$

The corresponding tangential equation is

$$d_{23}\,\mu\,\nu + d_{31}\,\nu\,\lambda + d_{12}\,\lambda\,\mu \, . \, = 0.$$

This equation together with (11) has four solutions,

$$(1:0:0),\ (0:1:0),\ (0:0:1)$$

and $$\lambda(Gd_{12} - Hd_{31}) = \mu\,(Hd_{23} - Fd_{12}) = \nu\,(Fd_{31} - Gd_{23}).$$

Hence the fourth concyclic focus is the point circle

$$\Sigma C_1/(Gd_{12} - Hd_{31}) = 0.$$

VIII. *The Circular Cubic.*

The equation $(a, b, c, f, g, h)(S_1, S_2, S_3)^2 = 0$ represents a circular cubic if $a + b + c + f + g + h = 0$. The point $S_1 = S_2 = S_3$ hence lies on the curve. So, in the canonical form $\alpha\,\Sigma_1^2 + \beta\,\Sigma_2^2 + \gamma\,\Sigma_3^2 = 0$, $\alpha/r_1^2 + \beta/r_2^2 + \gamma/r_3^2 = 0$, and the centres of the four base circles lie on the cubic. With this additional condition, the results of §§ III.—VI. hold for the circular cubic.

In the form $(a, b, c, -\sqrt{bc}, -\sqrt{ca}, -\sqrt{ab})(C_1, C_2, C_3)^2 = 0$ we have $a + b + c - 2\sqrt{bc} - 2\sqrt{ca} - 2\sqrt{ab} = 0$ or $\pm\sqrt{a} \pm \sqrt{b} \pm \sqrt{c} = 0$.

The equation to such cubic has the forms

$$(\sqrt{C_2} - \sqrt{C_3})/\sqrt{a} = \pm(\sqrt{C_3} - \sqrt{C_1})/\sqrt{b} = \pm(\sqrt{C_1} - \sqrt{C_2})/\sqrt{c}$$

If r_1, r_2, r_3 be the distances of any point of the cubic from any three foci, we have $\dfrac{r_2 - r_3}{\sqrt{a}} = \pm\dfrac{r_3 - r_1}{\sqrt{b}} = \pm\dfrac{r_1 - r_2}{\sqrt{c}}$

IX. *A System of Curves.*

Suppose we have two curves $(a, b, c, f, g, h)(S_1, S_2, S_3)^2 = 0$,

$$(a', b', c', f', g', h')(S_1', S_2', S_3')^2 = 0.$$

In general it will not be possible to refer these two curves to the same system of base circles, unless they each have four foci lying on the same circle, C.

In this case we have

$$(a, b, c, f, g, h)(S_1, S_2, S_3)^2 \equiv \alpha\Sigma_1^2 + \beta\Sigma_2^2 + \gamma\Sigma_3^2 \equiv U.$$
$$(a', b', c', f', g', h')(S_1, S_2, S_3)^2 \equiv \alpha'\Sigma_1^2 + \beta'\Sigma_2^2 + \gamma'\Sigma_3^2 \equiv V,$$

a unique transformation, in which $\Sigma_1, \Sigma_2, \Sigma_3$ are not in general

"focal" circles. The solutions of the finite points of intersection
of these two quartics are given by

$$\Sigma_1^2/(\beta\gamma' - \beta'\gamma) = \Sigma_2^2/(\gamma\alpha' - \gamma'\alpha) = \Sigma_3^2/\alpha\beta' - \alpha'\beta)$$

or $$\pm \Sigma_1/A = \pm \Sigma_2/B = \pm \Sigma_3/C\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

This gives the intersections of the two circles $\Sigma_1/A = \pm \Sigma_2/B$
with the two circles $\Sigma_2/B = \pm \Sigma_2/C$, and the two circles
$\Sigma_3/C = \pm \Sigma_1/A$ pass through these common points. There are thus
four pairs of such points, and therefore any two bicircular quartics
of the form

$$(a, b, c, f, g, h)(S_1, S_2, S_3)^2 = 0, \ (a', b', c', f', g', h')(S_1, S_2, S_3)^2 = 0,$$

will have eight such connected points in common, and all such bi-
circular quartics possessing these properties can be thrown into
the form $\alpha\Sigma_1^2 + \beta\Sigma_2^2 + \gamma\Sigma_3^2 = 0$.

The system is $U + t\ V$, where t is a parameter.

If $\alpha + \beta + \gamma = 0$, the curve is a circular cubic.

There corresponds for the case of two base cubics a system of
circular cubics, and for the finite common points of these cubics
condition (12) gives $\pm \Sigma_1 = \pm \Sigma_2 = \pm \Sigma_3$. These are the common
points of $\Sigma_1 = \pm \Sigma_2$ and $\Sigma_2 = \pm \Sigma_3$, through which pass also $\Sigma_3 = \pm \Sigma_1$.

This gives the point of concurrence of the three common chords
and points of intersection of each common chord with a circle of
the other systems, seven points in all.

The first system contains one circular cubic given by
$(\alpha + \beta + \gamma) + t(\alpha' + \beta' + \gamma') = 0$.

To the system $U + t\ V = 0$ corresponds a set of invariants
similar to those of the four point conic systems.

Casey has discussed these and obtained a more extended system
of invariants, from the geometrical point of view. An analytical
discussion of the invariant-theory from the modern point of view
is to be found in Prof. Turnbull's paper.*

X. *Inversion.*

Inversion with regard to any point leaves the ratios of the
coordinates $\dfrac{S_k}{r_k} \equiv \Sigma_k$, $(k = 1, 2, 3)$, unaltered. The inverses of the

* *Proc. Royal Soc., Edinburgh* (1924), 23-50.

base circles are the new mutually or orthogonal base circles $\Sigma_k' - \dfrac{S_k'}{r_k'}(k=1, 2, 3)$, and thus the inverse of the curve

$$\alpha\Sigma_1^2 + \beta\Sigma_2^2 + \gamma\Sigma_3^2 \text{ is } \alpha\Sigma_1'^2 + \beta\Sigma_2'^2 + \gamma\Sigma_3'^2 = 0 \quad \ldots\ldots\ldots(13)$$

Inverting with regard to a point on a bicircular quartic an exception arises, for the inverse of this point is the line at infinity, the 'S''' coordinates of *any* point of which are $(1 : 1 : 1)$.

The 'Σ'' coordinates $\left(\dfrac{1}{r_1'^2}, \dfrac{1}{r_2'^2}, \dfrac{1}{r_3'^2}\right)$ hence satisfy (13), *i.e.*

$$\alpha/r_1'^2 + \beta/r_1' + \gamma/r_3'^2 = 0$$

and the inverse curve is a circular cubic.

Inversion with regard to one of the base circles leaves the system, and hence by (12), the curve, unaltered.