# WEIERSTRASS POINTS ON RATIONAL NODAL CURVES

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(Received 12 December, 1985)

**Introduction.** C. Widland [14] has defined Weierstrass points on integral, projective Gorenstein curves. We show here that the Weierstrass points on a generic integral rational nodal curve have the minimal possible weights or, equivalently, that such a curve has the maximum possible number of distinct nonsingular Weierstrass points. Rational curves with $g$ nodes arise in degeneration arguments involving smooth curves of genus $g$ and they have also recently arisen in connection with $g$-soliton solutions to certain nonlinear partial differential equations [11], [13].

In the first section, we review Widland's definition and main results. Singularities are always Weierstrass points and have high Weierstrass weight. In the second section, we consider the set of smooth Weierstrass points of a family of Gorenstein curves. The situation is then analogous to the case of smooth curves as considered in our previous articles [8], [9]. In the final section, we present our results concerning Weierstrass points on rational nodal curves. Theorems 3 and 4 are similar to the main results in [8] and [9]. A key difference in the proofs is that we no longer have available the variational formula of Schiffer-Spencer-Rauch which specifies how holomorphic differentials vary as a smooth curve is deformed to another smooth curve. However, in the case of rational nodal curves we may explicitly compute how dualizing differentials vary as the curve is deformed to another rational nodal curve. In Theorem 5, we show that nodes on a generic rational nodal curve have the minimum possible Weierstrass weight. A key point in the argument is the fact that the minimum weight of a node is greater than the maximum weight of a smooth point.

It appears that Weierstrass points on nodal curves share many similarities with Weierstrass points on smooth curves. While Weierstrass points on a generic rational nodal curve all have the minimal possible weights, there are variations in the combinations of Weierstrass weights on special curves. For example, on rational curves with three nodes, there are at least 13 different combinations of Weierstrass weights possible [10]. This contrasts sharply with the case of cuspidal curves; indeed, Widland showed that on any rational curve with $g$ cusps, each cusp has Weierstrass weight $(g-1)(g+1)$ and there are no nonsingular Weierstrass points.

We work over an algebraically closed field $k$ of characteristic 0. By a "curve", we will mean a one-dimensional, *integral* projective $k$-scheme. A "point" of a scheme will refer to a closed point. By an abuse of language, the term "minor" of a matrix will refer to a square submatrix as well as to the determinant of that submatrix. If $X$ is a scheme, then $|X|$ will denote the underlying point set of $X$. We usually identify a vector bundle on a scheme with its locally free sheaf of sections.

We thank Bill Hoffman, Augusto Nobile, and Carl Widland for helpful conversations and Steven Kleiman for helpful correspondence.

**1. Widland's definition.** Let $X$ be a Gorenstein curve of arithmetic genus $g > 0$ and let $\omega$ denote the sheaf of dualizing differentials on $X$. Let $\sigma, \sigma_1, \ldots, \sigma_g$ be nonzero rational differentials on $X$. Widland defines the Wronskian of $\sigma_1, \ldots, \sigma_g$ with respect to $\sigma$, denoted $W_\sigma(\sigma_1, \ldots, \sigma_g)$, by

$$W_\sigma(\sigma_1, \ldots, \sigma_g) = \det[F_{ij}] \quad \text{for} \quad 1 \le i, j \le g,$$

where the $F_{ij}$ are rational functions defined by

$$\sigma_j = F_{1j}\sigma \quad \text{for} \quad j = 1, \ldots, g,$$

and for $i > 1$, the $F_{ij}$ are defined recursively by the formula

$$dF_{i-1,j} = F_{ij}\sigma \quad \text{for} \quad j = 1, \ldots, g.$$

Now suppose that $\sigma_1, \ldots, \sigma_g$ are a basis for $H^0(X, \omega)$. Define a section $\alpha \in H^0(X, \omega^{\otimes N})$, where $N = 1 + \ldots + g$, as follows. Suppose that $\{U_i; i \in I\}$ is an open cover of $X$ such that for each $i \in I$, $\Gamma(U_i, \omega)$ is a free rank one $\Gamma(U_i, \mathcal{O}_X)$-module with generator $\tau_i$. For each $i \in I$, define $\alpha_i \in \Gamma(U_i, \omega^{\otimes N})$ by

$$\alpha_i = W_{\tau_i}(\sigma_1, \ldots, \sigma_g)\tau_i^{\otimes N}.$$

It is not hard to see that the $\alpha_i$'s patch to give a section $\alpha \in H^0(X, \omega^{\otimes N})$.

For $P \in X$, let $\sigma$ generate $\omega_P$ and write $\alpha_P = f\sigma^{\otimes N}$, where $f \in \mathcal{O}_P$.

Definition. The *Weierstrass weight* of $P$, denoted $W(P)$, is defined by $W(P) = \operatorname{ord}_P f = \dim_k \mathcal{O}_P/(f)$. The point $P$ is called a *Weierstrass point* if $W(P) > 0$.

It is easy to see that $W(P)$ is independent of the choice of $\sigma$ and of the choice of the basis of $H^0(X, \omega)$. A computation of the degree of $\omega^{\otimes N}$ shows that the total of the Weierstrass weights of all points on $X$ is $g(g-1)(g+1)$.

As evidence that this definition is an appropriate generalization of the notion of Weierstrass point to the singular case, Widland proves the following key result.

Theorem 1. *Suppose $X$ is an integral, projective Gorenstein curve of arithmetic genus $g > 1$ and suppose $P \in X$. Then the following statements are equivalent.*

(1) $W(P) > 0$.

(2) *There is a nonzero $\sigma \in H^0(X, \omega)$ satisfying $\operatorname{ord}_P \sigma \ge g$.*

(3) *There is a* 1*-special subscheme with support $P$ and length equal to $g$.*

(4) *There is a* 1*-special subscheme with support $P$ and length at most $g$.*

The concept of an $r$-special subscheme was introduced by Kleiman [**6**]. In the case of a nonsingular curve, a multiple of a Weierstrass point is a special divisor. In the singular case, it becomes necessary to replace "special divisor" with "1-special subscheme", since there will now be non-principal subschemes supported at a singularity.

At a smooth point, one may define Weierstrass gaps and the semi-group of non-gaps and prove results completely similar to the classical case. In particular, if the gap sequence at a smooth point $P$ is $1, \gamma_2, \ldots, \gamma_g$ then the Weierstrass weight at $P$ is $\sum_{i=1}^{g} (\gamma_i - i)$.

However, the notions of Weierstrass gaps and semi-group of non-gaps appear not to extend to singular points. If $P$ is a singular point, then the objects of interest are not just the divisors $nP$, but rather all the subschemes supported at $P$.

Set $\delta_P = \text{length}\ (\bar{\mathcal{O}}_P/\mathcal{O}_P)$, where $\bar{\mathcal{O}}_P$ denotes the integral closure of $\mathcal{O}_P$. Then Widland proves the following.

THEOREM 2. $W(P) \geq \delta_P g(g-1)$. *In particular, if* $g > 1$, *then every singular point of* $X$ *is a Weierstrass point.*

We note that this Theorem may be viewed as a generalization of a result of S. Diaz **[1]** who showed that every non-separating node on a stable curve is a limit of Weierstrass points on nearby smooth curves and that the generic non-separating node on a uninodal stable curve is a limit of exactly $g(g-1)$ Weierstrass points on nearby smooth curves.

The fact that singular points must be Weierstrass points becomes clear if one looks more closely at the above Wronskian. Let $\sigma_1, \ldots, \sigma_g$ be a basis for $H^0(X, \omega)$. Suppose that $t$ is a rational function such that the differential $dt$ has order 0 at each point on the normalization of $X$ lying over $P$ and suppose that $h$ is a generator (in $\mathcal{O}_P$) of the conductor of $\mathcal{O}_P$ in $\bar{\mathcal{O}}_P$. Then $\sigma = dt/h$ generates $\omega_P$. Write $\sigma_i = f_i\sigma$ for $i = 1, 2, \ldots, g$. Then it is not hard to see that

$$\begin{aligned} W_\sigma(\sigma_1, \ldots, \sigma_g) &= \det(h^{i-1}f_j^{(i-1)}(t)) \quad \text{for} \quad i, j = 1, \ldots, g \\ &= h^{g(g-1)/2}W_t(f_1, \ldots, f_g), \end{aligned} \tag{1.1}$$

where $W_t(f_1, \ldots, f_g)$ is the usual Wronskian of rational functions. Since the function $h$ vanishes at $P$ if $P$ is a singular point, it is obvious that singularities are Weierstrass points of high weight.

We will need one more of Widland's results.

PROPOSITION 1. *Let $P$ be a node of $X$ with $\theta : Y \to X$ the partial normalization of $X$ at $P$. Put $\theta^{-1}(P) = \{Q_1, Q_2\}$. Then*

$$W(P) = g(g-1) + W(Q_1) + W(Q_2).$$

EXAMPLES

(1) Gorenstein curves of arithmetic genus 0 or 1 have no Weierstrass points.

(2) Suppose $X$ is a rational nodal curve of arithmetic genus 2. Then by Proposition 1 and (1) above, each node has Weierstrass weight 2. If $X$ is obtained from $\mathbb{P}^1_k$ by identifying 0 with $\infty$ and 1 with $b$ then $X$ has two nonsingular Weierstrass points of weight 1 at $\pm\sqrt{b}$.

(3) The situation becomes much more complicated with rational nodal curves of arithmetic genus 3. For example, the curve obtained from $\mathbb{P}^1_{\mathbb{C}}$ by identifying 0 with $\infty$, 1 with $-1$, and $i$ with $-i$, has no nonsingular Weierstrass points. Each node has Weierstrass weight 8.

**2. Spaces of smooth Weierstrass points on families of Gorenstein curves.** Let $S$ be an integral, noetherian scheme over $k$. Let $\pi : \mathscr{X} \to S$ be a family of Gorenstein curves of arithmetic genus $g \geq 2$. By this we mean that $\pi$ is a flat, projective morphism whose geometric fibers are integral Gorenstein curves of arithmetic genus $g$. By the theory of duality of coherent sheaves [**3**, V.9.7 and VII.4], there is a canonical invertible sheaf $\omega_{\mathscr{X}/S}$ on $\mathscr{X}$ whose restriction to a fiber $\mathscr{X}_s$ is the sheaf of dualizing differentials $\omega_{\mathscr{X}_s}$.

By Grauert's Theorem, the sheaf $\pi_*\omega_{\mathscr{X}/S}$ is locally free of rank $g$. Hence the sheaf

$$\mathscr{E} = \pi^*\pi_*\omega_{\mathscr{X}/S}$$

is locally free of rank $g$. The stalk of this sheaf at a point of $\mathscr{X}_s$ is $H^0(\mathscr{X}_s, \omega_{\mathscr{X}_s})$. Let $\iota : \mathscr{X}^0 \to \mathscr{X}$ denote the open subscheme of $\mathscr{X}$ consisting of smooth points and put $\mathscr{E}^0 = \iota^*\mathscr{E}$. As in [**8**] (also see [**12**]), there is a canonical map

$$u_k : \mathscr{E}^0 \to P^k_{\mathscr{X}/S}$$

which takes a section of $\mathscr{E}^0$ to its $k$-th principal part (or $k$-jet along the fiber).

Now assume that $S$ is smooth. If $\tilde{f}_j(s, t)\dfrac{dt}{h(s, t)}$, where $j = 1, \ldots, g$ and $s$ denotes a system of local coordinates on $S$, is a basis for $H^0(\mathscr{X}, \omega_{\mathscr{X}/S})$, then the map $u_k$ is given locally by the matrix

$$[\partial^{i-1}\tilde{f}_j/\partial t^{i-1}] \quad \text{for} \quad i = 0, \ldots, k \quad \text{and} \quad j = 1, \ldots, g.$$

Definition. Put $\mathscr{W}^r_k = Z^r(u_{k-1})$, the scheme where the map $u_{k-1}$ has rank at most $\min(k, g) - r$ (cf.[**7**]). We call $\mathscr{W}_k = \mathscr{W}^1_k$ the scheme of *relative smooth Weierstrass points of order* $k$ of $\mathscr{X}$ over $S$.

We note that we have the following obvious inclusions:

(1) $\emptyset = \mathscr{W}_1 \subseteq \mathscr{W}_2 \subseteq \ldots \subseteq \mathscr{W}_g$,
(2) $\mathscr{W}_g \supseteq \mathscr{W}_{g+1} \supseteq \mathscr{W}_{g+2} \supseteq \ldots \supseteq \mathscr{W}_{2g-1} = \emptyset$,
(3) $\mathscr{W}^{r+1}_k \subseteq \mathscr{W}^r_k$.

Since the notion of Weierstrass gaps at smooth points is virtually unchanged from the classical case, we have

Proposition 2.
(1) *If* $k \le g$, *then* $|\mathscr{W}^r_k| = \{P \in \mathscr{X}^0 \mid$ *in the gap sequence at* $P \in \mathscr{X}_{\pi(P)}$, *there are at least* $r$ *nongaps* $\le k\}$.
(2) *If* $k \ge g$, *then* $|\mathscr{W}^r_k| = \{P \in \mathscr{X}^0 \mid$ *in the gap sequence at* $P \in \mathscr{X}_{\pi(P)}$, *there are at least* $r$ *gaps* $> k\}$.

*Proof.* [**9**, prop. 3].

The following results are also analogous to the nonsingular case (cf. [**8**], [**9**]).

Proposition 3. *Either* $\mathscr{W}^r_k$ *is empty or each component has codimension at most* $r(|k-g| + r)$ *in* $\mathscr{X}^0$.

*Proof.* [**5**], [**2**, ch. 14].

Proposition 4. *Assume* $r > 0$. *Then the points of* $\mathscr{W}^{r+1}_k$ *are singular points of* $\mathscr{W}^r_k$.

*Proof.* The proof of [**7**, prop. **2**], with "analytic space" replaced by "scheme", is valid.

**3. Rational nodal curves.** We now define the family of "all" rational nodal curves of arithmetic genus $g$ and investigate the Weierstrass points of this family. Suppose that $g$ is at least two. Let $\mathscr{U}$ denote the open subscheme of $\mathbb{P}^{2g}$ whose closed points are $2g$-tuples of distinct points of $\mathbb{P}^1$. We will denote a point of $\mathscr{U}$ by $(a_1, b_1, \ldots, a_g, b_g)$.

Put $\mathscr{Y} = \mathscr{U} \times \mathbb{P}^1$. Let $\mathscr{X}$ denote the space obtained from $\mathscr{Y}$ by identifying the points $(a_1, b_1, \ldots, a_g, b_g, a_i)$ and $(a_1, b_1, \ldots, a_g, b_g, b_i)$ for $i = 1, 2, \ldots, g$. Then $\mathscr{X}$ is *a priori* an algebraic space, but it may be seen that $\mathscr{X}$ is in fact a projective scheme over $\mathscr{U}$. This construction is just a relativization over $\mathscr{U}$ of the construction of a single rational nodal curve by identifying pairs of points of $\mathbb{P}^1$. The fiber of $\mathscr{X}$ over a point $(a_1, b_1, \ldots, a_g, b_g)$ is the irreducible rational nodal curve obtained from $\mathbb{P}^1$ by identifying the points $a_i$ with $b_i$ for $i = 1, \ldots, g$.

Let $\pi: \mathscr{X} \to \mathscr{U}$ denote the obvious projection. We note that if $P_i$ is the point of $\mathscr{X}$ obtained by identifying $A_i = (a_i, b_i, \ldots, a_g, b_g, a_i)$ and $B_i = (a_1, b_1, \ldots, a_g, b_g, b_i)$, then the local ring at $P_i$ is

$$\mathcal{O}_{\mathscr{X}, P_i} = \{f \in \mathcal{O}_{\mathscr{Y}, A_i} \cap \mathcal{O}_{\mathscr{Y}, B_i} : \phi_i^*(f) = \psi_i^*(f)\},$$

where $\phi_i$ (resp. $\psi_i$) is the section of $\mathscr{Y} \to \mathscr{U}$ which takes $(a_1, b_1, \ldots, a_g, b_g)$ to $(a_1, b_1, \ldots, a_g, b_g, a_i)$ (resp. $(a_1, b_1, \ldots, a_g, b_g, b_i)$) and the upper star denotes comorphism.

At a singular point, $\mathscr{X}$ is analytically a complete intersection. Indeed, locally at a singular point, $\mathscr{X}$ looks analytically like the intersection of two $(2g+1)$-planes along a $2g$-plane. Hence $\mathscr{X}$ is a Cohen-Macauley scheme. Then $\mathscr{X}$ is flat over $\mathscr{U}$ ([**4**, p. 276]) and since all fibers are Gorenstein curves, we have

PROPOSITION 5. *$\mathscr{X}$ is a family of Gorenstein curves over $\mathscr{U}$.*

Note that $\mathscr{X}$ is not a "universal" family in the usual sense since there are many isomorphic copies of a given curve in the family (e.g. we have not taken the quotient of $\mathscr{X}$ under the obvious actions of the symmetric group or of $PGL(1)$).

We now wish to prove results analogous to the main theorems in [**8**] and [**9**]. We no longer have available the Schiffer-Spencer-Rauch variational formula which we used in the smooth case, but in the case of rational nodal curves we can explicitly compute how relative dualizing differentials vary.

THEOREM 3. *For $2 \le k \le g$, there exists a nonempty open subscheme $S_k$ of $\mathscr{U}$ such that the pullback of $\mathscr{W}_k - \mathscr{W}_k^2$ over $S_k$, if nonempty, is smooth of pure dimension $g + k$.*

*Proof.* Suppose $P \in \mathscr{W}_k - \mathscr{W}_k^2$. Let $\pi(P) = (a_1, b_1, \ldots, a_g, b_g)$. Without loss of generality we may assume that no $a_j$ or $b_j$ is $\infty$. Put $X = \mathscr{X}_{\pi(P)}$. Let $t$ be a local coordinate on $X$ centered at $P$. Then $dt$ generates $\omega_X$ and $\omega_{\mathscr{X}/\mathscr{U}}$ at $P$. A basis for $H^0(X, \omega_X)$ is

$$\sigma_j = \frac{dt}{(t-a_j)(t-b_j)} \quad \text{for} \quad j = 1, \ldots, g.$$

A basis for $H^0(\mathscr{X}, \omega_{\mathscr{X}/\mathscr{U}})$ is

$$\bar{\sigma}_j = \frac{dt}{(t-x_j)(t-y_j)} \quad \text{for} \quad j=1,\ldots,g,$$

where $x_j$ (resp. $y_j$) is a coordinate on $\mathscr{U}$ such that $x_j(a_1, b_1, \ldots, a_g, b_g) = a_j$ (resp. $y_j(a_1, b_1, \ldots, a_g, b_g) = b_j$). For coordinates on $\mathscr{X}$ centered at $P$, we will take $X_1, \ldots, X_g, Y_1, \ldots, Y_g, t$, where $X_j = x_j - a_j$ and $Y_j = y_j - b_j$ for $j = 1, \ldots, g$.

We will proceed as in [**8**] to compute the tangent space to $\mathscr{W}_k$ at $P$ as a subspace of the tangent space to $\mathscr{X}$ at $P$. Let $\xi$ denote a tangent vector to $\mathscr{X}$ at $P$. We view $\xi$ as a $k$-homomorphism of local rings

$$\xi : \mathcal{O}_{\mathscr{X},P} \to k[\varepsilon]/(\varepsilon^2).$$

Put $\xi(t) = u\varepsilon$, $\xi(X_j) = c_j\varepsilon$, and $\xi(Y_j) = d_j\varepsilon$, for $j = 1, \ldots, g$. We think of $c_1, d_1, \ldots, c_g, d_g, u$ as being "coordinates" for the tangent space to $\mathscr{X}$ at $P$. (In differential-geometric terms, one can think of $\xi$ as being the tangent vector $c_1 \dfrac{\partial}{\partial X_1} + d_1 \dfrac{\partial}{\partial Y_1} + \ldots + c_g \dfrac{\partial}{\partial X_g} + d_g \dfrac{\partial}{\partial Y_g} + u \dfrac{\partial}{\partial t}$.)

Put

$$\tilde{f}_j = 1/(t-x_j)(t-y_j) \quad \text{for} \quad j = 1, \ldots, g.$$

Then the matrix of $u_{k-1}$ locally at $P$ is

$$M = [\partial^i \tilde{f}_j / \partial t^i] \qquad i = 0, \ldots, k-1; \quad j = 1, \ldots, g.$$

As in [**8**], $\xi$ will be tangent to $\mathscr{W}_k$ if and only if all minors of order $k$ of the matrix

$$\xi(M) = [\xi(\partial^i \tilde{f}_j / \partial t^i)] \qquad i = 0, \ldots, k-1; \quad j = 1, \ldots, g$$

vanish.

By Taylor's Theorem, we have

$$\xi(\partial^i \tilde{f}_j/\partial t^i) = \partial^i \tilde{f}_j/\partial t^i(P) + \varepsilon u \partial^{i+1}\tilde{f}_j/\partial t^{i+1}(P) + \varepsilon \sum_{l=1}^{g} c_l \partial^{i+1}\tilde{f}_j/\partial X_l \partial t^i(P) + d_l \partial^{i+1}\tilde{f}_j/\partial Y_l \partial t^i(P).$$

Expanding $\tilde{f}_j$ in a power series in $t$ yields

$$\tilde{f}_j = \frac{1}{(Y_j + b_j) - (X_j + a_j)} \sum_{m=0}^{\infty} \left[ \frac{1}{(X_j + a_j)^{m+1}} - \frac{1}{(Y_j + b_j)^{m+1}} \right] t^m.$$

We then get the following expressions for the partial derivatives of $\tilde{f}_j$ evaluated at $P$:

$$\partial^i \tilde{f}_j / \partial t^i(P) = \frac{i!}{(b_j - a_j)(a_j^{i+1} - b_j^{i+1})}$$

$$\partial^{i+1}\tilde{f}_j/\partial X_j \partial t^i(P) = \frac{i!}{(b_j - a_j)(a_j^{i+1} - b_j^{i+1})} \left[ \frac{(i+1)a_j^i}{b_j^{i+1} - a_j^{i+1}} + \frac{1}{b_j - a_j} \right]$$

$$\partial^{i+1}\tilde{f}_j/\partial Y_j \partial t^i(P) = \frac{-i!}{(b_j - a_j)(a_j^{i+1} - b_j^{i+1})} \left[ \frac{(i+1)b_j^i}{b_j^{i+1} - a_j^{i+1}} + \frac{1}{b_j - a_j} \right]$$

$$\partial^{i+1}\tilde{f}_j/\partial X_n \partial t^i = \partial^{i+1}\tilde{f}_j/\partial Y_n \partial t^i = 0 \quad \text{if} \quad j \neq n$$

Now, since $P \in \mathscr{W}_k - \mathscr{W}_k^2$, we may assume, without loss of generality, that the leading minor of $M$ is nonzero. Then, as in [**8**], $\xi$ will be tangent to $\mathscr{W}_k$ if all maximal minors containing the leading minor of $\xi(M)$ vanish. Let $\bar{E}_j = \varepsilon E_j$, for $j = k, k+1, \ldots, g$, denote the maximal minor of $\xi(M)$ consisting of the first $k$ columns and the $j$th column. Our proof will be completed by showing that the equations $E_j = 0$ are linearly independent over a nonempty open subscheme of $\mathscr{U}$.

Note that $c_j$ and $d_j$ only occur in equation $E_j$ for $j = k, k+1, \ldots, g$. Expanding $E_j$ by its last column and using the partial derivatives above, we see that the coefficient of $c_j$ in the equation $E_j = 0$ is

$$\sum_{i=1}^{k} \frac{(-1)^{i+j} |\hat{\imath}|\, i!}{(b_j - a_j)(a_j^{i+1} - b_j^{i+1})} \left[ \frac{(i+1)a_j^i}{b_j^{i+1} - a_j^{i+1}} + \frac{1}{b_j - a_j} \right], \tag{3.1}$$

where $|\hat{\imath}|$ denotes the $(k-1) \times (k-1)$ minor of $M$ obtained by deleting the $i$th row from the first $k-1$ columns. To show the linear independence of the $E_j$ for $j = k, k+1, \ldots, g$ it then suffices to show that (3.1) is nonzero.

Let $F_j$ be the rational function on $\mathscr{U}$ obtained by replacing $a_j$ by $x_j$ and $b_j$ by $y_j$ in (3.1). We claim that $F_j$ is not identically zero. Indeed, if we set $x_j = 0$ then $F_j$ becomes

$$F_j = (-1)^j \left[ \frac{|\hat{1}|\, y_j^{k-1} - 2!\, |\hat{2}|\, y_j^{k-2} + - \ldots + (-1)^k k!\, |\hat{k}|}{y_j^{k+3}} \right].$$

Since $|\hat{k}|$ is nonzero, we see that $F_j$ is not identically zero.

Thus there exist nonempty open subschemes $U_j$ for $j = k, k+1, \ldots, g$ of $\mathscr{U}$ such that the coefficient of $c_j$ in $E_j = 0$ is nonzero over $U_j$. Hence the equations $E_j = 0$ for $j = k, k+1, \ldots, g$ are linearly independent over

$$S_k = \bigcap_{j=k}^{g} U_j.$$

The dimension of the tangent space to $\mathscr{W}_k - \mathscr{W}_k^2$ at any point lying over $S_k$ is then $g + k$, so these points are smooth by Proposition 3.

The following result is much weaker than Theorem 2 of [**9**], but is the best result we have been able to obtain for rational nodal curves with these methods. We will be able to use it to show that nonsingular Weierstrass points on a generic rational nodal curve have weight one.

THEOREM 4. *For $1 \le l \le g-2$, there exists a nonempty open subscheme $T_l$ of $\mathscr{U}$ such that the pullback of $\mathscr{W}_{g+l} - \mathscr{W}_{g+l}^2$ over $T_l$ has dimension less than $2g$.*

*Proof.* Suppose that $P \in \mathscr{W}_{g+l} - \mathscr{W}_{g+l}^2$. Note that $P \in \mathscr{W}_{g+l}$ if and only if there is at least one gap at $P$ greater than $g + l$ and, since $P \notin \mathscr{W}_{g+l}^2$, the $g$th gap at $P$ must be $g + l + 1$. Using Theorem 3, it suffices to prove that the dimension of the tangent space to $\mathscr{W}_{g+l}$ at $P$ is less than $2g$ when the first non-gap in the gap sequence at $P$ is $g$. This gap sequence is then $1, 2, \ldots, g-1, g+l+1$.

Let $\xi$ denote a tangent vector to $\mathscr{X}$ at $P$ as in the proof of Theorem 3. Let $\bar{E}_i = \varepsilon E_i$ for $i = 1, \ldots, l+1$ denote the $g \times g$ minor of $\xi(M)$ consisting of the first $g-1$ rows and the

$(g-1+i)$th row. We will show that the equations $E_1=0$ and $E_2=0$ are linearly independent over a nonempty open subscheme of $\mathscr{U}$.

Let $A_{1,g}$ and $A_{2,g}$ denote the coefficients of $c_g$ in the equations $E_1=0$ and $E_2=0$, respectively and let $B_{1,g}$ and $B_{2,g}$ denote the analogous coefficients of $d_g$, where $\xi(X_g)=\varepsilon c_g$ and $\xi(Y_g)=\varepsilon d_g$. We may compute these coefficients by using the partial derivatives in the proof of Theorem 3. Form the determinant

$$C=\begin{vmatrix} A_{1,g} & B_{1,g} \\ A_{2,g} & B_{2,g} \end{vmatrix}.$$

As in the proof of Theorem 3, replace $a_g$ by $x_g$ and $b_g$ by $y_g$ in $C$, obtaining a rational function on $\mathscr{U}$. We must show that this function is not identically zero. Setting $x_g=0$, this function becomes $C(y_g)=q(y_g)/y^{g+4}$, where $q$ is a polynomial whose constant term may be seen to be $-m^2g!(g+1)!$, where $m$ is the leading minor of the matrix $M$. But this minor is nonzero since the gap sequence at $P$ is $1, 2, \ldots, g-1, g+l+1$. Hence, $C(y_g)$ is not identically zero and there exists a nonempty open subscheme $T_l$ of $\mathscr{U}$ such that the equations $E_1=0$ and $E_2=0$ are linearly independent over $T_l$. Thus the tangent space to $\mathscr{W}_{g+l}$ at $P$ has dimension at most $2g+1-2=2g-1$.

COROLLARY. *There exists a nonempty open subscheme $T_1$ of $\mathscr{U}$ such that the pull-back of $\mathscr{W}_{g+1}-\mathscr{W}^2_{g+1}$ over $T_1$, if nonempty, is smooth of pure dimension $2g-1$.*

THEOREM 5. *There exists a nonempty open subscheme $U$ of $\mathscr{U}$ such that if $(a_1, b_1, \ldots, a_g, b_g)$ is a point of $U$, then every node on the curve $\mathscr{X}_{(a_1,b_1,\ldots,a_g,b_g)}$ has Weierstrass weight $g(g-1)$.*

*Proof.* We consider the Wronskian $\alpha$ of §1 formed from a basis $\sigma_1, \ldots, \sigma_g$ of $H^0(\mathscr{X}, \omega_{\mathscr{X}/\mathscr{U}})$ (the definition in §1 clearly extends to the case of relative dualizing differentials) as a section of the bundle $\omega_{\mathscr{X}/\mathscr{U}}^{\otimes N}$, where $N=g(g+1)/2$. Put

$$Y=\{P\in\mathscr{X}\mid W(P)>g(g-1)\}=\{P\in\mathscr{X}\mid \mathrm{ord}_P\,\alpha>g(g-1)\}.$$

Then $Y\subset\mathscr{X}_{sing}$, since at a smooth point one may show, as in the classical case, that the Weierstrass weight is at most $g(g-1)/2$. In fact, $Y$ is a closed subset of $\mathscr{X}_{sing}$. To see this, note that locally at $P$ the section $\alpha$ may be written (cf. 1.1)

$$\alpha=(\tilde{h}(x_1, y_1, \ldots, x_g, y_g, t))^{g(g-1)/2}W_t(\tilde{f}_1, \ldots, \tilde{f}_g)\left(\frac{dt}{\tilde{h}}\right)^{\otimes N}$$

and it follows that a node $P$ has weight greater than $g(g-1)$ if and only if $W_t(\tilde{f}_1, \ldots, \tilde{f}_g)$ vanishes at a point lying over $P$ on the normalization of $\mathscr{X}$. Then $\pi(Y)$ is a closed subset of $\mathscr{U}$ and it suffices to show that $\pi(Y)\neq\mathscr{U}$.

But this is clear, since one may recursively construct a rational nodal curve of arithmetic genus $g$ such that all the nodes have Weierstrass weight $g(g-1)$ by beginning with $\mathbb{P}^1$ and at each step identifying two points which are not Weierstrass points. By Proposition 1, the resulting nodes will always have the minimum weight.

We note that although it was a trivial matter in the last proof to construct a rational nodal curve such that all the nodes have the minimum Weierstrass weight, there is no guarantee that the nonsingular Weierstrass points on such a curve will all have weight one. We have not been able to construct for every arithmetic genus a rational nodal curve such that all Weierstrass points have the minimal weights (i.e. weight 1 for nonsingular Weierstrass points and weight $g(g-1)$ for nodes). However, we may use the previous results to prove:

THEOREM 6. *On a generic rational nodal curve all Weierstrass points have the minimal possible weights.*

*Proof.* Let $Z$ denote the intersection of the open subschemes $S_k$, $T_l$, and $U$ of the three preceding theorems where $k=2,\ldots,g-1$ and $l=1,\ldots,g-2$. Let $\pi_Z:\mathscr{X}_Z\to Z$ denote the pullback of the family $\mathscr{X}$ to $Z$. Then, by Theorem 5, every node on a curve over $Z$ has Weierstrass weight $g(g-1)$. By Theorems 3 and 4, the constructible sets $\pi_Z(\mathscr{W}_k-\mathscr{W}_k^2)$ and $\pi_Z(\mathscr{W}_{g+l}-\mathscr{W}_{g+l}^2)$ have codimension at least one in $Z$, for $k=2,\ldots,g-1$ and $l=1,\ldots,g-2$. Also, note that if $V$ is any component of $\mathscr{W}_k^2$, for $k=3,\ldots,g$, then there exists $k'<k$ such that $V\subseteq\mathscr{W}_{k'}$ and $V\nsubseteq\mathscr{W}_{k'}^2$. Hence it follows from Theorem 3 that every component of $\mathscr{W}_k^2$ lying over $Z$ has dimension less than $2g$, so cannot dominate $Z$. Similarly, if $V'$ is any component of $\mathscr{W}_{g+l}^2$, for $l=0,\ldots,g-3$, then there exists $l'>l$ such that $V'\subseteq\mathscr{W}_{g+l'}$ and $V'\nsubseteq\mathscr{W}_{g+l'}^2$. Hence it follows from Theorem 4 that every component of $\mathscr{W}_{g+l}^2$ lying over $Z$ has dimension less than $2g$, so cannot dominate $Z$.

Therefore, there exists a dense open subset $Z'$ of $Z$ such that all nonsingular Weierstrass points on curves over $Z'$ have gap sequence $1,2,\ldots,g$ and hence have weight one. Hence, all Weierstrass points on any rational nodal curve lying over a point of the nonempty open subset $Z'$ have the minimal possible weights.

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