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A property of quartic curves with two cusps and one node

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1. A method of characterising plane curves was given by Plücker, who showed that the degree (n), the class (m), and the numbers of nodes (δ), bitangents (τ), cusps (κ), and inflexional tangents (ι) are connected by three independent relations. He showed also that n and m, δ and τ, κ and ι are dual pairs.

In general a curve is not self-dual, but some special curves are. In particular, a quartic curve with two cusps and one node is self-dual, since its Plücker numbers are

$$n = m = 4, \ \delta = \tau = 1, \ \kappa = \iota = 2.$$

The object of this paper is to show that such a curve is self-polar with respect to each of two conics. The corresponding problem for a quintic curve with five cusps was solved by R. Apéry (Comptes Rendus (Paris), Vol. 213 (1941), pp. 674-5).

2. If A is the node, B and C are the cusps, and the cuspidal tangents meet at I, we take ABC as triangle of reference and I as unit point. Since the curve is rational, we may choose a parameter t/u such that $u = 0$ and $t = 0$ correspond to B and C, and the roots of $t^2 + 2rtu + u^2 = 0$ correspond to $A : r$ is arbitrary, and the choice of unity for the ratio of the coefficients of t^2 and u^2 in this equation is equivalent to choosing the point corresponding to $t = u$. Then x and y

must contain a factor t^2, x and z must contain a factor u^2, and y and z must contain a factor $t^2 + 2rtu + u^2$. Hence

$$x = at^2u^2,\ y = bt^2\ (t^2 + 2rtu + u^2),\ z = cu^2(t^2 + 2rtu + u^2).$$

Now the tangent at B is $x/a = z/c$ and the tangent at C is $x/a = y/b$. Hence $a = b = c$, so that, in terms of a non-homogeneous parameter, we may write

$$x = t^2,\ y = t^2\ (t^2 + 2rt + 1),\ z = t^2 + 2rt + 1. \tag{1}$$

We shall need the values of t which correspond to the inflexions; these are found to be given by

$$3rt^2 + 2\ (r^2 + 2)\ t + 3r = 0. \tag{2}$$

We shall also need the line co-ordinates of the tangent to the quartic at the point whose parameter is t'; these are found to be

$$l = -\ (t'^2 + 2rt' + 1)^2,\ m = rt' + 1,\ n = t'^3\ (t' + r). \tag{3}$$

We must rule out the cases $r = 0$, when x, y, z are quadratic functions of t^2 and therefore the quartic degenerates, $r = \pm 1$, when the quartic has three cusps, and $r = \pm 2$, when the inflexions coincide. (In the latter case, it may be shown that the bitangent coincides with the inflexional tangents, so that this is a triple line of the envelope).

3. We now suppose that there is a conic with respect to which the quartic is self-polar. Its equation must be symmetrical in y and z, and is therefore of the form

$$ax^2 + b\ (y^2 + z^2) + 2fyz + 2gx\ (y + z) = 0. \tag{4}$$

Hence the polar of the point (1) with respect to the conic (4) must have line co-ordinates of the form (3), for some value of t'. This sets up a (1, 1) algebraic correspondence between t and t', and since the cusps must correspond to the inflexional tangents and the inflexions to the cuspidal tangents, the correspondence is one of the two involutions in which $t = 0$ and $t = \infty$ correspond, in either order, to the roots of (2), which we call $t = \alpha$ and $t = 1/\alpha$, where

$$3r\alpha^2 + 2\ (r^2 + 2)\alpha + 3r = 0. \tag{5}$$

The equation of such an involution is $t' = (t - \alpha)/(t\alpha - 1)$.
Substituting this value for t' in (3) and multiplying by a suitable factor, we have

$$l = -\ [t^2\ (1 + 2r\alpha + \alpha^2) - 2t\ (r + 2\alpha + r\alpha^2) + (1 + 2r\alpha + \alpha^2)]^2,$$

$$m = [(r + \alpha)\ t - (r\alpha + 1)]\ (t\alpha - 1)^3,$$

$$n = [(r\alpha + 1)\ t - (r + \alpha)]\ (t - \alpha)^3.$$

Comparing these co-ordinates with the coefficients of x, y, z in the equation of the polar of (1) with respect to (4), we have

$$at^2 + gt^2(t^2 + 2rt + 1) + g(t^2 + 2rt + 1) = -[t^2(1 + 2r\alpha + \alpha^2) - 2t(r + 2\alpha + r\alpha^2) + (1 + 2r\alpha + \alpha^2)]^2,$$

$$gt^2 + bt^2(t^2 + 2rt + 1) + f(t^2 + 2rt + 1) = [(r+\alpha)t - (r\alpha + 1)](t\alpha - 1)^3,$$

$$gt^2 + ft^2(t^2 + 2rt + 1) + b(t^2 + 2rt + 1) = [(r\alpha + 1)t - (r + \alpha)](t - \alpha)^3.$$

It is found that the equations are all satisfied identically if

$$a = -4(r + 2\alpha + r\alpha^2)^2,$$
$$b = \alpha^3(r + \alpha),$$
$$f = r\alpha + 1,$$
$$g = -(1 + 2r\alpha + \alpha^2)^2.$$

Hence the quartic is self-polar with respect to each of the conics (4), where a, b, f, g have the above values and α is one of the roots of (5).

Corollary 1. Since the bitangent is the polar of the node, and since two polar triangles with respect to a conic are perspective, it follows that the triangle formed by the node and the cusps is perspective, in two different ways, with the triangle formed by the bitangent and the inflexional tangents.

Corollary 2.[1] If the equation of the conic corresponding to one of the values of α is

$$S_1 \equiv ax^2 + b(y^2 + z^2) + 2fyz + 2gx(y + z) = 0,$$

it is easily shown that the equation of the other conic is

$$S_2 \equiv ax^2 + f(y^2 + z^2) + 2byz + 2gx(y + z) = 0.$$

Hence $S_1 - S_2$ is a multiple of $(y - z)^2$. It follows that the conics have double contact, the line joining the points of contact being $y = z$.

It follows also from the above equations that the conics coincide if and only if $b = f$, which gives $r^2 = 1$ or 4, some of the exceptional cases already mentioned.

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