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HEWITT REALCOMPACTIFICATIONS OF PRODUCTS

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1. Introduction. The Hewitt realcompactification υX of a completely regular Hausdorff space X has been widely investigated since its introduction by Hewitt [**17**]. An important open question in the theory concerns when the equality $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is valid. Glicksberg [**10**] settled the analogous question in the parallel theory of Stone-Čech compactifications: for infinite spaces X and Y, $\beta(X \times Y) = \beta X \times \beta Y$ if and only if the product $X \times Y$ is pseudocompact. Work of others, notably Comfort [**3**; **4**] and Hager [**13**], makes it seem likely that Glicksberg's theorem has no equally specific analogue for $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. In the absence of such a general result, particular instances may tend to be attacked by *ad hoc* techniques resulting in much duplication of effort. Our goals in this paper are threefold: to present a useful technique for dealing with particular instances, to illustrate the unific nature of the technique, and to pose and partially answer some general questions which are less ambitious than the quest for a Glicksberg analogue.

That the question of when $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is susceptible to attack by both uniform-theoretic and purely topological methods is evidenced in [**14**; **3**; **4**; **5**]. Our technique, introduced here as Proposition 3.3, might best be described as "hybrid" since its statement is topological and its flavour uniform-theoretic. We indicate in § 4 how our technique may be applied to prove the sufficiency of Glicksberg's theorem and the important Comfort-Negrepontis theorem [**5**, Theorem 5.3]. In § 5 we consider three classes of topological spaces which are defined in terms of the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$, feeling that single spaces are more manageable than pairs of spaces. Although we fall short of a complete characterization for each of the classes, we are able to characterize one of them (barring measurable cardinals). Also, through sufficient conditions for membership and examples of members and non-members, we are able to give some indication of the scope of each class. Full knowledge of the membership of each of the three classes would provide much of the general information on the equality $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ that we lack due to the absence of an analogue of Glicksberg's theorem.

2. Preliminaries. All topological spaces discussed in this work are assumed to be completely regular Hausdorff spaces. For a space X, $C(X)$ denotes the ring of all continuous real-valued functions on X and $C^*(X)$ denotes the sub-

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ring of bounded functions. A subset A of X is said to be C-embedded in X if every function in $C(A)$ extends to a function in $C(X)$. C^*-embedding is defined analogously. For a topological space X, βX denotes the Stone-Čech compactification of X, which is characterized as a compact space in which X is densely C^*-embedded. υX denotes the Hewitt realcompactification of X and is characterized as a realcompact space in which X is densely C-embedded. Both of the spaces βX and υX are unique up to a homeomorphism which extends the identity on X. The equality $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is to be interpreted to mean that $X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$. For details see [**9**, especially Chapters 6 and 8].

3. The rectangle condition. In this section we introduce our technique for dealing with Hewitt realcompactifications of products. For economy of verbiage in later results, we begin with some definitions. As is usual, for a non-void subset A of X and a function $f \in C(X)$,

$$(\operatorname{osc} f)(A) = \sup\{|f(x) - f(y)|: x, y \in A\}.$$

3.1. *Definition.* A filter base $\mathscr{F}$ on a topological space X is said to have property Ω if for every f in $C(X)$ and $\epsilon > 0$ there is a set F in $\mathscr{F}$ with $(\operatorname{osc} f)(F) \leqq \epsilon$.

It is easily seen that a filter base $\mathscr{F}$ has property Ω if and only if it is a Cauchy filter base in the uniformity on X generated by $C(X)$.

3.2. *Definition.* A pair of spaces (X, Y) is said to have the rectangle condition if whenever $\mathscr{F}$ is a filter base on X with property Ω and $\mathscr{G}$ is a filter base on Y with property Ω, then the filter base

$$\mathscr{F} \times \mathscr{G} = \{F \times G: F \in \mathscr{F}, G \in \mathscr{G}\}$$

has property Ω on $X \times Y$.

We now present our fundamental tool for dealing with

$$\upsilon(X \times Y) = \upsilon X \times \upsilon Y.$$

3.3. PROPOSITION. *$\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ if and only if the pair (X, Y) satisfies the rectangle condition.*

Proof. (i) *Necessity.* Let $\mathscr{F}$ and $\mathscr{G}$ be filter bases on X and Y, respectively, each having property Ω. Since υX may be viewed (topologically) as the completion of X with respect to the uniformity generated by $C(X)$, there is a point p in υX such that $\mathscr{F}$ converges to p. Similarly, there is a point q in υY such that $\mathscr{G}$ converges to q. Hence, the filter base $\mathscr{F} \times \mathscr{G}$ converges to the point (p, q) of $\upsilon X \times \upsilon Y$. Let f be in $C(X \times Y)$. Then, by hypothesis, f extends continuously to $\bar{f}$ on $\upsilon X \times \upsilon Y$. Thus, the filter base $\bar{f}(\mathscr{F} \times \mathscr{G})$ converges to the point $\bar{f}(p, q)$. Then, $\bar{f}(\mathscr{F} \times \mathscr{G}) = f(\mathscr{F} \times \mathscr{G})$ must be Cauchy with respect to the usual metric on the real line. This implies that for

$\epsilon > 0$, there exists $F \times G$ in $\mathscr{F} \times \mathscr{G}$ such that $(\operatorname{osc} f)(F \times G) \leqq \epsilon$. Hence, $\mathscr{F} \times \mathscr{G}$ has property Ω and (X, Y) satisfies the rectangle condition.

(ii) *Sufficiency.* Let f be in $C(X \times Y)$ and (p, q) in $\upsilon X \times \upsilon Y$. Let $\mathscr{N}(p)$ and $\mathscr{N}(q)$ be the neighbourhood filters of p and q in υX and υY, respectively, and let $\mathscr{F}_p$ be the trace of $\mathscr{N}(p)$ on X and $\mathscr{G}_q$ the trace of $\mathscr{N}(q)$ on Y. Then, $\mathscr{F}_p$ and $\mathscr{G}_q$ have property Ω on X and Y, respectively. Thus, by hypothesis, the filter base $\mathscr{F}_p \times \mathscr{G}_q$ has property Ω. Then, the filter base $f(\mathscr{F}_p \times \mathscr{G}_q)$ is Cauchy with respect to the usual metric on the real line. Thus, there is a real number $\bar{f}(p, q)$ such that $f(\mathscr{F}_p \times \mathscr{G}_q)$ converges to $\bar{f}(p, q)$. This defines an extension $\bar{f}$ of f on $\upsilon X \times \upsilon Y$. Let (p, q) be in $\upsilon X \times \upsilon Y$ and $\epsilon > 0$. Then, there is a set $F \times G$ in $\mathscr{F}_p \times \mathscr{G}_q$ such that

$$f(F \times G) = \bar{f}(F \times G) \subset (\bar{f}(p, q) - \epsilon, \bar{f}(p, q) + \epsilon).$$

Note that $S = (F \times G) \cup \{(p, q)\}$ is a neighbourhood of (p, q) in the space $(X \times Y) \cup \{(p, q)\}$. But,

$$\bar{f}(S) = \bar{f}(F \times G) \cup \{\bar{f}(p, q)\} \subset (\bar{f}(p, q) - \epsilon, \bar{f}(p, q) + \epsilon).$$

Thus, $\bar{f}\,|_{(X \times Y) \cup \{(p,q)\}}$ is continuous. Since $X \times Y$ is dense in $\upsilon X \times \upsilon Y$, we may conclude that $\bar{f}$ is continuous on $\upsilon X \times \upsilon Y$.

Proposition 3.3 admits a "local" form. We begin with another definition.

3.4. *Definition.* A function f in $C(X \times Y)$ is said to be weakly uniformly continuous if whenever $\mathscr{F}$ and $\mathscr{G}$ are filter bases on X and Y, respectively, both having property Ω, then for every $\epsilon > 0$, there exists $F \times G \in \mathscr{F} \times \mathscr{G}$ with $(\operatorname{osc} f)(F \times G) \leqq \epsilon$.

It is not difficult to see that the following result holds.

3.5. Proposition. *A function f in $C(X \times Y)$ is extendable over $\upsilon X \times \upsilon Y$ if and only if f is weakly uniformly continuous.*

For a topological space X, $\mathscr{C}(X)$ denotes the uniformity generated by $C(X)$. Since $\upsilon X \times \upsilon Y$ may be viewed as the completion of $X \times Y$ with respect to the product uniformity $\mathscr{C}(X) \times \mathscr{C}(Y)$ generated by $\mathscr{C}(X)$ and $\mathscr{C}(Y)$, it follows that every function in $C(X \times Y)$ which is uniformly continuous with respect to $\mathscr{C}(X) \times \mathscr{C}(Y)$ is weakly uniformly continuous. On the other hand, as a result of the theorem in [**20**], we know that there is a function f in $C(R \times R)$ (R the real line) such that f is not uniformly continuous with respect to $\mathscr{C}(R) \times \mathscr{C}(R)$; but, every function in $C(R \times R)$ is weakly uniformly continuous since $R \times R = \upsilon(R \times R) = \upsilon R \times \upsilon R$. Finally, every weakly uniformly continuous function on $X \times Y$ is uniformly continuous with respect to $\mathscr{C}(X \times Y)$.

The following result is quoted from [**5**].

3.6. Theorem (Comfort-Negrepontis). *If $X \times Y$ is C^*-embedded in $\upsilon X \times \upsilon Y$, then $X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$.*

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Thus, from 3.5 and 3.6 we immediately obtain the following result.

3.7. COROLLARY. *$\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ if and only if every function f in $C^*(X \times Y)$ is weakly uniformly continuous.*

It is useful for applications to note that a space X is realcompact if and only if every filter base on X with property Ω converges to some point of X.

4. Applications to known results. We now indicate the usefulness of our technique for unifying the theory of Hewitt realcompactifications of products.

The projection π_X from $X \times Y$ onto X is said to be z-closed if for every zero-set Z on $X \times Y$, $\pi_X(Z)$ is a closed subset of X. It is unusual for π_X to be z-closed; in fact, if π_X is z-closed for every space Y, then X is discrete and if π_X is z-closed for all spaces X, then Y is compact (see [**11**, § 2.6 for details]).

An infinite cardinal m is said to be measurable if a set X of cardinality m admits a non-trivial two-valued, countably additive measure defined on all subsets of X. The existence of measurable cardinals is an open question of set theory; however, the class of non-measurable cardinals in known to be closed under the usual cardinal operations and to contain $\aleph_0$. Our present interest in measurable cardinals lies in the fact that a discrete space X is realcompact if and only if card X is non-measurable. Also, a metric space of non-measurable cardinal is realcompact.

4.1. *Definition.* For f in $C^*(X \times Y)$, let

$$\Xi_f\colon X \to C^*(Y) \text{ be defined by } (\Xi_f(x))(y) = f(x, y)$$

and

$$\Xi^f\colon Y \to C^*(X) \text{ be defined by } (\Xi^f(y))(x) = f(x, y).$$

The following result can be found in the literature.

4.2. THEOREM. *The following are equivalent:*
(i) *$X \times Y$ is C^*-embedded in $X \times \beta Y$;*
(ii) *π_X is z-closed;*
(iii) *Ξ_f is continuous for each f in $C^*(X \times Y)$.*

Proof. (i) $\Leftrightarrow$ (ii) is due to Hager and Mrowka and is proved in [**14**, Proposition 6.1].

(ii) $\Rightarrow$ (iii) is due to Tamano and is proved in [**21**].

(i) $\Leftrightarrow$ (iii) is due to Glicksberg and is proved in [**10**].

Theorem 4.2 has a local formulation which is given in 4.3 and proved in [**11**, § 2.4].

4.3. PROPOSITION. *For f in $C^*(X \times Y)$, the following are equivalent:*
(i) *f extends to $X \times \beta Y$;*
(ii) *Ξ_f is continuous.*

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Let f be in $C^*(X \times Y)$ and let $\mathscr{F}$ be a filter base on X which has property Ω. It is natural to wonder if there is an x in X such that the behaviour of the function $\Xi_f(x)$ on a filter base $\mathscr{G}$ on Y determines the behaviour of f on $\mathscr{F} \times \mathscr{G}$. We are motivated to consider the following situation.

4.4. LEMMA. *Let* card X *or* card Y *be non-measurable and let f be in* $C^*(X \times Y)$. *If either of the functions Ξ_f or Ξ^f is continuous, then f is weakly uniformly continuous.*

Proof. Let $\mathscr{F}$ and $\mathscr{G}$ be filter bases on X and Y, respectively, both having property Ω. We assume for definiteness that Ξ_f is continuous. Let $\epsilon > 0$. Let $M = \Xi_f(X)$. Since card $M \leqq \min\{\text{card } X, 2^{\aleph_0 \text{card } Y}\}$, card M is non-measurable (any cardinal smaller than a non-measurable cardinal is non-measurable [**9**, § 12.5]). Thus, the metric space M is realcompact. Now, the filter base $\Xi_f(\mathscr{F})$ has property Ω on M since Ξ_f is continuous. Thus, there is a point x_0 in X such that the filter base $\Xi_f(\mathscr{F})$ converges to the function $\Xi_f(x_0)$. Let $\|\cdots\|$ denote the usual sup-norm on $C^*(Y)$ and let

$$U = \{\Xi_f(x) \in M \colon \|\Xi_f(x) - \Xi_f(x_0)\| < \epsilon/3\}.$$

Then, there is a set F in $\mathscr{F}$ such that $\Xi_f(F) \subset U$. Thus, for x in F,

$$\sup_{y \in Y} |f(x, y) - f(x_0, y)| < \epsilon/3.$$

Choose G in $\mathscr{G}$ such that $(\text{osc } \Xi_f(x_0))(G) \leqq \epsilon/3$. Then, for points (x_1, y_1) and (x_2, y_2) in $F \times G$,

$$\begin{aligned} |f(x_1, y_1) - f(x_2, y_2)| &\leqq |f(x_1, y_1) - f(x_0, y_1)| \\ &\quad + |f(x_0, y_1) - f(x_0, y_2)| + |f(x_0, y_2) - f(x_2, y_2)| \\ &\leqq \|\Xi_f(x_1) - \Xi_f(x_0)\| + (\text{osc } \Xi_f(x_0))(G) + \|\Xi_f(x_0) - \Xi_f(x_2)\| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Thus, $(\text{osc} f)(F \times G) \leqq \epsilon$. Hence, f is weakly uniformly continuous.

The next result is immediate from 3.7, 4.3, and 4.4.

4.5. THEOREM (Comfort-Negrepontis). *If* card X *or* card Y *is non-measurable and $X \times Y$ is C^*-embedded in $X \times \beta Y$, then $v(X \times Y) = vX \times vY$.*

Theorem 4.5 was announced in [**5**] in a slightly different form and has been the source of many of the known sufficient conditions for the relation $v(X \times Y) = vX \times vY$ to hold (see, for example, [**3**; **4**]). The asymmetry of the statement of 4.5 is only apparent. For, suppose that card X or card Y is non-measurable and that for each f in $C^*(X \times Y)$ at least one of the functions Ξ_f or Ξ^f is continuous. Then, by 4.4 and 3.7, $v(X \times Y) = vX \times vY$. Let A denote the subalgebra of $C^*(X \times Y)$ consisting of those functions which extend to $\beta X \times Y$ and let B denote the subalgebra of functions extendable to $X \times \beta Y$. By 4.3, for this particular pair (X, Y), $C^*(X \times Y)$ is equal

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to the set-theoretic union of A and B. It is well known that an algebra cannot be realized as the set-theoretic union of two proper subalgebras. Hence, $A = C^*(X \times Y)$ or $B = C^*(X \times Y)$ and we find ourselves back in the context of 4.5.

Part (iii) of the following is known from [**5**], but the proof presented here is new.

4.6. PROPOSITION. *Let D be a discrete space of measurable cardinal. Let Y be any of the spaces*

(i) D,

(ii) υD,

(iii) βD,

Then, the relation $\upsilon(D \times Y) = \upsilon D \times \upsilon Y$ fails.

Proof. Since D has measurable cardinal, there is a filter base $\mathscr{F}$ on D such that $\mathscr{F}$ has property Ω and $\mathscr{F}$ is free (i.e. $\bigcap \mathscr{F} = \emptyset$). Then, $\mathscr{F}$ has Ω on Y since $D \subset Y$. Let $\Delta = \{(d, d): d \in D\}$. Then, Δ is open and closed in $D \times Y$. Define f in $C^*(D \times Y)$ by $f(\Delta) = \{1\}$ and $f(D \times Y \backslash \Delta) = \{0\}$. Let F_1 and F_2 belong to $\mathscr{F}$ and let F be in $\mathscr{F}$ such that $F \subset F_1 \cap F_2$. Then, $F \times F \subset F_1 \times F_2$. Let p be a point in F. Since $\mathscr{F}$ is free, there is a point q in F such that $q \neq p$. Then, the points (p, p) and (p, q) both belong to $F \times F$. But,

$$|f(p, p) - f(p, q)| = 1.$$

Thus, $\mathscr{F} \times \mathscr{F}$ does not have property Ω. Hence, $\upsilon(D \times Y) \neq \upsilon D \times \upsilon Y$.

Thus, we see that the cardinality conditions imposed in 4.4 and 4.5 are necessary.

We continue with a theorem of Comfort which appears in [**4**]. The proof we give here is no shorter than that offered in [**4**], but is presented here because of its consistency with the spirit of our preceding results.

4.7. THEOREM. *Let X be a locally compact realcompact space with the property that each point of X admits a neighbourhood with non-measurable cardinal. Then, $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for every space Y.*

Proof. Let $\mathscr{F}$ and $\mathscr{G}$ be filter bases on X and Y, respectively, both having property Ω. Since X is realcompact, there is a point x_0 in X such that $\mathscr{F}$ converges to x_0. By hypothesis, x_0 admits a compact neighbourhood K which has non-measurable cardinal. Let $\mathscr{U} = \{F \in \mathscr{F}: F \subset K\}$. Then, $\mathscr{U}$ is a filter base on K with property Ω (K is C-embedded in X [**9**, § 3.11]). By 4.5, $\mathscr{U} \times \mathscr{G}$ has property Ω on $K \times Y$, and hence on $X \times Y$. Thus, $\mathscr{F} \times \mathscr{G}$ has property Ω on $X \times Y$. Then, $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$.

We turn now to the sufficiency of Glicksberg's theorem. We shall need the following result of Frolík which appears in [**7**], to which the reader is referred for proof.

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4.8. LEMMA (Frolík). *If $X \times Y$ is pseudocompact, then for $f \in C^*(X \times Y)$, the function*

$$F(x) = \sup_{y \in Y} f(x, y)$$

is continuous on X.

The following result places us in the context of 4.4.

4.9. LEMMA. *If $X \times Y$ is pseudocompact, then for $f \in C^*(X \times Y)$, Ξ_f is continuous.*

Proof. Let $x_0 \in X$ and $\epsilon > 0$. Define g on $X \times Y$ by

$$g(x, y) = |f(x, y) - f(x_0, y)|.$$

Then $g \in C^*(X \times Y)$. Hence, the function

$$G(x) = \sup_{y \in Y} g(x, y)$$

is continuous on X by 4.8. Thus, there is a neighbourhood U of x_0 such that $x \in U$ implies that $|G(x) - G(x_0)| < \epsilon$. But,

$$\begin{aligned} |G(x) - G(x_0)| &= \left| \sup_{y \in Y} g(x, y) - \sup_{y \in Y} g(x_0, y) \right| \\ &= \left| \sup_{y \in Y} |f(x, y) - f(x_0, y)| - \sup_{y \in Y} |f(x_0, y) - f(x_0, y)| \right| \\ &= \left| \sup_{y \in Y} |f(x, y) - f(x_0, y)| \right| \\ &= ||\Xi_f(x) - \Xi_f(x_0)||. \end{aligned}$$

Thus, Ξ_f is continuous at x_0.

A study of the proof of 4.4 shows that the critical fact needed to conclude that f is weakly uniformly continuous is that the metric space $\Xi_f(X)$ is realcompact. It is well known that a pseudocompact metric space is compact and hence realcompact. Hence, if X is pseudocompact and Ξ_f is continuous, then $\Xi_f(X)$ is realcompact. These remarks together with 4.9 and the fact that $\upsilon X = \beta X$ if X is pseudocompact yield the following result immediately.

4.10. THEOREM (Glicksberg). *If $X \times Y$ is pseudocompact, then*

$$\beta(X \times Y) = \beta X \times \beta Y.$$

5. The classes $\mathscr{R}$, $\mathscr{M}$, and $\mathscr{P}$. In this section we consider three classes of spaces which are defined in terms of the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$, and we show by example that these classes are distinct. In passing, we mention analogous classes defined by the relation $\beta(X \times Y) = \beta X \times \beta Y$. Recall from [**10**] that $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact or X or Y is finite.

5.1. *Definition.* Let $\mathscr{R}$ denote the class of all spaces X such that for every space Y, $v(X \times Y) = vX \times vY$. Let $\mathscr{R}^*$ denote the class of all spaces X such that for every space Y, $\beta(X \times Y) = \beta X \times \beta Y$.

An easy application of Glicksberg's theorem shows that the class $\mathscr{R}^*$ consists precisely of the finite spaces. From 4.7 we have that if X is locally compact realcompact of non-measurable cardinal, then X is a member of $\mathscr{R}$. We have not been able to completely characterize the class $\mathscr{R}$; but at least we have been able to prove the following.

5.2. THEOREM. *If X is a member of $\mathscr{R}$, then X is realcompact.*

Proof. Suppose that X is not realcompact. Then, there is a point p in the set $vX \backslash X$. Let $\mathscr{N}(p)$ denote the filter base of open (in vX) neighbourhoods of p. Let q be an ideal point and set $Y = \mathscr{N}(p) \cup \{q\}$ with topology defined by the rules:

(i) members of $\mathscr{N}(p)$ are isolated points of Y,

(ii) for a set U in $\mathscr{N}(p)$, the set $\{q\} \cup \{V \in \mathscr{N}(p) \colon V \subset U\}$ is a neighbourhood of the point q.

For a set U in $\mathscr{N}(p)$, let $f_U \colon vX \to [0, 1]$ be continuous and satisfy $f_U(p) = 0$ and $f_U(vX \backslash U) \subset \{1\}$. For (x, y) in $X \times Y$, let

$$f(x, y) = \begin{cases} 1, & \text{if } y = q, \\ f_U(x), & \text{if } y = U. \end{cases}$$

Then, f is continuous on $X \times Y$. Let $\mathscr{F}$ denote the trace of $\mathscr{N}(p)$ on X (i.e. $\mathscr{F} = \{U \cap X \colon U \in \mathscr{N}(p)\}$). Then, the filter base $\mathscr{F}$ has property Ω on X since X is C-embedded in vX and $\mathscr{N}(p)$ has property Ω on vX. Let $\mathscr{G}$ be the neighbourhood filter of the point q in Y. Then, $\mathscr{G}$ has property Ω on Y. Let F be a set in $\mathscr{F}$ and G a set in $\mathscr{G}$. Choose U in $\mathscr{N}(p)$ satisfying:

(i) $U \cap X \subset F$,

(ii) $\{q\} \cup \{V \in \mathscr{N}(p) \colon V \subset U\} \subset G$.

Choose V_1, V_2 in $\mathscr{N}(p)$ satisfying $V_1 \subset \mathrm{cl}_{vX} V_1 \subset V_2 \subset U$, $\mathrm{cl}_{vX} V_1 \neq V_2$. Pick x_1 in $V_1 \cap X$ such that $f_{V_1}(x_1) < 1/2$ and pick x_2 in $(V_2 \backslash \mathrm{cl}_{vX} V_1) \cap X$. Then, we have that x_1 and x_2 are points of F and V_1 is a point of G. Then, the points (x_1, V_1) and (x_2, V_1) both lie in the set $F \times G$. But,

$$|f(x_2, V_1) - f(x_1, V_1)| = |f_{V_1}(x_2) - f_{V_1}(x_1)| > |1 - 1/2| = 1/2.$$

Thus, $\mathscr{F} \times \mathscr{G}$ does not have property Ω on $X \times Y$. Thus

$$v(X \times Y) \neq vX \times vY$$

and X is not a member of $\mathscr{R}$.

A space similar to the space Y constructed in the proof of 5.2 is used in **[14]** to show that if X is not compact, then for some space Y, the projection π_Y is not z-closed.

We suspect that if card X is non-measurable and X is a member of the class $\mathscr{R}$ then X is necessarily locally compact, but we have not yet been able to prove (or disprove) this conjecture. In the following example we see that even a countable space (hence Lindelöf and hence realcompact [**9**, § 8.2]) may fail to be a member of $\mathscr{R}$.

5.3. *Example.* Let N denote the discrete space of positive integers. Let $T = N \cup \{p\}$, with p in $\beta N \backslash N$, and $S = \beta N \backslash \{p\}$. Now T is countable and S is pseudocompact (as is well known). Now let $f = 1$ on $\Delta = \{(n, n): n \in N\}$, $f = 0$ elsewhere on $T \times S$. Since Δ is open and closed in $T \times S$, f is continuous on $T \times S$. To see that f cannot extend continuously to (p, p) in $\upsilon T \times \upsilon S = T \times \beta N$, notice that (p, p) is in $\operatorname{cl} f^{-1}(0) \cap \operatorname{cl} f^{-1}(1)$. Thus,

$$\upsilon(T \times S) \neq \upsilon T \times \upsilon S.$$

It is easy to see that the class $\mathscr{R}$ is closed under finite products. Note that the space N is a member of $\mathscr{R}$ and the space T of 5.3 is a continuous one-to-one image of N; thus, $\mathscr{R}$ is not closed under continuous mappings.

5.4. *Definition.* Let $\mathscr{M}$ denote the class of all spaces X such that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for every realcompact space Y. Let $\mathscr{M}^*$ denote the class of all spaces X such that $\beta(X \times Y) = \beta X \times \beta Y$ for every compact space Y.

Glicksberg's theorem characterizes $\mathscr{M}^*$ as precisely the class of pseudocompact spaces. It is a trivial observation that $\mathscr{R} \subset \mathscr{M}$, but the space T of 5.3 provides an example of a space in $\mathscr{M}$ which is not in $\mathscr{R}$. With a restriction on cardinalities, we have been able to characterize $\mathscr{M}$.

5.5. THEOREM. *If* card X *is non-measurable, then* X *is in* $\mathscr{M}$ *if and only if* X *is realcompact.*

Proof. (i) If X is realcompact, then clearly X is in $\mathscr{M}$.

(ii) Suppose that X is not realcompact. Let Y be the space constructed in 5.2. Since

$$\operatorname{card} Y \leqq 2^{\operatorname{card} \upsilon X} \leqq 2^{\operatorname{card} \beta X} \leqq 2^{2^{2^{\operatorname{card} X}}} \text{ (see [9, § 8.16])},$$

card Y is non-measurable. Thus, the discrete space $\mathscr{N}(p) \subset Y$ is realcompact. Then, Y is the union of a realcompact space together with the compact space $\{q\}$. Hence, by [**9**, § 8.16], Y is realcompact. But, as shown in 5.2,

$$\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y.$$

Thus, X is not in $\mathscr{M}$.

5.6. *Definition.* Let $\mathscr{P}$ denote the class of all spaces X such that $\upsilon(X \times \upsilon X) = \upsilon X \times \upsilon X$. Let $\mathscr{P}^*$ denote the class of all spaces X such that $\beta(X \times \beta X) = \beta X \times \beta X$.

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Glicksberg's theorem tells us that $\mathscr{P}^*$ is precisely the class of pseudocompact spaces. Hence, $\mathscr{P}^* = \mathscr{M}^*$. Obviously, $\mathscr{M} \subset \mathscr{P}$. It is also clear that $\mathscr{P}$ contains all of the pseudocompact spaces. Thus, the space S of 5.3 is a member of $\mathscr{P}$ but is not in $\mathscr{M}$. One might be tempted to call the members of $\mathscr{P}$ *realpseudocompact* spaces. If D is discrete with measurable cardinal, then D is not in $\mathscr{P}$ by 4.6. We now discuss some spaces that are members of $\mathscr{P}$.

Following [**11**], we say that a pair of spaces (X, Y) has condition Σ if

$$X = \bigcup_{n \in N} X_n, \qquad Y = \bigcup_{n \in N} Y_n,$$

each $X_n \times Y_n$ is pseudocompact, each X_n is completely separated from $X \backslash X_{n+1}$, and each Y_n is completely separated from $Y \backslash Y_{n+1}$. Hager [**11**] has shown that if (X, Y) has condition Σ, then $v(X \times Y) = vX \times vY$. If $X = \bigcup_{n \in N} X_n$ with each X_n pseudocompact and completely separated from $X \backslash X_{n+1}$, it is easily seen that (X, vX) has condition Σ. Thus, the following result holds.

5.7. PROPOSITION. *Suppose that $X = \bigcup_{n \in N} X_n$ with each X_n pseudocompact and completely separated from $X \backslash X_{n+1}$. Then, X is a member of $\mathscr{P}$.*

A space X is said to be the sum of its subsets $A_1, A_2, \ldots, A_n$, and is written

$$X = A_1 + A_2 + \ldots + A_n = \sum_{i=1}^{n} A_i$$

if $X = \bigcup_{i=1}^{n} A_i$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and each A_i is open in X. If $X = \sum_{i=1}^{n} A_i$ and $Y = \sum_{j=1}^{m} B_j$, it is easy to see that

$$X \times Y = \sum_{i,j} (A_i \times B_j).$$

Equally easy is that

$$v\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} vA_i.$$

Thus, if $X = \sum_{i=1}^{n} A_i$ and $Y = \sum_{j=1}^{m} B_j$, then $v(X \times Y) = vX \times vY$ if and only if $v(A_i \times B_j) = vA_i \times vB_j$ for all $1 \leqq i \leqq n$ and $1 \leqq j \leqq m$. Hence, for the spaces S and T of 5.3 it is seen that the space $S + T$ (see 5.8 below for the definition of this construction) is not a member of $\mathscr{P}$. We now give an example of a non-pseudocompact, non-realcompact space X which does not fall under the hypotheses of 5.7 such that X is in the class $\mathscr{P}$.

5.8. *Example.* Let A be any pseudocompact, non-compact space (the space S of 5.3 for example). Then, A is not realcompact [**9**, 5H]. Let B be any locally compact, realcompact space of non-measurable cardinal which is not σ-compact (e.g. the discrete space of cardinality $2^{\aleph_0}$). Let $X = A + B$ (we can construct this sum "externally" by taking A and B to be disjoint, letting $X = A \cup B$, and topologizing X by stipulating that a subset D of X is open

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if and only if the set $D \cap A$ is open in A and the set $D \cap B$ is open in B). Then, X is not realcompact since A is a closed subset which is not realcompact [**9**, § 8.10]. Also, X is not pseudocompact since B is not pseudocompact and is C-embedded in X. It also follows easily that X does not satisfy the hypotheses of 5.7. However,

$$\begin{aligned} \upsilon(X \times \upsilon X) &= \upsilon((A + B) \times \upsilon(A + B)) \\ &= \upsilon((A \times \beta A) + (A \times B) + (B \times \beta A) + (B \times B)) \\ &= \upsilon(A \times \beta A) + \upsilon(A \times B) + \upsilon(B \times \beta A) + \upsilon(B \times B) \\ &= (\beta A \times \beta A) + (\beta A \times B) + (B \times \beta A) + (B \times B) \\ &= \upsilon X \times \upsilon X. \end{aligned}$$

Thus, X is in the class $\mathscr{P}$.

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