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# EXTREME POINTS OF POSITIVE FUNCTIONALS AND SPECTRAL STATES ON REAL BANACH ALGEBRAS

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ABSTRACT. Extreme points of positive functionals and spectral states on real commutative Banach algebras are investigated and characterized as multiplicative functionals extending the well-known results from complex to real Banach algebras. As an application a new and short proof of the existence of the Shilov boundary of a real commutative Banach algebra with nonempty maximal ideal space is given.

**Introduction.** Let $A$ be a complex, commutative Banach* algebra with isometric involution and bounded approximate identity. It is well-known that the extreme points of the set $\{f \in A' : f \geq 0, \|f\| \leq 1\}$ are multiplicative functionals (1).

The result is valid for complex Banach algebras with unit and arbitrary involutions [16]. G. Niestegge showed that the concept of positive functionals can be generalized to complex, commutative Banach algebras considering the so called spectral states [14]. It turns out that the extreme points of spectral states are multiplicative and on a complex, commutative Banach* algebras with a symmetric involution spectral states are precisely the positive functionals.

Both characterizations have nice applications. The extreme-point characterization of positive functionals yields a short proof of the classical Riesz-Bochner-Herglotz theorem [5] while the analogous characterization of spectral states gives a new proof of the existence of the Shilov boundary of a complex, commutative Banach algebra [14].

But all these results are valid only for complex algebras. Since N. L. Alling [1] showed in 1970 that Banach algebras over Klein surfaces are real but *not* complex, it is a natural and interesting question to ask for generalizations of such theorems for real algebras.

In the first section of this paper the extreme-point characterization of positive functionals is extended to real commutative Banach* algebras with an isometric involution and bounded approximate identity. The crucial point is that we have to consider real-linear, positive functionals which are in addition hermitian and satisfy the Cauchy-Schwarz inequality, the so called positive Schwarz functionals. In the second section real-linear spectral states are introduced and it is shown that their extreme points are also multiplicative.

Our characterization of real-linear spectral states gives a new proof of the existence of the Shilov boundary of a real, commutative Banach algebra with non-void maximal ideal space.

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In the last section the results of Section 1 and 2 are used to obtain extension of positive functionals and spectral states on real Banach algebras.

**Preliminaries.** Let $A$ be a real Banach algebra and $x \in A$. $\sigma_A(x)$ (or $\sigma(x)$) denotes the spectrum of $x$ with respect to $A$ and is defined as the spectrum of $x$ regarded as an element in the complexification $A_{\mathbb{C}}$ of $A$ [10], [11]. The spectral radius $r(x)$ of $x$ is defined as $r(x) = \inf_n \|x^n\|^{1/n}$. A real Banach algebra with a unit $e$ is called unital, if $\|e\| = 1$.

$A$ is called strictly real, if $(-x^2)$ is quasi-invertible for all $x \in A$. If $A$ is commutative, then the Gelfand space (or carrier space) of $A$ is the set $\Phi_A$ of non-trivial, continuous, $\mathbb{R}$-linear, complex-valued and multiplicative functionals on $A$. The Gelfand space can be separated into a real part $\Phi_A^R$ and a complex part $\Phi_A^C$ with

$$\Phi_A^R = \{h \in \Phi_A : \bar{h} = h\} \text{ and } \Phi_A^C = \{h \in \Phi_A : \bar{h} \neq h\}.$$

We say, $h, g \in \Phi_A$ are equivalent if $h = g$ or $\bar{h} = g$ (the bar denotes complex conjugation). Let $[\Phi_A]$ be the set of all such equivalence classes, then $[\Phi_A]$ is isomorphic to the space of maximal ideals $\Delta_A$ of $A$.

Note that for a complex commutative Banach algebra $A$, $\Delta_A$ is isomorphic to the set $\{h \in A' : h \not\equiv 0,\ h \text{ multiplicative}\}$ and each maximal ideal is the kernel of such a non-trivial, multiplicative functional. Therefore in the complex case we want not to distinguish between these two spaces. Let $j\colon \Phi_A \longrightarrow \Delta_{A_{\mathbb{C}}}$ defined by

$$j(h)(x + iy) := h(x) + ih(y) \quad (x, y \in A, h \in \Phi_A)$$

and

$$c\colon \Delta_{A_{\mathbb{C}}} \longrightarrow \Delta_A$$

defined by

$$c(M) = M \cap A \quad (M \in \Delta_{A_{\mathbb{C}}}).$$

For $x \in A$ and $N \in \Delta_A$ define $\hat{x}(N)$ as the image of $x$ in $A/N$. By the Gelfand-Mazur theorem $A/N$ is an extension of $\mathbb{R}$ with degree 1 or 2 and is mapped into $\mathbb{C}$ by exactly two distinct homomorphism either $\alpha$ or $\beta$ with $\beta(t) = \overline{\alpha(t)}$ for all $t \in A/N$. Endow $\Delta_A$ with the weak topology such that all functions $|\hat{x}|\colon \Delta_A \longrightarrow \mathbb{R}$ with $|\hat{x}|(N) = |\hat{x}(N)|$ are continuous. If $\Delta_{A_{\mathbb{C}}}$ has the usual weak topology, then $c$ is a continuous and surjective map.

An important fact is the following. Let $h \in \Delta_{A_{\mathbb{C}}}$ and define $\tau(h)$ by $\tau(h)(x + iy) = h(x) - ih(y)$ $(x, y \in A)$. Then $\tau(h) \in \Delta_{A_{\mathbb{C}}}$ and for each $N \in \Delta_A$ there is an $h \in \Delta_{A_{\mathbb{C}}}$ such that $c^{-1}(N) = \{h, \tau(h)\}$.

Now let $A$ be a real Banach* algebra. The sets $A_H = \{x \in A : x^* = x\}$ and $A_J = \{x \in A : x^* = -x\}$ denote the hermitian and skew-hermitian part of $A$. $A_H$ and $A_J$ are real vector spaces and each $x \in A$ has a unique decomposition $x = u + v$ with $u \in A_H$ and $v \in A_J$.

$A$ is called symmetric, if for all $x \in A$, $\sigma(x^*x) \subseteq \mathbb{R}_0^+$, hermitian if for all $x \in A_H$, $\sigma(x) \subseteq \mathbb{R}$ and skew-hermitian if $\sigma(x)$ contains no non-zero real number for all $x \in A_J$.

By a famous theorem of J. W. M. Ford, T. W. Palmer and S. Shirali $A$ is symmetric if and only if $A$ is hermitian and skew-hermitian. For a detailed discussion of real Banach* algebras and their representations see P. Fijalkowski [8], K. R. Goodearl [9], L. Ingelstam [10], S. H. Kulkarni and B. V. Limaye [11], and A. Srivastav [17], [18].

**I. Positive real-linear functionals.** We say that a real-linear functional $f: A \longrightarrow \mathbb{C}$ on a real Banach* algebra $A$ is

positive, if $f(x^*x) \geq 0$ for all $x \in A$,
hermitian, if $f(x^*) = \overline{f(x)}$ for all $x \in A$, and
weak hermitian, if $f(x^*y) = \overline{f(yx^*)}$ for all $x \in A$.

If $A$ is a complex Banach* algebra and $f$ is a $\mathbb{C}$-linear positive functional on $A$, then $f$ is automatically weak hermitian and satisfies the Cauchy-Schwarz inequality $|f(x^*y)|^2 \leq f(x^*x)f(y^*y)$ for all $x, y \in A$ ([16], Theorem 11.31).

But this is false for real Banach* algebras and real-linear functionals, even if $A$ possesses an identity: Take for example the set of complex numbers as a real Banach* algebra with unit and define functionals $f, g$ by

$$f(x+iy) := x + ity \text{ and } g(x+iy) := y \quad (x, y \in \mathbb{R}, t \geq 0).$$

Obviously $f, g$ are positive, but $g$ is not hermitian and $f$ does not satisfy the Cauchy-Schwarz inequality.

Therefore we say that a real-linear, weak-hermitian functional $f$ on a real Banach* algebra, $f: A \longrightarrow \mathbb{C}$, is a Schwarz functional, if $f$ satisfies the Cauchy-Schwarz inequality

$$|f(x^*y)|^2 \leq f(x^*x)f(y^*y) \text{ for all } x, y \in A.$$

It would be of interest to know whether a real-linear, positive functional on a real Banach* algebra satisfying the Cauchy-Schwarz inequality is automatically weak hermitian or not. For real-valued $f$ this is true. If $f: A \longrightarrow \mathbb{R}$ is a real-linear, positive and weak-hermitian functional, then $f$ is a Schwarz functional and if in addition the algebra has a unit $e$ and is commutative, then $\|f\| = f(e)$ ([9], Proposition 14.3). For a Schwarz functional $f$ the number $f(x^*x)$ is real and by the Cauchy-Schwarz inequality either $f$ or $(-f)$ is positive.

Let $A$ be a real Banach algebra and define

$$F(A, \mathbb{C}) := \{f: A \longrightarrow \mathbb{C} : f \text{ real-linear and continuous}\}.$$

Denote by $A'$ the norm dual of $A$, i.e. the space of all real-linear, real-valued and continuous functionals on $A$.

On $F(A, \mathbb{C})$ we consider the weak topology $\sigma\big(F(A, \mathbb{C}), A\big)$ and on $A'$ the usual weak* topology $\sigma(A', A)$.

Define the sets

$$P_S^1 = \{f: A \longrightarrow \mathbb{C} : f \text{ is a positive Schwarz functional with } \|f\| \leq 1\}$$
$$M_S = \{f: A \longrightarrow \mathbb{C} : f \text{ is } \mathbb{R}\text{-linear, positive, weak-hermitian and multiplicative}\}$$

For a complex Banach* algebra $A$ define

$$P_A = \{f : A \longrightarrow \mathbb{C} : f \text{ if } \mathbb{C}\text{-linear and positive}\}$$

and

$$P_A^1 = \{f \in A' \cap P_A : \|f\| \leq 1\}.$$

The first lemma states that a positive Schwarz functional can be extended to a positive functional on the complexification. This fact constitutes an important step in the proof of the main theorem of this section.

LEMMA 1.1. *Let $A$ be a real Banach* algebra and $f : A \longrightarrow \mathbb{C}$ a positive Schwarz functional. Then there is a complex-linear, positive and unique extension $\tilde{f}$ of $f$ to the complexification $A_{\mathbb{C}}$ of $A$ defined by*

$$\tilde{f}(x+iy) = f(x) + if(y) \quad (x, y \in A)$$

PROOF. Obviously $\tilde{f}$ is well-defined, complex-linear and unique. Let $x, y \in A$ and define

$$\alpha := \mathrm{Re} f(x^*y),\ \beta := \mathrm{Im} f(x^*y),\ r := f(x^*x) \text{ and } s := f(y^*y).$$

$f$ is a Schwarz functional, hence by the Cauchy-Schwarz inequality

$$0 \leq \alpha^2 + \beta^2 \leq rs.$$

It follows

$$\beta \leq r^{1/2}s^{1/2} \leq \frac{r+s}{2}$$

and by (1) (see Introduction)

$$\tilde{f}\big((x+iy)^*(x+iy)\big) = r + s - 2\beta \geq 0 \qquad \blacksquare$$

To ensure that there are extreme points of $P_S$ we prove (with respect to the Krein-Milman theorem) that $P_S^1$ is a non-empty, convex and compact set.

LEMMA 1.2. *Let $A$ be a real Banach* algebra.*

*(i) Let $F_S \subseteq F(A, \mathbb{C})$ the set of all positive functionals for which the Cauchy-Schwarz inequality holds. Then $F_S$ is convex.*

*(ii) If $\sigma\big(F(A, \mathbb{C}), A\big)$ is the weak topology on $F(A, \mathbb{C})$, then $P_S^1$ is a non-empty, convex and compact subset of $F(A, \mathbb{C})$.*

PROOF. (i) Let $f, g \in F_S$, $0 < t < 1$ and $h := tf + (1-t)g$. Obviously $h$ is positive and continuous. We show that $h$ satisfies the Cauchy-Schwarz inequality. Let $x, y \in A$ and

$$\alpha_1 + i\alpha_2 := f(x^*y),\ \beta_1 + i\beta_2 := g(x^*y),\ \alpha_x := f(x^*x),$$
$$\alpha_y := f(y^*y),\ \beta_x := g(x^*x),\ \beta_y := g(y^*y)$$
$$\text{with } \alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_x, \alpha_y, \beta_x, \beta_y \in \mathbb{R}.$$

Since $f, g$ satisfy the Cauchy-Schwarz inequality, we have

$$|h(x^*y)|^2 \leq t^2\alpha_x\alpha_y + (1-t)^2\beta_x\beta_y + 2t(1-t)[\alpha_1\beta_1 + \alpha_2\beta_2].$$

Furthermore $2(\alpha_1\beta_1 + \alpha_2\beta_2) \leq \alpha_x\beta_y + \alpha_y\beta_x$ and the assertion is proved.

(ii) Compactness of $P_S^1$ is a consequence of the Banach-Alaoglu theorem. $P_S^1$ is nonempty since $0 \in P_S^1$. ∎

THEOREM 1.3. *Let $A$ be a real, commutative Banach* algebra with isometric involution and bounded approximate identity $(u_\lambda)$. Then*

$$\mathrm{ex}(P_S^1) = M_S.$$

PROOF. Let $\|\cdot\|_C$ be the usual complexification norm on $A_{\mathbb{C}}$ defined as in C. E. Rickart ([15], Theorem 1.3.2). $(A_{\mathbb{C}}, \|\cdot\|_C)$ is a complex Banach* algebra and the embedding of $A$ into $A_{\mathbb{C}}$ is an isometry. Define a second norm $\|\cdot\|_1$ on $A_{\mathbb{C}}$ by

$$\|x+iy\|_1 := \max\{\|x+iy\|_C : \|x^* - iy^*\|_C\}.$$

Then $(A_{\mathbb{C}}, \|\cdot\|_1)$ is a Banach* algebra with an isometric involution and the two norms $\|\cdot\|_{\mathbb{C}}$ and $\|\cdot\|_1$ are equivalent. Obviously $(u_\lambda)$ is a bounded approximate identity for $(A_{\mathbb{C}}, \|\cdot\|_1)$. Let $\pi$ be the map $\pi\colon P_{A_{\mathbb{C}}}^1 \longrightarrow P_S^1$ with

$$\pi(f) := f/A \quad (f \in P_{A_{\mathbb{C}}}^1).$$

We show that $\pi$ is bijective. It is trivial that $\pi$ is one-to-one. By Lemma 1.1 each $f \in P_S^1$ possesses a complex-linear, positive extension $\tilde{f}$ on $A_{\mathbb{C}}$ and $\|\tilde{f}\| = \lim_\lambda f(u_\lambda)$ ([7], Proposition 2.15). Since $\|u_\lambda\| \leq 1$ for all $\lambda$, and $\|f\| \leq 1$, we have $\|\tilde{f}\| \leq 1$. Hence $\tilde{f} \in P_{A_{\mathbb{J}}}^1$ and $\pi$ is surjective.

The convexity of $P_S^1$ implies that $\pi$ is an affine bijection. By Theorem (1) in the introduction

$$\mathrm{ex}(P_{A_{\mathbb{C}}}^1) = \{f \in P_{A_{\mathbb{C}}}^1 : f \text{ is multiplicative}\},$$

and therefore

$$\mathrm{ex}(P_S^1) = \mathrm{ex}\big(\pi(P_{A_{\mathbb{C}}}^1)\big) = \pi(\{f \in P_{A_{\mathbb{C}}}^1 : f \text{ is multiplicative}\}) \subset M_S.$$

Let $h \in M_S \backslash \{0\}$. Then $\tilde{h}$ with $\tilde{h}(x+iy) := h(x) + ih(y)$, $x, y \in A$, defines a multiplicative, non-trivial extension of $h$ on $A_{\mathbb{C}}$ and by Bonsall-Duncan ([3], Chapter II, §16, Proposition 3) $\tilde{h}$ is continuous with $\|\tilde{h}\| \leq 1$.

Hence $M_S \subset \pi(\{f \in P_{A_{\mathbb{C}}}^1 : f \text{ is multiplicative}\})$. ∎

Let $A$ be a real Banach* algebra with unit $e$ and set

$$P^1 := \{f\colon A \longrightarrow \mathbb{C} : f \text{ is a Schwarz functional with } 0 \leq f(e) \leq 1.\}$$

Then we can deduce from Theorem 1.3 the following corollary for algebras with a unit.

COROLLARY 1.4. *Let A be a real, commutative, unital Banach* algebra. Then we have*

$$\operatorname{ex}(P^1) = M_S.$$

PROOF. Since $f(e) \geq 0$, each $f \in P^1$ is positive. Define a new norm $\|\cdot\|_1$ on $A$ by

$$\|x\|_1 := \max\{\|x^*\|, \|x\|\} \quad (x \in A).$$

Then $\|\cdot\|_1$ is a Banach algebra norm equivalent to the original norm and the involution is an isometry with respect to the $\|\cdot\|_1$-norm. According to Theorem 1.3 the assertion is proved if we can show that $P^1 = P^1_S$. Let $f \in P^1$ and $\|f\|_1$ the norm of $f$ with respect to the $\|\cdot\|_1$-norm. By ([16], Theorem 11.31) the positive extension $\tilde{f}$ of $f$ to $A_{\mathbb{C}}$ satisfies $\|\tilde{f}\| = \tilde{f}(e) = f(e)$. Hence $\|f\|_1 \leq \|f\| \leq \|\tilde{f}\| = f(e) \leq 1$ and $f \in P^1_S$. Since the inclusion $P^1_S \subseteq P^1$ is trivial, the Corollary is proved. ∎

If the algebra is symmetric, then the set of extreme points of $P^1$ is the union of the carrier space and the trivial functional. Corollary 1.4 gives

COROLLARY 1.5. *Let A be a real, commutative, symmetric and unital Banach* algebra. Then we have*

$$\operatorname{ex}(P^1) = \Phi_A \cup \{0\}.$$

Finally, for strictly-real commutative Banach algebras with unit we have an involution-free characterization of positive functionals.

COROLLARY 1.6. *Let A be a strictly-real, commutative and unital Banach algebra. If P is the set of real-linear, real-valued functionals f which satisfy $f(e) \leq 1$ and $f(x^2) \geq 0$ for all $x \in A$, then*

$$\operatorname{ex}(P) = \Phi_A^{\mathbb{R}} \cup \{0\}.$$

PROOF. Take the identity map on $A$ as an involution. With this involution $A_J = \{0\}$ and every real-linear, real-valued functional on $A$ is hermitian. Let $f$ be a real-linear and real-valued functional with $f(x^2) \geq 0$ for all $x \in A$. By Lemma 1.1 $f$ is a Schwarz functional with $\|f\| = f(e)$. This fact implies $P \subset P^1$ ($P^1$ defined as above). Let $x \in A$. The decomposition $4x = (x+e)^2 - (x-e)^2$ implies that $f(A) \subset \mathbb{R}$ for every real-linear and positive functional; hence $P = P^1$. Since the algebra $A$ is strictly-real, $\Phi_A = \Phi_A^{\mathbb{R}}$ and every $x \in A$ has real spectrum ([10], Proposition 6.7 (b)). Hence $A$ is symmetric and the corollary follows with Corollary 1.5. ∎

Let us proceed to the analysis of spectral states extending the concept of positive functionals to real commutative Banach algebra without involutions.

**II. Spectral states on real Banach algebras.** For complex commutative Banach algebras the theory of spectral states was founded by R. T. Moore, F. F. Bonsall and J. Duncan [2].

G. Niestegge investigated in [14] the extreme points of spectral states, showed that they are multiplicative and gave a new proof of the existence of the Shilov boundary of complex commutative Banach algebras. The existence of the Shilov boundary for real commutative Banach algebras was not proved until N. L. Alling claimed it in 1970 in his work on real Banach algebras over compact Klein surfaces [1]. But the proof of Alling, which is the same as the proof for the complex case given by C. E. Rickart [15], could not be justified. Five years later B. V. Limaye and R. R. Simha gave a correct proof for the existence of the Shilov boundary [12].

In the following we will extend the definition of spectral states to real commutative Banach algebras. With methods from convex analysis (see [14]) we will prove an extreme point characterization of real spectral states which yields the existence of the Shilov boundary for general real commutative Banach algebras with non-empty maximal ideal space. In particular the spectral states on a real Banach* algebra with bounded approximate identity and symmetric and isometric involution are exactly the positive Schwarz functionals and therefore satisfy the Cauchy-Schwarz inequality.

Let $A$ be a real, commutative Banach algebra. A real-linear functional $f: A \longrightarrow \mathbb{C}$ is called a *spectral state* if $f(x) \in \operatorname{co}\sigma(x)$ for all $x \in A$ where $\operatorname{co}\sigma(x)$ is the convex hull of $\sigma(x)$. Obviously a spectral state $f$ is continuous with $\|f\| \leq 1$. Let $\Omega(A)$ be the set of all spectral states on $A$. This definition extends the usual notation of spectral states on complex, commutative Banach algebras as introduced by R. T. Moore, F. F. Bonsall and J. Duncan [2].

DEFINITION 2.1. Let $A$ be a real, commutative Banach algebra. A subset $S \subseteq \Delta_A$ is called a boundary for $A$, iff for all $x \in A$

$$\|\hat{x}\|_\infty = \sup_{N\in S} |\hat{x}(N)|.$$

Since a maximal ideal $N \in S$ is the equivalence class $[g]$, where $g \in \Phi_A$, the canonical extension $\tilde{g}$ of $g$ to $A_{\mathbb{C}}$ and $\tau\tilde{g}$ are in $\Delta_{A_{\mathbb{C}}}$. Hence for $x, y \in A$ we have

$$\max\big(|g(x)|, |g(y)|\big) \leq |\tilde{g}(x+iy)| \leq |g(x)| + |g(y)|.$$

Using this fact B. V. Limaye and R. R. Simha showed:

LEMMA 2.2 ([12], PROOF OF PROPOSITION 1.0). *Let $A$ be a real, commutative Banach algebra and $S \subseteq \Delta_A$ a boundary of $A$. Then $c^{-1}(S)$ is a boundary for $A_{\mathbb{C}}$.*

Notice that the map $c: \Delta_{A_{\mathbb{C}}} \longrightarrow \Delta_A$ is continuous, and hence $c^{-1}(S)$ is closed, if $S$ is closed.

LEMMA 2.3. *Let $A$ be a real, commutative Banach algebra and $S \subseteq \Delta_A$ a closed boundary. Then we have for all $x \in A$*

*(i) If $A$ has a unit, $\partial\sigma(x) \subseteq \{h(x) : h \in c^{-1}(S)\}$*

*(ii) In general, $\partial\sigma(x) \subseteq \{h(x) : h \in c^{-1}(S)\} \cup \{0\}$. ($\partial\sigma(x)$ denotes the toplogical boundary of $\sigma(x)$).*

PROOF. Without using the existence of the Shilov boundary for $A_{\mathbb{C}}$ one can prove as in ([15], pp. 142–143) that $\partial\sigma(z) \subseteq \hat{z}(S') \cup \{0\}$ for any closed boundary $S'$ of $A_{\mathbb{C}}$ and $z \in A_{\mathbb{C}}$. Lemma 2.3 now follows from Lemma 2.2.

THEOREM 2.4. *Let $A$ be a real, commutative Banach algebra.*

*(i) If $S$ is any closed boundary of $A$, then $\Omega(A) = \overline{\mathrm{co}\big(c^{-1}(S)/_A \cup \{0\}\big)}$ and $\mathrm{ex}\,\Omega(A) \subseteq c^{-1}(S)/_A \cup \{0\}$.*

*(ii) If $A$ possesses a unit, then $\Omega(A) = \overline{\mathrm{co}\big(c^{-1}(S)/_A\big)}$ and $\mathrm{ex}\,\Omega(A) \subseteq c^{-1}(S)/_A$.*

*(iii) If $\Delta_A$ is non-void, then $(c \circ j)(\mathrm{ex}\,\Omega(A))$ is non-void and is the Shilov boundary of $A$.*

PROOF. We modify the arguments of [14]:

(i) Let $S$ be a closed boundary of $A$ and put $S' := c^{-1}(S)/_A \cup \{0\}$. Since $c$ is continuous, $c^{-1}(S)$ is closed. Furthermore $\Omega(A)$ is compact and convex in $F(A, \mathbb{C})$, hence $\overline{\mathrm{co}(S')} \subseteq \Omega(A)$.

Assume that there is a $g \in \Omega(A) \backslash \overline{\mathrm{co}(S')}$. The Hahn-Banach theorem gives $x_0 \in A$ such that $\max\{\mathrm{Re}\, h(x_0) : h \in \overline{\mathrm{co}(S')}\} < \mathrm{Re}\, g(x_0)$. By Lemma 2.3 we get the contradiction $\mathrm{Re}\, g(x_0) > \max\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{co}\,\sigma(x_0)\}$. Then $\Omega(A) = \overline{\mathrm{co}(S')}$ and Milman's theorem ([6], Corollary 25.14) yields $\mathrm{ex}\,\Omega(A) \subset S'$.

(ii) Define $S' := c^{-1}(S)/_A$ and argue as in (i).

(iii) Since $c^{-1}(\Delta_A)/_A \subseteq \Omega(A)$ the assumption $\Delta_A \cap (c \circ j)\big(\mathrm{ex}\,\Omega(A)\big) = \emptyset$ and the Krein-Milman theorem imply $\Omega(A) = \{0\}$ in contradiction to the fact that each element of $\Delta_{A_{\mathbb{C}}} = c^{-1}(\Delta_A)$ is non-zero.

We prove that $S'' := \Delta_A \cap (c \circ j)\big(\mathrm{ex}\,\Omega(A)\big)$ is a boundary for $A$. Let $x \in A$. We can assume $\|\hat{x}\|_\infty \neq 0$ (otherwise $|\hat{x}|$ possesses a maximum point in $S''$). Define a function $H\colon \Omega(A) \longrightarrow \mathbb{R}$ by

$$H(f) = |f(x)|, \quad f \in \Omega(A).$$

$H$ is convex and continuous and by the maximum principle of H. Bauer ([6], Theorem 25.9) there is a $f_0 \in \mathrm{ex}\,\Omega(A)$ such that

$$|f_0(x)| = \max_{f \in \Omega(A)} |f(x)| \geq \|\hat{x}\|_\infty,$$

hence

$$|f_0(x)| = \|\hat{x}\|_\infty.$$

But with $N := c\big(j(f_0)\big)$ we have by definition

$$|f_0(x) = |\hat{x}(N)|,$$

and $S''$ is a boundary of $A$. By (i) and (ii) $S''$ is contained in any boundary of $A$ and this proves the assertion of (iii). ∎

Let $A$ be a real, commutative Banach* algebra with symmetric involution. Each $f \in \Omega(A)$ is positive and hermitian. The fact that $f$ is a Schwarz functional can be proved with Theorem 2.4. It would be interesting to give a direct and elementary proof.

LEMMA 2.5. *Let $A$ be a real, commutative Banach* algebra. Then $\Omega(A) \subseteq P_S^1$, i.e. each spectral state is a Schwarz functional.*

PROOF. By Theorem 2.4 the extreme points of $\Omega(A)$ are multiplicative. Let $h \in \operatorname{ex}\Omega(A)$. $h$ is positive, hermitian and, since multiplicative, a Schwarz functional. Lemma 1.2 ensures that each $h \in \operatorname{co}\big(\operatorname{ex}\Omega(A)\big)$ satisfies the Cauchy-Schwarz inequality. But since $\Omega(A)$ is the closure of $\operatorname{co}\big(\operatorname{ex}\Omega(A)\big)$ in the weak topology $\sigma\big(F(A,\mathbb{C}),A\big)$, each spectral state satisfies the Cauchy-Schwarz inequality.

THEOREM 2.6. *Let $A$ be a real, commutative Banach* algebra with symmetric involution and unit $e$. Then $\Omega(A) = P_S^1$.*

PROOF. Due to Lemma 2.5 we have only to show that $P_S^1 \subseteq \Omega(A)$. By Corollary 1.5 $\operatorname{ex} P_S^1 \subseteq \Phi_A$, hence $\operatorname{ex} P_S^1 \subseteq \Omega(A)$ and by the Krein-Milman theorem

$$P_S^1 = \overline{\operatorname{co}\operatorname{ex} P_S^1} \subseteq \Omega(A).$$

REMARK. The result of Theorem 2.6 can be extended to real, commutative Banach* algebras with an isometric and symmetric involution and approximate identity in the same way.

In the last section we consider the extension of positive functionals and spectral states.

**III. Extensions of positive functionals.** Let $A$ be a real, commutative Banach* algebra with isometric involution and bounded approximate identity, and $A_0$ a closed *subalgebra of $A$. With $P_S^1(A)$ (resp $M_S(A)$) we indicate that the sets $P_S^1$ and $M_S$ are over $A$.

Then the following extension result holds.

THEOREM 3.1. *Let $A$ be a real, commutative Banach* algebra with isometric involution and bounded approximate identity $(u_\lambda)$ and $A_0$ be a closed *subalgebra of $A$ containing $(u_\lambda)$. A positive Schwarz functional on $A_0$ can be extended to a positive Schwarz functional on $A$ if and only if a multiplicative, positive and weak-hermitian functional on $A_0$ can be extended to a multiplicative, positive and weak hermitian functional on $A$.*

PROOF. First let us suppose that each $h \in M_S(A_0)$ has an extension $H \in M_S(A)$. By Theorem 1.3 $\operatorname{ex}\big(P_S^1(A_0)\big) = M_S(A_0)$ and by the Krein-Milman theorem

$$P_S^1(A_0) = \overline{\operatorname{co} M_S(A_0)}.$$

Let $f \in P_S^1(A_0)$ and $h_\alpha \in \operatorname{co} M_S(A_0)$ with $f = \lim_\alpha h_\alpha$ (pointwise convergence). Define $F := \lim_\alpha H_\alpha$ with $H_\alpha \in M_S(A)$ and $H_\alpha/_{A_0} = h_\alpha$. Then $F \in P_S^1(A)$ is an extension of $f$.

The reverse conclusion can be proved as in ([13], Corollary p. 503). ∎

Theorem 3.1 generalizes the well-known result in the complex case [13]. The method given in the proof of Theorem 3.1 can also be used to prove extension properties of spectral states.

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THEOREM 3.2. *Let $A$ be a real, commutative Banach algebra with identity $e$ and $A_0$ a closed subalgebra of $A$ with $e \in A_0$. A spectral state on $A_0$ can be extended as a spectral state to $A$ if and only if a maximal ideal in the Shilov boundary of $A_0$ can be embedded into a maximal ideal in the Shilov boundary of $A$.*

PROOF. Let $S$ (resp. $S_0$) be the Shilov boundary of $A$ (resp. $A_0$), $S' = c^{-1}(S)/_A$ and $S_0' = c^{-1}(S_0)/_{A_0}$. By Rickart ([15], p. 33) we have $\operatorname{co}\big(\sigma_A(x)\big) = \operatorname{co}\big(\sigma_{A_0}(x)\big)$ for all $x \in A_0$ $(*)$.

Define for $h \in S_0'$, $X_h := \{g \in \Omega(A) : g/_{A_0} = h\}$.

The same arguments as in the proof of Theorem 3.1 and the property $(*)$ show that $h$ has an extension $H \in S'$ if and only if each $f \in \Omega(A_0)$ has an extension $F \in \Omega(A)$.

But $(c \circ j)(h)$ (resp. $(c \circ j)(H)$) are maximal ideals in $S_0$ (resp. $S$); so the extension property of $S_0'$ and $S'$ implies the extension property for $S_0$ and $S$. Since the reverse conclusion is immediate, the proof is complete. ∎

If $A$ is a complex commutative Banach algebra then each spectral state on $A_0$ admits an extension to a spectral state on $A$ ([14], Theorem 3(i)). With the help of Theorem 2.4 we have the same property in the real case.

LEMMA 3.3. *Let $A$ be a real commutative Banach algebra and $f \in \Omega(A)$. Then $\tilde{f}$ with $\tilde{f}(x+iy) = f(x) + if(y)$ $(x, y \in A)$ is a $\mathbb{C}$-linear functional on $A_{\mathbb{C}}$ with $\tilde{f} \in \Omega(A_{\mathbb{C}})$.*

PROOF. By Theorem 2.4 $f$ is the limit of convex sums of multiplicative functionals and since $\operatorname{co}\sigma(x+iy)$ is compact for all $x, y \in A$, $\tilde{f}(x+iy) \in \operatorname{co}\sigma(x+iy)$. ∎

With Lemma 3.3, Theorem 3.2 and the extension property of complex linear spectral state ([14], Theorem 3(i)) we get

COROLLARY 3.4. *Let $A$ be a real commutative Banach algebra with identity $e$ and $A_0$ a closed subalgebra of $A$ with $e \in A$. Then we have*

*(i) Each spectral state on $A_0$ admits an extension to a spectral state on $A$.*

*(ii) Each maximal ideal in the Shilov boundary of $A_0$ can be extended to a maximal ideal in the Shilov boundary of $A$.*

With some more work Corollary 3.4 can be proved for real commutative Banach algebras without unit. We omit the proof, because it would be the same as in [14].

REMARK. In this paper we considered only commutative algebras. But it seems that the results of Section 1 can be extended to non-commutative, real Banach* algebras if the set of multiplicative functionals is replaced by the set of pure states.

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