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TOPICS IN DIRECT DIFFERENTIAL GEOMETRY

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PREFACE

In the theory of curves, one often makes differentiability assumptions in order that analytic methods can be used. Then one tries to weaken these assumptions as much as possible. The theory of curves which is presented here uses geometric methods, such as central projection, rather than analysis. In this way, no analytic assumptions are needed and a purely geometric theory results. Since this theory is not so well known as the analytic one, I have tried to make the treatment as self-contained as possible. It is hoped that this paper will form a quick introduction for a reader who has had no previous acquaintance with the subject.

We assume that our curves satisfy a condition, which we call direct differentiability. Roughly this condition is that, at each point of the curve, all the osculating spaces exist. In particular, the line through a fixed point and a neighbouring point of the curve tends to a limit, called the tangent at the point. Under central projection, a curve may acquire many types of singularities which are often not admitted in a theory of curves. We do admit such singularities and thus obtain a rather wide class of curves.

In [**7**], a general theory of geometric orders is developed. The geometric order of a curve is the maximum number of points in which a hyperplane can meet the curve. Many of our theorems are related, in some way, to the geometric order of a curve. For example, the tangent of a curve of order n in real projective n-space depends continuously on the point of contact.

Theorems about algebraic curves, which can be given a purely geometric formulation, provide a source of conjectures for our curves. For example, Scherk and Derry conjectured that the k-th rank of a curve of order n is $(k+1)(n-k)$; cf. [**7**, p. 396]. As yet, this has been proved only in special cases.

This paper has developed out of my doctoral thesis *On Barner arcs and curves*, which I wrote at the University of Toronto under the supervision of Professor Peter Scherk. In that thesis, we investigated a condition on curves which Barner, and later Haupt, had studied. In the course of that investigation, arcs with tower were discovered. By giving these arcs with tower a central role in the present work, a number of simplifications have resulted.

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CONTENTS

1. DIRECTLY DIFFERENTIABLE ARCS

In this section, we give the precise definition of the arcs which we shall be studying. More general arcs could be studied; however, the theory would become much more complicated. Many of the proofs in later sections are based on induction by dimension. In all of these proofs, one uses the fact that the projection of an arc from a point—be it on the arc or not—is again an arc; cf. Theorem 1.3.1.

We define the characteristic of a point and the order of an arc. One of the main themes in later sections is the relationship between these two concepts; cf. [**9**].

1.1. Real projective n-space. A *real projective n-space*, $n \geqq 1$, is a set $\mathscr{P}^n$ of objects called spaces along with a 1–1 mapping of $\mathscr{P}^n$ onto the set of all subspaces of a real $(n + 1)$-dimensional vector space. As usual, one defines the inclusion relation $L \subset M$ between spaces, the intersection $L \cap M$ of spaces and the span LM of spaces. For a collection of spaces, the intersection is denoted by $\bigcap_i L_i$ and the span by $\bigvee_i L_i$.

A space of $\mathscr{P}^n$ is said to have dimension k if the vector subspace corresponding to it has dimension $k + 1$. Let $\mathscr{P}_k^n$ denote the set of all k-spaces, $-1 \leqq k \leqq n$. The unique (-1)-space is denoted by $\emptyset$. The elements of $\mathscr{P}_0^n$ and $\mathscr{P}_{n-1}^n$ are called points and hyperplanes of $\mathscr{P}^n$, respectively.

We give $\mathscr{P}^n$ a topology in the usual way. Thus, each $\mathscr{P}_k^n$ is compact and connected, $-1 \leqq k \leqq n$, and every sequence in $\mathscr{P}^n$ has a convergent subsequence. If (L_i) and (M_i) are any two convergent sequences of spaces and $L_i \subset M_i$, for $i = 1, 2, \ldots$, then $\lim L_i \subset \lim M_i$.

If L is a k-space, $-1 \leqq k \leqq n - 2$, then the set $\mathscr{P}^{n-k-1}(L)$ of all spaces containing L is a projective space in a natural way. A sequence in $\mathscr{P}^{n-k-1}(L)$ converges in $\mathscr{P}^{n-k-1}(L)$ if and only if it converges in $\mathscr{P}^n$.

1.2. Direct differentiability. Let J be an ordered topological space which is isomorphic with the ordered topological space of the real numbers. A set $X \subset J$ is called an *interval* if there exist $p, q \in J$ with $p < q$ such that

$$X = (p, q), [p, q], (p, q], \text{ or } [p, q].$$

Thus, $[p, q) = \{r \in J | p \leqq r < q\}$. By a two-sided (deleted, right, left) *neighbourhood* of $p \in J$ we mean a set $U(p) = (q,r)$ containing p

$$(U'(p) = (q, p) \cup (p, r), \quad U^+(p) = (p, r), \quad U^-(p) = (q, p));$$

here, $q < p < r$. If X is a finite subset of J, we write $|X|$ for the number of elements of X.

Let a mapping $A: J \rightarrow \mathscr{P}^n$ and a k-space L be given. For $p \in J$, it may happen that $\lim_{q \to p} A(q)L$, $q \neq p$, exists in which case we denote it by $A(p)|L$; in particular, $A(p)|\emptyset = \lim_{q \to p} A(q)$, $q \neq p$.

The mapping A is said to be *directly differentiable* at $p \in J$ if there exist spaces

$$A_k(p) \in \mathscr{P}_k^n, \ -1 \leqq k \leqq n,$$

such that $A_0(p) = A(p)$ and $A_k(p) = A(p)|A_{k-1}(p)$, $0 \leqq k \leqq n$. Putting $k = 0$, we obtain that A is continuous at p. Putting $k = n$, we obtain that there is a $U'(p)$ such that $A(q) \not\subset A_{n-1}(p)$, if $q \in U'(p)$. If A is directly differentiable at each $p \in J$, then A is a (*directly differentiable*) *arc*. By a *point of an arc* A we mean an element p of J. If A is an arc then $A_k(p)$ is called *the osculating k-space* of A at p.

THEOREM 1.2.1. *Let A be an arc and let L be a hyperplane. For each $p \in J$, there is a $U'(p)$ such that $A(q) \not\subset L$, if $q \in U'(p)$. In particular, if $n \geqq 2$ the image of an arc cannot contain a straight line segment.*

Proof. Given $p \in J$, let $A_k(p)$ be the largest osculating space at p contained in L. If no such $U'(p)$ exists, then there is a sequence $p_i \rightarrow p$ with $p_i \neq p$ and $A(p_i) \subset L$, for all i. But then $A_{k+1}(p) \subset L$, contradicting our definition of $A_k(p)$.

1.3. Projection. The position of a point P relative to the osculating spaces of an arc A at a point p is indicated by $\pi(P, p)$, the dimension of the largest osculating space at p which does not contain P. Thus, $-1 \leqq \pi(P, p) \leqq n - 1$. If $\pi(P, p) = -1$ then $P = A(p)$ and P lies on the arc.

THEOREM 1.3.1. *Suppose that* $n \geqq 2$. *Put*

$$\bar{A}_k(p) = \begin{cases} A_k(p)P, \text{ if } -1 \leqq k \leqq \pi(P, p) \\ A_{k+1}(p), \text{ if } \pi(P, p) \leqq k \leqq n - 1. \end{cases}$$

Then $\bar{A} = \bar{A}_0$ *is an arc in* $\mathscr{P}^{n-1}(P)$ *with* $\bar{A}_k(p)$ *as its osculating* k-*space at* p, $-1 \leqq k \leqq n - 1$.

Proof. One has

$$\bar{A}_k(p) \in \mathscr{P}_k^{n-1}(P), \; -1 \leqq k \leqq n - 1.$$

By Theorem 1.2.1, there is a $U'(p)$ such that $A(q) \neq P$ if $q \in U'(p)$. Thus, $\bar{A}(q) = A(q)P$, for all $q \in U'(p)$.

If $0 \leqq k \leqq \pi(P, p)$, then $P \not\subset A_k(p)$ and

$$\bar{A}(p)|\bar{A}_{k-1}(p) = \lim_{\substack{q \to p \\ q \neq p}} (A(q)P)(A_{k-1}(p)P) = A_k(p)P = \bar{A}_k(p).$$

If $\pi(P, p) < k \leqq n - 1$, then $P \subset A_k(p)$ and

$$\bar{A}(p)|\bar{A}_{k-1}(p) = \lim_{\substack{q \to p \\ q \neq p}} (A(q)P)A_k(p) = A_{k+1}(p) = \bar{A}_k(p).$$

The arc $\bar{A}$ is called the projection of A from P. If $P = A(q)$ we write $\pi(P, p) = \pi(q, p)$, $\bar{A} = A|P = A|q$ and speak of the projection of A from q.

For $L \in \mathscr{P}_k^n$, $-1 \leqq k \leqq n - 2$, the projection $A|L$ of A from L is defined by

$$(A|L)(p) = A(p)|L, \text{ for all } p \in J.$$

THEOREM 1.3.2. $A|L$ *is an arc in* $\mathscr{P}^{n-k-1}(L)$.

Proof. If $k = -1$, $A|L = A$. If $k = 0$, $A|L$ is an arc, by Theorem 1.3.1.

Let $1 \leqq k \leqq n - 2$ and assume that the theorem is true for $k - 1$. Take a point $P \subset L$. Then $A|P$ is an arc in $\mathscr{P}^{n-1}(P)$. Since $L \in \mathscr{P}_{k-1}^{n-1}(P)$, $(A|P)|L$ is an arc in $\mathscr{P}^{n-k-1}(L)$. Now

$$\begin{aligned} ((A|P)|L)(p) &= \lim_{\substack{q \to p \\ q \neq p}} (A(q)P)L \\ &= A(p)|L = (A|L)(p). \end{aligned}$$

1.4. The characteristic of a point. The dimension $\delta(p, L)$ of the largest osculating space of an arc A at a point p which is contained in a k-space L, $-1 \leqq k \leqq n$, will occur frequently. One has $-1 \leqq \delta(p, L) \leqq k$.

LEMMA 1.4.1. *Let P be a point of the k-space L; $\bar{A} = A|P$. Then*

$$\bar{\delta}(p, L) = \begin{cases} \delta(p, L), \text{ if } -1 \leqq \delta(p, L) < \pi(P, p), \\ \delta(p, L) - 1, \text{ if } \pi(P, p) < \delta(p, L) \leqq k. \end{cases}$$

Proof. Use Theorem 1.3.1.

We say that a hyperplane L *supports* A at p if there is a hyperplane H_∞ with $H_\infty \neq L$, $A(p) \not\subset H_\infty$, and a $U'(p)$ such that $A(U'(p))$ is contained in one of the two open half spaces determined by L and H. If L supports A at p, then any hyperplane M with $M \neq L$, $A(p) \not\subset M$, can be taken as H_∞. When L does not support A at p, we say that L *cuts* A at p; cf. Theorem 1.2.1.

LEMMA 1.4.2. (Scherk's lemma.) *Let $-1 \leqq k \leqq n - 1$. Let S_k be the set of all hyperplanes with $\delta(p, L) = k$. Either all hyperplanes of S_k support A at p or all cut A at p.*

Proof. Suppose that L_i, $L \in S_k$, $L_i \to L$. There is a $U'(p)$ such that $A(q) \not\subset L_i$, $A(q) \not\subset L$, for all $q \in U'(p)$. For, otherwise, there exist points q_i with $q_i \to p$, $q_i \neq p$, and integers $j(i)$ with $A(q_i) \subset L_{j(i)}$. Then,

$$A(q_i)A_k(p) \subset L_{j(i)} \quad \text{and} \quad A_{k+1}(p) \subset L,$$

contradicting $L \in S_k$. Hence, if all L_i cut (support) A at p, then L cuts (supports) A at p. Thus, the set of hyperplanes of S_k which cut (support) A at p is a closed subset of S_k. The lemma now follows from the connectedness of S_k.

Let p be a point of an arc. For $-1 \leqq k \leqq n - 1$, define $\sigma_k(p) = 0$ or 1, according as the hyperplanes of S_k support or cut A at p; cf. Lemma 1.4.2. Thus, $\sigma_{-1}(p) = 0$. The *characteristic* $(\alpha_0(p), \ldots, \alpha_{n-1}(p))$ of p is defined by taking $\alpha_i(p)$ to be 1 or 2 and requiring that

$$\alpha_0(p) + \ldots + \alpha_k(p) \equiv \sigma_k(p) \pmod 2, \; 0 \leqq k \leqq n - 1.$$

We also define numbers

$$\beta_k(p) = \sum_{i=0}^{k} \alpha_i(p), \text{ for } -1 \leqq k \leqq n - 1.$$

Thus, $\beta_k(p) \equiv \sigma_k(p) \pmod 2$ and $\alpha_k = 2 - |\sigma_k - \sigma_{k-1}|$, if $0 \leqq k \leqq n - 1$.

THEOREM 1.4.3. *Suppose that $n \geqq 2$. Let P be a point; $\bar{A} = A|P$. Then*

$$\bar{\sigma}_k(p) \equiv \begin{cases} \sigma_0(p) + \sigma_{k+1}(p), \text{ if } P = A(p), \; -1 \leqq k \leqq n - 2 \\ \sigma_k(p), \text{ if } P \neq A(p), \; -1 \leqq k < \pi(P, p) \\ \sigma_{k+1}(p), \text{ if } P \neq A(p), \; \pi(P, p) \leqq k \leqq n - 2, \end{cases}$$

$$\bar{\alpha}_k(p) \equiv \begin{cases} \alpha_k(p), \text{ if } 0 \leqq k < \pi(P, p) \\ \alpha_k(p) + \alpha_{k+1}(p), \text{ if } k = \pi(P, p) \\ \alpha_{k+1}(p), \text{ if } \pi(P, p) < k \leqq n - 2 \end{cases}$$

(mod 2) *and*

$$\bar{\beta}_k(p) = \begin{cases} \beta_{k+1}(p) - \beta_0(p), \text{ if } P = A(p), & -1 \leqq k \leqq n-2 \\ \beta_k(p), \text{ if } P \neq A(p), & -1 \leqq k < \pi(P, p). \end{cases}$$

Proof. Suppose that $P = A(p)$, $-1 \leqq k \leqq n - 2$. Let L and H_∞ be distinct hyperplanes satisfying $\delta(p, L) = k + 1$, $\delta(p, H_\infty) = 0$. Some $U'(p)$ is contained in one of the two open half spaces determined by L and H if and only if $\sigma_0(p) + \sigma_{k+1}(p) \equiv 0$ (mod 2). Hence, $\bar{\sigma}_k(p) \equiv \sigma_0(p) + \sigma_{k+1}(p)$ (mod 2).

Suppose that $P \neq A(p)$, $-1 \leqq k < \pi(P, p)$. Considering distinct hyperplanes L and H_∞ with $P \subset L \cap H_\infty$, $\delta(p, L) = k$, and $\delta(p, H_\infty) = -1$, one has $\bar{\sigma}_k(p) = \sigma_k(p)$.

Suppose that $P \neq A(p)$, $\pi(P, p) \leqq k \leqq n - 2$. Considering hyperplanes L and H_∞ with $P \subset L \cap H_\infty$, $\delta(p, L) = k + 1$, and $\delta(p, H_\infty) = -1$, one has $\bar{\sigma}_k(p) = \sigma_{k+1}(p)$.

The remaining relations now follow.

1.5. Order and rank. Let A be an arc and let X be a subset of J. If the set

$$S(X, L) = \{p \in X | A_k(p) \cap L \neq \emptyset\}$$

is finite for every $(n - k - 1)$-space L, we say that k-th rank of X is *finite*, $0 \leqq k \leqq n - 1$. If, in addition,

$$r = \sup_{L \in \mathscr{P}_{n-k-1}{}^n} |S(X, L)|$$

is finite, we say that X has *bounded* k-th rank r. By the k-th *rank* of A we mean the k-th rank of J. By the *order* of X we mean the 0-rank; this is at least n.

THEOREM 1.5.1. *Let an arc A be given. Any compact set $X \subset J$ is of finite order. In particular, A is locally of finite order.*

Proof. Use Theorem 1.2.1.

2. SECANTS

We define secants in such a way that multiplicities are taken into account. For example, in $\mathscr{P}^3$ the concept of a 2-secant of an arc includes not only a plane which is spanned by three points of the arc, but also a plane which is spanned by a point of the arc and the tangent at another point. A k-secant is called independent if it meets the arc in only $k + 1$ points, multiplicites being included. An arc is k-independent if all its k-secants are independent.

As will be seen in § 3, an arc is $(n - 1)$-independent if and only if it is of order n; cf. Theorem 3.1.1. The condition of $(n - 2)$-independence has been studied for more general arcs than ours by Haupt; cf. [**6**]. For $n = 3$, the condition that an arc be $(n - 2)$-independent is that no line meet the arc in 3 or more points, multiplicities being included.

In this section, we develop some rather formal properties of k-independent arcs.

2.1. Connectivity of secants. Let A be an arc and let L be a k-space, $-1 \leqq k \leqq n$. Then

$$\bigvee_{p \in X} A_{\delta(p,L)}(p) \subset L,$$

for any $X \subset J$. If the inclusion is improper, we say that L is a *k-secant* of X.

THEOREM 2.1.1. *The set of all k-secants of a connected set $X \subset J$ is pathwise connected*, $-1 \leqq k \leqq n$.

Proof. We may assume that $n \geqq 2$, $1 \leqq k \leqq n$, and that the theorem is true for $k - 1$.

Suppose that $p \in X$. By Theorem 1.3.1, a k-space L with $A(p) \subset L$ is a $(k - 1)$-secant of X on $A|p$ if and only if it is a k-secant of X on A. Thus, the set of k-secants of X containing $A(p)$ is pathwise connected.

Let L, M be k-secants of X. Take $p, q \in X$ with $A(p) \subset L$, $A(q) \subset M$ and let N be a k-secant of X containing $A(p)$ and $A(q)$. Then construct a path from L to M by constructing a path from L to N and then from N to M.

2.2. Independence. Let L be a k-secant of $X \subset J$. Then

$$k \leqq \sum_{p \in X} (\delta(p, L) + 1) - 1.$$

We say that L is an *independent* k-secant of X if equality holds. We say that X (the arc A) is *k-independent* if every k-secant of $X(J)$ is independent.

THEOREM 2.2.1. *Let $n \geqq 2$. If X is k-independent and $p \in X$, then X is $(k - 1)$-independent on $A|p$.*

Proof. Let L be a $(k - 1)$-secant of X on $\bar{A} = A|p$. By Theorem 1.3.1, L is a k-secant of X on A with $A(p) \subset L$. Hence, $k = \sum_{q \in X} (\delta(q, L) + 1) - 1$. We have $A(p) \not\subset A_{\delta(q,L)}(q)$, for all $q \in X$, $q \neq p$; hence, $\bar{\delta}(q, L) = \delta(q, L)$, for such q. Since $\bar{\delta}(p, L) = \delta(p, L) - 1$, $k - 1 = \sum_{q \in X} (\bar{\delta}(q, L) + 1) - 1$. Thus, L is an independent $(k - 1)$-secant of X on $\bar{A}$.

THEOREM 2.2.2. *Suppose that $-1 \leqq h \leqq k \leqq n - 1$. If X is k-independent, it is also h-independent.*

Proof. Let L be an h-secant of X. Choose $p \in X$ and take i such that $LA_i(p)$ is a k-secant of X. With $LA_i(p)$, L must be independent.

THEOREM 2.2.3. *If X is k-independent, then a k-space can meet X in at most $k + 1$ points. The converse is not true.*

Proof. The converse does not hold, for, if $n = 2$, we may take $X = [p, q]$ to be of order 2 with $A(p) \subset A_1(q)$; then $L = A_1(q)$ is not independent.

2.3. The mapping A^k. We define inductively, mappings $A^k: J^{k+1} \to \mathscr{P}_k^n$, $-1 \leqq k \leqq n$, by requiring that $A^{-1}(\quad) \in \mathscr{P}_{-1}^n$ and that $A^k(p_0, \ldots, p_k) = A(p_k)|A^{k-1}(p_0, \ldots, p_{k-1})$, $0 \leqq k \leqq n$. Note that $A^k(p, \ldots, p) = A_k(p)$.

THEOREM 2.3.1. *$A^k(X^{k+1})$ is the set of all k-secants of X, $-1 \leqq k \leqq n$.*

Proof. For $k = -1$ and $k = 0$, this is obvious. Assume that it is true for $k - 1$, where $1 \leqq k \leqq n$. We may also assume that $n \geqq 2$.

Let $p_0, \ldots, p_k \in X$ be given. If one projects A from p_0, then

$$\bar{A}^{k-1}(p_1, \ldots, p_k) = A^k(p_0, \ldots, p_k).$$

Hence, $A^k(p_0, \ldots, p_k)$ is a $(k-1)$-secant of X on $\bar{A}$ and, by Theorem 1.3.1, also a k-secant of X on A.

Conversely, let a k-secant L of X be given. Projecting A from a point p with $A(p) \subset L$ one has, by Theorem 1.3.1, that L is a $(k-1)$-secant of X on $\bar{A}$. Thus,

$$L = \bar{A}^{k-1}(p_1, \ldots, p_k) = A^k(p, p_1, \ldots, p_k),$$

and Theorem 2.3.1 is proved.

Suppose that $p \in J$, $x = (p_0, \ldots, p_k) \in J^{k+1}$, $-1 \leqq k \leqq n$. Put

$$\gamma(p, x) = \sum_{p_i = p} 1 - 1.$$

Thus, $\gamma(p, x) \leqq \delta(p, A^k(x))$.

THEOREM 2.3.2. *Suppose that $X \subset J$, $x \in X^{k+1}$. Then $A^k(x)$ is an independent k-secant of X if and only if $\delta(p, A^k(x)) = \gamma(p, x)$, for all $p \in X$.*

Proof. $A^k(x)$ is an independent k-secant of X if and only if

$$k = \sum_{p \in X} (\delta(p, A^k(x)) + 1) - 1.$$

Since $k = \sum_{p \in X} (\gamma(p, x) + 1) - 1$, the statement follows.

Suppose that $x = (p_0, \ldots, p_k) \in J^{k+1}$, $-1 \leqq k \leqq n$. Then

$$\bigvee_{p \in J} A_{\gamma(p,x)}(p) \subset A^k(x).$$

We say that x is *independent* if the inclusion is improper. If y is a permutation of x, then

$$\bigvee_{p \in J} A_{\gamma(p,x)}(p) = \bigvee_{p \in J} A_{\gamma(p,y)}(p).$$

Thus, if x is independent, so is y and $A^k(x) = A^k(y)$.

THEOREM 2.3.3. *Let $-1 \leqq k \leqq n - 1$. Then $X \subset J$ is k-independent if and only if every $x \in X^{k+2}$ is independent.*

Proof. Suppose that X is k-independent. If $x \in X^{k+2}$, then $L = \bigvee_{p \in X} A_{\gamma(p,x)}(p)$ is an h-secant of X, $h \leqq k + 1$. Suppose that $h < k + 1$. By Theorem 2.2.2,

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X is h-independent, so $h = \sum_{p \in X} (\delta(p, L) + 1) - 1$. Since, by the definition of L, $\gamma(p, x) \leqq \delta(p, L)$, for all $p \in X$, we have

$$k + 1 = \sum_{p \in X} (\gamma(p, x) + 1) - 1 \leqq \sum_{p \in X} (\delta(p, x) + 1) - 1 = h,$$

which is a contradiction. Hence, $h = k + 1$.

Next, suppose that $L = A^k(y)$, $y \in X^{k+1}$, is a dependent k-secant of X. By Theorem 2.3.2, there is a $q \in X$ with $\gamma(q, y) < \delta(q, L)$. Put $x = (y, q)$. Then $\gamma(p, x) \leqq \delta(p, L)$, for all $p \in X$, and

$$\bigvee_{p \in X} A_{\gamma(p,x)}(p) \subset L \neq A^{k+1}(x).$$

Thus, $x \in X^{k+2}$ is dependent.

THEOREM 2.3.4. *Let $X \subset J$ and let $L = A^k(x)$ be a k-secant of X, where $x \in X^{k+1}$. Then L is an independent secant of X if and only if the $y \in X^{k+1}$ for which $L = A^k(y)$ holds are exactly the permutations y of x.*

Proof. Firstly, suppose that L is an independent secant of X. Then by Theorem 2.3.2,

$$\bigvee_{p \in J} A_{\gamma(p,x)}(p) = \bigvee_{p \in X} A_{\delta(p,A^k(x))}(p) = A^k(x).$$

Hence, x is independent and $L = A^k(y)$ for all permutations y of x.

If $L = A^k(y)$, where $y \in X^{k+1}$, then by Theorem 2.3.2,

$$\gamma(p, y) = \delta(p, L) = \gamma(p, x),$$

for all $p \in X$ and y is a permutation of x.

Secondly, suppose that $L = A^k(y)$ holds for all permutations y of x and L is a dependent secant of X. By Theorem 2.3.2, there is a $p \in X$ with $\gamma(p, x) < \delta(p, L)$. We may assume that $p_i = p$, for $0 \leqq i \leqq \gamma(p, x)$. Take j such that

$$\gamma(p, x) = \delta(p, A^j(p_0, \ldots, p_j)) < \delta(p, A^{j+1}(p_0, \ldots, p_{j+1})).$$

Then $A^{j+1}(p_0, \ldots, p_{j+1}) = A^{j+1}(p_0, \ldots, p_j, p)$. Thus, $L = A^k(y)$ where

$$y = (p_0, \ldots, p_j, p, p_{j+2}, \ldots, p_k) \in X^{k+1}$$

is not a permutation of x.

THEOREM 2.3.5. *Let A be k-independent. Then $A^k(x) = A^k(y)$ exactly when x is a permutation of y.*

3. ARCS OF ORDER n

Compared with arcs of higher order, arcs of order n are well behaved. For example, if $k + 1$ points of an arc of order n converge to a point then the k-secant through them converges to $A_k(p)$; cf. [**10**] and Theorem 3.4.1. Arcs of higher order may not have this property. Because of the relative simplicity of arcs of order n we define a singularity as a point which has no neighbourhood of order n. Later we will see this is justified in the sense that the singularities of an arc are nowhere dense in J; cf. Theorem 9.1.2.

Arcs of order n are important in the study of elementary arcs, i.e., arcs whose points have right and left neighbourhoods of order n. By assuming an arc to be elementary one is able to avoid, or at least postpone, the consideration of pathological behaviour.

3.1. Projection of arcs of order n.

THEOREM 3.1.1. *If (p, q) has order n, then $[p, q)$ and $(p, q]$ are $(n - 1)$-independent; cf. Theorem* 2.2.3.

Proof. This is true for $n = 1$; assume that it is true for $n - 1$.

If $p < r < s < q$, then $A(r) \not\subset A_{n-1}(s)$. For, (p, s) has order $n - 1$ on $A|s$ and hence $(p, s]$ is $(n - 2)$-independent on $A|s$.

Let L be an $(n - 1)$-secant of (p, q) and let r be the first point of (p, q) with $A(r) \subset L$. Since (r, q) has order $n - 1$ on $A|r$, $[r, q]$ is $(n - 2)$-independent on $A|r$. In particular, L is an independent secant of $[r, q]$ on $A|r$. Since $A(r) \not\subset A_{n-1}(s)$ for $s \in (r, q)$, it follows that projecting from r, one has $\bar{\delta}(s, L) = \delta(s, L)$, if $s \in (r, q)$. Thus, L is an independent secant of $[r, q)$ on A. Hence, (p, q) is $(n - 1)$-independent.

Continuing our proof, we next verify that $A(p) \not\subset A_{n-1}(s)$, if $s \in (p, q)$. For, by Theorem 2.2.1, (p, q) is $(n - 2)$-independent on $A|s$; by Theorem 2.2.3, (p, q) is of order $n - 1$ on $A|s$; by our induction assumption, $[p, q)$ is $(n - 2)$-independent on $A|s$.

Furthermore, $[p, q)$ has order n. For, if $p < p_1 < \ldots p_n < q$ and if $p, p_1, \ldots, p_n$ were to lie in a hyperplane, then $[p, p_n)$ would not have order $n - 1$ on $A|p_n$. Now, since (p, q) has order $n - 1$ on $A|p$, $[p, q)$ is $(n - 2)$-independent on $A|p$.

Finally, let L be any $(n - 1)$-secant of $[p, q)$ which contains $A(p)$. Then L is an independent secant of $[p, q)$ on $A|p$. By the paragraph before the previous one, if one projects from p, $\bar{\delta}(s, L) = \delta(s, L)$, for all $s \in (p, q)$. Thus, L is an independent secant of $[p, q)$ on A.

The symmetric argument holds for $(p, q]$.

THEOREM 3.1.2. *If (p, q) has order n on A and $r \in [p, q]$, then (p, q) has order $n - 1$ on $A|r$.*

Proof. Use Theorems 3.1.1, 2.2.1, and 2.2.3.

3.2. Properties of points. The *order* of a point p is the minimum order which a neighbourhood of p can possess. By Theorem 1.5.1, the order of a point is finite although perhaps not bounded. The point p is *ordinary* if it is of order n; otherwise, it is a *singularity*. The point p is *elementary* if there exist

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$U^+(p)$ and $U^-(p)$ of order n. An arc is *ordinary* (*elementary*) if each of its points is ordinary (elementary).

THEOREM 3.2.1. *The set of singularities of an arc is closed. An elementary singularity is an isolated singularity but the converse is not in general true.*

Proof. Consider the graph of the parabola $y = x^2$. Through each of the points $(1/n, 1/n^2)$, $n = 1, 2, \ldots$, construct the circle of diameter $1/n^2$ which lies above the parabola and has the same tangent as the parabola at $(1/n, 1/n^2)$. Introducing a parameter, one obtains an arc for which the origin is a non-elementary isolated singularity.

We say that a point p is *regular* if $\alpha_i(p) = 1, 0 \leqq i \leqq n - 1$. We say that p is an *inflection* if $\alpha_i(p) = 1$, $0 \leqq i \leqq n - 2$, and $\alpha_{n-1}(p) = 2$. An arc is *regular* (*has at most inflections*) if each point is regular (regular or an inflection).

THEOREM 3.2.2. *An ordinary point is regular.*

Proof. This is true for $n = 1$; assume that it is true for $n - 1$. Let p be ordinary and take $U(p)$ of order n. Let $q \in U(p)$, $q \neq p$. For $A|p$, $\bar{\alpha}_k(p) = \alpha_{k+1}(p)$, $0 \leqq k \leqq n - 2$. For $A|q$, $\bar{\alpha}_k(p) = \alpha_k(p)$, $0 \leqq k \leqq n - 2$, since $A(q) \not\subset A_{n-1}(p)$ by Theorem 3.1.1. By Theorem 3.1.2, $U(p)$ is of order $n - 1$ on $A|p$ $(A|q)$. Thus, p is regular on $A|p$ $(A|q)$, so $\alpha_{k+1}(p) = \alpha_k(p) = 1$, if $0 \leqq k \leqq n - 2$. Hence, p is regular on A.

3.3. Monotonicity. Let A be an arc of order n, L an oriented line and suppose that no $(n-2)$-secant of A meets L. Then for each $x \in J^n$, $A^{n-1}(x) \cap L$ is a point $\varphi(x)$ of L. We assume that there is a point $P_\infty \subset L$ such that $\varphi(x) \neq P_\infty$, for all $x \in J^n$. Put

$$(p_0, \ldots, p_{n-1}) \leqq (q_0, \ldots, q_{n-1}),$$

if $p_i \leqq q_i$, for all i, $0 \leqq i \leqq n - 1$.

THEOREM 3.3.1. *φ is (strictly) monotone.*

Proof. We first show that if $n \geqq 2$ and $\{p_0, \ldots, p_{n-3}, p_{n-2}, q_{n-2}\} \subset J$, then the mappings $p \to \varphi(p_0, \ldots, p_{n-3}, p_{n-2}, p)$ and $p \to \varphi(p_0, \ldots, p_{n-3}, q_{n-2}, p)$ are monotone with the same sense.

Suppose that $n = 2$. By Theorem 3.1.2 and projection from p_0, $p \to \varphi(p_0, p)$ is of order 1 and hence monotone. Take r_1, r_2 with $p_0 < r_1 < r_2$. We may assume that the orientation of L is such that $\varphi(p_0, r_1) < \varphi(p_0, r_2)$. There is a $U(p_0)$ such that, if $q_0 \in U(p_0)$, then $\varphi(q_0, r_1) < \varphi(q_0, r_2)$. Thus, $p \to \varphi(q_0, p)$ has the same sense for all $q_0 \in U(p_0)$. Since J is connected, the case $n = 2$ follows. The general case now follows by induction using projection from p_0.

Since φ is monotone for $n = 1$, we may assume that it is monotone for $n - 1$. By Theorems 3.1.1 and 2.3.5, $A^{n-1}(x)$ is symmetric. Hence, $\varphi(x)$ is symmetric.

Suppose that $(p_0, \ldots, p_{n-1}) \leqq (q_0, \ldots, q_{n-1})$. We may assume that $p \to \varphi(p_0, \ldots, p_{n-2}, p)$ is increasing. Then

$$\begin{aligned}\varphi(p_0, \ldots, p_{n-2}, p_{n-1}) &\leqq \varphi(p_0, \ldots, p_{n-2}, q_{n-1})\\ &\leqq \varphi(p_0, \ldots, q_{n-2}, q_{n-1})\\ &\quad\vdots\\ &\leqq \varphi(q_0, \ldots, q_{n-2}, q_{n-1}).\end{aligned}$$

3.4. Continuity of A^k.

THEOREM 3.4.1. *If (p, q) is of order n, then A^k is continuous on $[p, q)^{k+1}$, $-1 \leqq k \leqq n$.*

Proof. The theorem is true for $n = 1$; assume that it is true for $n - 1$.

Case 1. $x = (p_0, \ldots, p_k) \in [p, q)^{k+1}$, $-1 \leqq k \leqq n - 2$. Take q_1, q_2 such that $p_i < q_1 < q_2 < q$, for all i. Then by Theorem 3.1.1,

$$A^k(p_0', \ldots, p_k') = \bigcap_{i=1}^{2} A^{k+1}(q_i, p_0', \ldots, p_k'),$$

for all $(p_0', \ldots, p_k') \in [p, q_1)^k$, and the continuity of A^k at x follows by projection from q_1 and from q_2.

Case 2. $x = (p_0, \ldots, p_{n-1}) \in [p, q)^n$; not all p_i are equal. By Theorems 2.3.1 and 2.3.2,

$$A^{n-1}(x) = \bigvee_i A_{\gamma(p_i,x)}(p_i).$$

Choose neighbourhoods U_i of the points p_i relative to $[p, q)$ such that $p_j \in U_i$ only if $p_j = p_i$. Put $V_i = U_i^{\gamma(p_i,x)+1}$. By Case 1, U_i may be chosen such that $A^{\gamma(p_i,x)}(z)$ is in any given neighbourhood of $A_{\gamma(p_i,x)}(p_i)$ if $z \in V_i$. Thus, it is possible to choose the U_i such that $A^{n-1}(y)$ is in any given neighbourhood of $A^{n-1}(x)$ if $y \in U_0 \times \ldots \times U_{n-1}$.

Case 3. $x = (r, \ldots, r) \in (p, q)^n$. Consider $x_i \in (p, q)^n$ such that $x_i \to x$, $A^{n-1}(x_i) \to L$. By Case 1, $A_{n-2}(r) \subset L$.

We may assume that L meets $[p, q]$ only in r and that there is a hyperplane H_∞ which does not meet $[p, q]$. Then p, q lie in the same open half space determined by $M = A^{n-1}(x_i)$ and H_∞ exactly when $\sum_{p<s<q} \sigma_{\delta(s,M)}(s)$ is even. By Theorems 3.2.2 and 3.1.1,

$$\begin{aligned}\sum_{p<s<q} \sigma_{\delta(s,M)}(s) &\equiv \sum_{p<s<q} \sum_{i=0}^{\delta(s,M)} \alpha_{i(s)} \pmod 2\\ &= \sum_{p<s<q} (\delta(s, M) + 1)\\ &= \dim M + 1\\ &= n.\end{aligned}$$

Thus, L supports A at r exactly when n is even.

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By Theorem 3.2.2, $\sigma_{n-2}(r) \equiv n - 1 \pmod 2$), Thus, a hyperplane M with $\delta(r, M) = n - 2$ supports A at r exactly when n is odd. Hence, $L = A_{n-1}(r)$.

Case 4. $x = (p, \ldots, p) \in [p, q)^n$. Suppose that $x_i \in [p, q)^n$, $x_i \to x$, $x_i \neq x$, and $A^{n-1}(x_i) \to L$. By Case 1, $A_{n-2}(p) \subset L$. By Case 2, there exist $y_i \in (p, q)^n$ such that $y_i \to x$ and $A^{n-1}(y_i) \to L$. Let $P \in A_{n-1}(p) \backslash A_{n-2}(p)$. Put $M = PA(r)$, where $r \in (p, q)$. By Theorem 3.1.1, $A_{n-2}(p) \cap M = \emptyset$. By Case 1, there is a $U^+(p)$ such that no $(n-2)$-secant of $U^+(p)$ meets M. Put

$$\varphi(y) = A^{n-1}(y) \cap M,$$

for $y \in (U^+(p))^n$. Since $\varphi(y) \neq A(r)$, Theorem 3.3.1 applies and φ is monotone. Since there exist $z_i \in (U^+(p))^n$ such that $z_i \to x$ and $A^{n-1}(z_i) \to A_{n-1}(p)$, one has $\lim_{i\to\infty} \varphi(y_i) = P$; cf. Theorem 5.3.1. Hence, $P \subset L$ and $L = A_{n-1}(p)$.

THEOREM 3.4.2. *If A is elementary, then A_k is continuous*, $0 \leqq k \leqq n - 1$.

4. ARCS WITH TOWER

The purpose of this section is to prove Theorem 4.1 which is a fundamental result in our development. For $n = 2$, the condition of continuity of the osculating spaces can be removed; cf. Theorems 8.1.2 and 9.1.1. For $n \geqq 3$, it is an open question whether the continuity condition can be removed.

A set $\{H_i | -1 \leqq i \leqq n\}$ of spaces is called a *tower* if $H_i \in \mathscr{P}_i^{\,n}$ and $H_{-1} \subset \ldots \subset H_n$. An *arc with tower* is an arc A for which there exists a tower satisfying

$$A_k(p) \cap H_{n-k-1} = \emptyset,$$

for all $p \in J$, $-1 \leqq k \leqq n$. Any arc in the affine plane which has no vertical tangent is an arc with tower.

THEOREM 4.1. *If A is a regular arc with tower and A_k is continuous*, $0 \leqq k \leqq n - 1$, *then A is of order n.*

LEMMA 4.2. *If p is a point of an arc A satisfying the hypothesis of Theorem* 4.1, *then*

$$\{H_{-1}, H_0, H_0A(p), \ldots, H_{n-1}A(p)\}$$

is a tower for each of the components of A determined by p.

Proof. To prove Theorem 4.1 for $n = 1$, suppose that there are points $p < q$ such that $A(p) = A(q)$. Since the image of A is not all of $\mathscr{P}_0^{\,1}$, there is an inflection in (p, q), which is a contradiction. Lemma 4.2(1) is also true. We assume that Theorem 4.1$(n - 1)$ and Lemma 4.2$(n - 1)$ are true.

Proof of Lemma 4.2(n). Consider $\bar{A} = A|H_0$. Since $H_0 \not\subset A_{n-1}(q)$, for all $q \in J$, $\bar{A}$ is regular. Put $\bar{H}_k = H_{k+1}$, $-1 \leqq k \leqq n - 1$. Since

$$\bar{A}_k(q) \cap \bar{H}_{n-k-2} = A_k(q)H_0 \cap H_{n-k-1} = H_0,$$

$\{\bar{H}_k\}$ is a tower for $\bar{A}$. Since $\bar{A}_k(q) = A_k(q)H_0$, $\bar{A}_k$ is continuous.

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By Lemma 4.2$(n-1)$,

$$\{\bar{H}_{-1}, \bar{H}_0, \bar{H}_0\bar{A}(p), \ldots, \bar{H}_{n-2}\bar{A}(p)\}$$

is a tower for each component of $\bar{A}$ determined by p. If $q \neq p$ and $-1 \leqq k \leqq n-3$, then

$$A_k(q)H_0 \cap H_{n-k-2}A(p) = \bar{A}_k(q) \cap \bar{H}_{n-k-3}\bar{A}(p) = H_0.$$

Thus, $A_k(q) \cap H_{n-k-2}A(p) = \emptyset$.

By Theorem 4.1$(n-1)$, $\bar{A}$ is of order n. By Theorem 3.1.1,

$$A_{n-2}(q)H_0 \cap H_0A(p) = \bar{A}_{n-2}(q) \cap \bar{A}(p) = H_0,$$

$q \neq p$. Thus, $A_{n-2}(q) \cap H_0A(p) = \emptyset$ and the proof of Lemma 4.2(n) is complete.

In Lemmas 4.3 and 4.4 we shall assume that the hypothesis of Theorem 4.1 holds.

LEMMA 4.3. *For any point $P \subset H_1$ there is at most one $p \in J$ such that $P \subset A_{n-1}(p)$.*

Proof. Put $\varphi(p) = A_{n-1}(p) \cap H_1$. φ is continuous and $\varphi(p) \neq H_0$, for all $p \in J$. Suppose that there exist points $p_1 < p_2$ such that $\varphi(p_1) = \varphi(p_2)$. Then there exists a $q \in (p_1, p_2)$ such that $\varphi((p_1, p_2))$ lies in one of the closed segments of H_1 with end points H_0 and $Q = \varphi(q)$, say S. By projection from $A_{n-2}(q)$ and the regularity of q, there is a point $r \in (p_1, p_2)$ such that $R = A(r)A_{n-2}(q) \cap H_1 \notin S$. Now (p_1, p_2) is regular on $\bar{A} = A|R$ and has tower $\{R, H_1, \ldots, H_n\}$. Since $\bar{A}_k(r) = A_k(r)H_0$, $\bar{A}_k$ is continuous. By Theorem 4.1$(n-1)$, (p_1, p_2) is of order $n-1$ on $\bar{A}$. But $\bar{A}(r) \subset \bar{A}_{n-2}(q)$, contradicting Theorem 3.1.1.

LEMMA 4.4. *If $p < q$, then $A(p) \not\subset A_{n-1}(q)$.*

Proof. Put $P = A_{n-1}(q) \cap H_1$. By Lemma 4.3, $P \subset A_{n-1}(r)$ only if $r = q$. Hence, $X = \{r | r < q\}$ is regular on $\bar{A} = A|P$. Now $\{P, H_1, \ldots, H_n\}$ is a tower for X on $\bar{A}$ and $\bar{A}_k$ is continuous. By Theorem 4.1$(n-1)$, X has order $n-1$ on $\bar{A}$. By Theorem 3.1.1, $\bar{A}(p) \not\subset \bar{A}_{n-2}(q)$. Hence, $A(p) \not\subset A_{n-1}(q)$.

Proof of Theorem 4.1(n). Suppose there are points $p_0 < p_1 < \ldots < p_n$ which lie in a hyperplane L.

Let $X = \{p | p > p_0\}$. By Lemma 4.4, X is regular on $\bar{A} = A|p_0$. By Lemma 4.2(n),

$$\{H_{-1}, H_0, H_0A(p_0), \ldots, H_{n-1}A(p_0)\}$$

is a tower for X. Hence,

$$\{A(p_0), H_0A(p_0), \ldots, H_{n-1}A(p_0)\}$$

is a tower for X on $\bar{A}$. Since $\bar{A}_k$ is continuous, Theorem 4.1$(n-1)$ applies and X has order $n-1$ on $\bar{A}$. This contradicts $\bar{A}(p_1), \ldots, \bar{A}(p_n) \subset L$.

5. FINITENESS

We introduce the condition that a point be strongly finite. This is weaker than the condition that the point be elementary; cf. Theorem 5.2.1. In Theorems 5.2.2–5.2.4, we develop properties of elementary points which will be needed later. Perhaps the main result of this section is Theorem 5.5.1. It describes the behaviour of the k-secants of a one-sided neighbourhood of a strongly finite point. We will use Theorem 5.5.1 in the proof of Theorem 6.2.4.

5.1. Finiteness and strong finiteness. A point p of an arc is *right finite* (*strongly right finite*) if, for every $(n - k - 1)$-space L, $0 \leqq k \leqq n - 1$, there is a $U^+(p)$ such that no osculating k-space (no k-secant) of $U^+(p)$ meets L. *Left finiteness* (*strong left finiteness*) is defined similarly. The point p is *finite* (*strongly finite*) if it is both right and left finite (both strongly right and left finite). The arc is *finite* (*strongly finite*) if each of its points is finite (strongly finite).

THEOREM 5.1.1. *Suppose that $n \geqq 2$. A finite (strongly finite) point p is finite (strongly finite) on any projection $\bar{A} = A|P$.*

Proof. Let $L \in \mathscr{P}_{n-k-2}{}^{n-1}(P)$, $0 \leqq k \leqq n - 2$. Then L is an $(n - k - 1)$-space with $P \subset L$. Take $U^+(p)$ such that no osculating k-space (no k-secant) of $U^+(p)$ meets L. An osculating k-space (a k-secant) of $U^+(p)$ on $\bar{A}$ is spanned by an osculating k-space (a k-secant) of $U^+(p)$ on A and P. Since such a space meets L only in P, the theorem follows.

5.2. Elementary points.

THEOREM 5.2.1. *An elementary arc is strongly finite.*

Proof. This is true for $n = 1$; assume that it is true for $n - 1$. Let $U^+(p)$ be of order n.

Let P be a point. If $P = A(p)$, then no $(n - 1)$-secant of $U^+(p)$ contains P, by Theorem 3.1.1. Therefore, assume that $P \neq A(p)$ and put $L = PA(p)$. By projection from p, there is a $U_1{}^+(p) \subset U^+(p)$ such that $A^{n-2}(y) \cap L = \emptyset$, for all $y \in (U_1{}^+(p))^{n-1}$. By Theorem 3.3.1, $\varphi(x) = A^{n-1}(x) \cap L$ is monotone for $x \in (U_1{}^+(p))^n$. If $q \in U_1{}^+(p)$ then

$$\lim_{r \to p+} \varphi(q, \ldots, q, r) = A(p).$$

Thus, there exists $U_2{}^+(p) \subset U_1{}^+(p)$ such that $\varphi(x) \neq P$, for all $x \in (U_2{}^+(p))^n$. Hence, no $(n - 1)$-secant of $U_2{}^+(p)$ contains P.

Let L be an $(n - k - 1)$-space, $0 \leqq k \leqq n - 2$. Let P be a point on L. Choose $U_2{}^+(p)$ as above. Suppose that there is a hyperplane H through P and there are n distinct points $p_1, \ldots, p_n$ in $U_2{}^+(p)$ such that $A(p_1), \ldots, A(p_n) \subset H$. Since $U_2{}^+(p)$ is of order n, $H = \bigvee_{i=1}^{n} A(p_i)$; thus, H is an $(n - 1)$-secant containing P, which is a contradiction. Hence, $U_2{}^+(p)$ has order $n - 1$ on $A|P$.

By the induction hypothesis, there is a $U_3^+(p) \subset U_2^+(p)$ such that no k-secant of $U_3^+(p)$ on $A|P$ meets L. Thus, no k-secant of $U_3^+(p)$ on A meets L.

THEOREM 5.2.2. *Suppose that $n \geqq 2$. An elementary point p of an arc A is elementary on any projection $\bar{A} = A|P$.*

Proof. By Theorem 5.2.1, there is a $U^+(p)$ of order n such that no $(n-1)$-secant of $U^+(p)$ contains P. As in the proof of Theorem 5.2.1, $U^+(p)$ is of order $n-1$ on $\bar{A}$.

THEOREM 5.2.3. *A regular elementary point p is ordinary.*

Proof. Let $\{H_i\}$ be a tower of spaces such that $A_k(p) \cap H_{n-k-1} = \emptyset$, $-1 \leqq k \leqq n$. By Theorem 3.4.2, there exist $U^+(p)$ and $U^-(p)$ of order n such that $A_k(q) \cap H_{n-k-1} = \emptyset$, for all $q \in U(p) = U^+(p) \cup \{p\} \cup U^-(p)$, $-1 \leqq k \leqq n$. By Theorem 3.2.2, $U(p)$ is regular; by Theorem 4.1, it is of order n. Thus, p is ordinary.

THEOREM 5.2.4. *Suppose that $n \geqq 2$. Let p be an ordinary point or an elementary inflection. If $P \not\subset A_{n-1}(p)$, then p is ordinary on $A|P$.*

Proof. Use Theorems 1.4.3, 5.2.2, and 5.2.3.

5.3. Continuity properties.

THEOREM 5.3.1. *Let p be a point of an arc A. Given $U^+(p)$ and a neighbourhood $U \subset \mathscr{P}_k^n$ of $A_k(p)$, there is a k-secant L of $U^+(p)$ with $L \in U(A_k(p))$, $0 \leqq k \leqq n-1$.*

Proof. This is true for $k = 0$. Assume that it is true for $k-1$, where $1 \leqq k \leqq n-1$. Assume that U is open and take $q \in U^+(p)$ such that $A(q) \not\subset A_{k-1}(p)$ and $A(q)A_{k-1}(p) \in U$. Let U' be a neighbourhood of $A_{k-1}(p)$ such that $A(q)M \in U$, for all $M \in U'$. By the induction assumption, there is a $(k-1)$-secant M of $U^+(p)$ with $M \in U'$. Put $L = A(q)M$.

THEOREM 5.3.2. *Let p be a strongly right finite point of an arc A. Then*

$$A^k(p_0, \ldots, p_k) \to A_k(p)$$

as $(p_0, \ldots, p_k) \to (p, \ldots, p)$ $p_0 > p, \ldots, p_k > p$. Hence, A_k is continuous at a strongly finite point, $0 \leqq k \leqq n-1$; cf. Theorem 3.4.1.

Proof. By Theorem 2.3.1, we must show that given a neighbourhood $U(A_k(p))$ of $A_k(p)$, there is a $U^+(p)$ every k-secant of which is in $U(A_k(p))$. This is true for $n = 1$; assume that it is true for $n-1$.

Case 1. $0 \leqq k \leqq n-2$. Let Q_1, Q_2 be points with $Q_1, Q_2 \not\subset A_k(p)$ and $Q_1A_k(p) \neq Q_2A_k(p)$. Let $U_1^+(p)$ be such that no k-secant of $U_1^+(p)$ contains Q_1 or Q_2. For $i = 1, 2$, let $\bar{U}(Q_iA_k(p))$ be a neighbourhood of $Q_iA_k(p)$ in $A|Q_i$ such that $M_1 \cap M_2 \in U(A_k(p))$, if $M_i \in \bar{U}(Q_iA_k(p))$. By projection from Q_i

and the induction assumption, there is a $U^+(p)$ such that every k-secant of $U^+(p)$ on $A|Q_i$ is in $\bar{U}(Q_iA_k(p))$, $i = 1, 2$. If M is a k-secant of $U^+(p)$ on A then Q_iM is a k-secant of $U^+(p)$ on $A|Q_i$, $i = 1, 2$. Since $M = Q_1M \cap Q_2M$, Case 1 follows.

Case 2. $k = n - 1$. Let L be a line with $A_{n-2}(p) \cap L = \emptyset$. Put $P = A_{n-1}(p) \cap L$. Let $U(p)$ be a neighbourhood of P on L, say with end points Q_1, Q_2, and let $U(A_{n-2}(p))$ be a nieghbourhood of $A_{n-2}(p)$ such that $QM \in U(A_{n-1}(p))$, for all $Q \in U(P)$, $M \in U(A_{n-2}(p))$. Take $U^+(p)$ such that no $(n - 1)$-secant of $U^+(p)$ contains Q_1 or Q_2 and every $(n - 2)$-secant of $U^+(p)$ is in $U(A_{n-2}(p))$. By Theorem 5.3.1, there is an $(n - 1)$-secant of $U^+(p)$ which meets $U(P)$. By Theorem 2.1.1, every $(n - 1)$-secant of $U^+(p)$ meets $U(P)$.

5.4. Regular finite arcs.

THEOREM 5.4.1. *A finite arc with at most inflections is strongly finite.*

THEOREM 5.4.2. *A regular finite arc with tower is of order n.*

Proofs of Theorems 5.4.1. *and* 5.4.2. Theorem 5.4.1(1) is true; Theorem 5.4.2(1) is the same as Theorem 4.1(1). Assume that Theorems 5.4.1$(n - 1)$ and 5.4.2$(n - 1)$ are true.

We first prove Theorem 5.4.1(n). Let P be a point and let $\{H_i\}$ be a tower with $H_0 = P$. Take $U^+(p)$ such that $A_k(q) \cap H_{n-k-1} = \emptyset$, for all $q \in U^+(p)$, $0 \leqq k \leqq n - 1$. Then $U^+(p)$ is regular on $\bar{A} = A|P$ and has tower $\{H_0, \ldots, H_n\}$. By Theorems 5.1.1 and 5.4.2$(n - 1)$, $U^+(p)$ is of order $n - 1$ on $\bar{A}$. By Theorem 3.1.1, it is $(n - 2)$-independent on $\bar{A}$. Let M be a hyperplane through P. Since $P \not\subset A_{n-1}(q)$, $\bar{\delta}(q, M) = \delta(q, M)$, for all $q \in U^+(p)$. Thus,

$$\begin{aligned} \dim \bigvee_{q \in U^+(p)} A_{\delta(q,M)}(q) &\leqq \sum_{q \in U^+(p)} (\delta(q, M) + 1) - 1 \\ &= \sum_{q \in U^+(p)} (\bar{\delta}(q, M) + 1) - 1 \\ &\leqq n - 2. \end{aligned}$$

Hence, M is not an $(n - 1)$-secant of $U^+(p)$. Thus, no $(n - 1)$-secant of $U^+(p)$ contains P.

Let L be an $(n - k - 1)$-space, $0 \leqq k \leqq n - 2$. Let P be a point on L. Take $U^+(p)$ such that no $(n - 1)$-secant of $U^+(p)$ contains P and $U^+(p)$ is of order $n - 1$ on $\bar{A} = A|P$. By Theorem 5.2.1, there is a $U_1^+(p) \subset U^+(p)$ such that no k-secant of $U_1^+(p)$ on $\bar{A}$ meets L. Hence, no k-secant of $U_1^+(p)$ on A meets L and the proof of Theorem 5.4.1(n) is complete.

By Theorem 5.4.1(n), a regular finite arc is strongly finite. By Theorem 5.3.2, A_k is continuous, $0 \leqq k \leqq n - 1$, and Theorem 5.4.2(n) follows from Theorem 4.1.

THEOREM 5.4.3. *A regular finite arc is ordinary.*

Proof. Let p be a point of the arc and $\{H_i\}$ be any tower. Take $U^+(p)$ such that $\{H_i\}$ is a tower for $U^+(p)$. By Theorem 5.4.2, $U^+(p)$ is of order n. Hence, p is elementary and Theorem 5.2.3 applies.

5.5. Behaviour of secants. Let p be a point of an arc A. Let $P_i \in A_i(p)\setminus A_{i-1}(p)$, $0 \leqq i \leqq n$. The points P_i are independent and are the vertices of 2^n open n-simpleces. Let S^+ be that open n-simplex which contains some $U^+(p)$. If $1 \leqq i \leqq n$ and i is odd (even), let E_i^+ be the open segment of P_0P_i with ends P_0 and P_i which is (is not) an edge of S^+. S^- and E_i^-, $1 \leqq i \leqq n$, are defined using $U^-(p)$ instead of $U^+(p)$.

Let p be a strongly right finite point of an arc. Take $U^+(p)$ such that no k-secant of $U^+(p)$ meets an $(n-k-1)$-space spanned by points P_i, $0 \leqq k \leqq n-1$. Consider the $(n-k)$-space $P_0P_{m(1)} \ldots P_{m(n-k)}$, where $0 < m(1) < \ldots < m(n-k) \leqq n$. Let $S^+(m(1), \ldots, (n-k))$ be the open $(n-k)$-simplex with the vertices $P_0, P_m(1), \ldots, P_{m(n-k)}$ which the k-secants of $U^+(p)$ meet; cf. Theorem 2.1.1.

THEOREM 5.5.1. *$E_{m(i)}^+$ is an edge of $S^+(m(1), \ldots, m(n-k))$ if and only if i is odd, $1 \leqq i \leqq n-k$, $0 \leqq k \leqq n-1$.*

Proof. If $k = 0$, then $S^+(m(1), \ldots, m(n)) = S^+$, and the theorem follows from the definition of $E_{m(i)}^+ = E_i^+$. Hence, it is true for $n = 1$.

Suppose that $n = 2$, $k = 1$. Take $q \in U^+(p)$ such that (q, p) does not meet the line $A(p)A(q)$. S^+ is divided into two open triangles by $A(p)A(q)$; (q, p) is contained in the triangle which has E_1^+ as an edge. Let $r \in (q, p)$ be such that $A(q)A(r)$ meets E_1^+, E_2^+. Thus a 1-secant of $U^+(p)$, viz. $A(q)A(r)$, meets E_1^+, E_2^+.

Assume that the theorem is true for $n-1$, where $n \geqq 3$.

Case 1. $1 \leqq k \leqq n-2$. The projections of $P_0, P_{m(2)}, \ldots, P_{m(n-k)}$ from $P_{m(1)}$ are $\bar{P}_0, \bar{P}_{m(2)-1}, \ldots, \bar{P}_{m(n-k)-1}$, respectively. The projection of S^+ is $\bar{S}^+$. Hence, the projection of $E_{m(i)}^+$ is the open segment of $\bar{P}_0\bar{P}_{m(i)-1}$ different from $\bar{E}_{m(i-1)}^+$, $2 \leqq i \leqq n-k$. But the projection of $S^+(m(1), \ldots, m(n-k))$ is $\bar{S}^+(m(2)-1, \ldots, m(n-k)-1)$. Since $\bar{E}^+_{m(i)-1}$ is an edge of

$$\bar{S}^+(m(2)-1, \ldots, m(n-k)-1)$$

if and only if i is even, it follows that $E^+_{m(i)}$ is an edge of

$$S^+(m(1), \ldots, m(n-k))$$

if and only if i is odd, $2 \leqq i \leqq n-k$. This remains true for $i = 1$, by projection from $P_{m(n-k)}$ instead of $P_{m(1)}$.

Case 2. $k = n-1$. We have to show that $S^+(m) = E_m^+$, for $1 \leqq m \leqq n$. Suppose that $m > 1$. Project A from $A_{n-3}(q)$ into the plane $P_0P_1P_m$. By Case 1, $E_1^+(E_m^+)$ is (is not) an edge of $S^+(1, m)$. Also, $\bar{S}^+ = \bar{S}^+(1, m)$. Hence, $\bar{E}_1^+ = E_1^+$, $\bar{E}_2^+ = E_m^+$. Since $\bar{A}_1(q)$ meets $\bar{E}_1^+$ and $\bar{E}_2^+$, $A_{n-1}(q)$ meets E_1^+, E_m^+. Thus, $S^+(m) = E_m^+$, for all m.

6. DUAL DIFFERENTIABILITY

We introduce the condition that an arc be dually differentiable. This condition is weaker than the condition that the arc be strongly finite; cf. Theorem 6.1.3. In [**11**; **12**], pathological examples of points on dually differentiable arcs in the plane are given. A reader desiring some motivation for our σ_k^+ and σ_k^- should refer to these papers. The main results of this section are Theorems 6.2.4 and 6.3.1. The result, Theorem 6.4.2, is an application of dual differentiability.

6.1. Dually differentiable arcs. In our definition of a projective space $\mathscr{P}^n$, each k-space L corresponds to a $(k+1)$-dimensional subspace of an $(n+1)$-dimensional vector space V. If we associate to L the corresponding $(n-k)$-dimensional subspace of V^*, then $\mathscr{P}^n$ is again a projective space, called the *dual* $\mathscr{P}^{n*}$ of $\mathscr{P}^n$. One has $\mathscr{P}_k^{n*} = \mathscr{P}^n_{n-k-1}$, $-1 \leqq k \leqq n$.

Let A be an arc in $\mathscr{P}^n$. Define a mapping $A^*: J \to \mathscr{P}_0^{n*}$ by $A^*(p) = A_{n-1}(p)$, for all $p \in J$. We say that A is *dually differentiable* if A^* is an arc in $\mathscr{P}^{n*}$ and $A_k^* = A_{n-k-1}$, $-1 \leqq k \leqq n$. By the definition of an arc, A is dually differentiable if and only if

$$A_k(p) = \lim_{\substack{q \to p \\ q \neq p}} A_{n-1}(q) \cap A_{k+1}(p),$$

for all $p \in J$, $-1 \leqq k \leqq n-1$.

THEOREM 6.1.1. *If A is a dually differentiable arc, then A_{n-1} is continuous. If $P \in \mathscr{P}_0^n$ and $p \in J$, then there is a $U'_{(0)}$ such that $P \not\subset A_{n-1}(q)$, for all $q \in U'(p)$.*

Proof. Use Theorem 1.2.1.

THEOREM 6.1.2. *Let p be a strongly right finite point of an arc A. Let P_i be a point on $A_i(p)$ but not on $A_{i-1}(p)$, $0 \leqq i \leqq n$. Then*

$$\lim_{q \to p+} A_{n-1}(q) \cap P_k P_{k+1} = P_k,$$

$0 \leqq k \leqq n-1$.

Proof. Let $U(P_k)$ be a neighbourhood of P_k on P_kP_{k+1}. Take $U^+(p)$ such that no h-secant of $U^+(p)$ meets an $(n-h-1)$-space spanned by points P_i, $0 \leqq h \leqq n-1$, and further such that no $(n-1)$-secant of $U^+(p)$ contains an end point of $U(P_k)$. Let L be an $(n-k-2)$-secant of $U^+(p)$. By the choice of $U^+(p)$, $A_{k+1}(p) \cap L = \emptyset$. Thus, $A_k(p)L$ is a hyperplane. Also,

$$A_k(p)L \cap P_kP_{k+1} = P_k;$$

for, if $P_{k+1} \subset A_k(p)L$, then $A_{k+1}(p) \subset A_k(p)L$ and $A_{k+1}(p) \cap L \neq \emptyset$, which is a contradiction. Let $U(A_k(p))$ be a neighbourhood of $A_k(p)$ such that $L \cap M = \emptyset$ and LM meets $U(P_k)$, for all $M \in U(A_k(p))$. By Theorem, 5.3.1

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there is a k-secant M of $U^+(p)$ in $U(A_k(p))$. Since the $(n-1)$-secant LM meets $U(P_k)$, all $(n-1)$-secants of $U^+(p)$ meet $U(P_k)$ by Theorem 2.1.1. Thus, Theorem 6.1.2 is proved.

THEOREM 6.1.3. *A strongly finite arc A is dually differentiable.*

Proof. Let p be a point of A and take points P_i as in Theorem 6.1.2. It is sufficient to show that

$$\lim_{q\to p+} A_{n-1}(q) \cap A_{k+1}(p) = A_k(p),$$

$-1 \leqq k \leqq n-1$. Take $U^+(p)$ such that $P_0 \not\subset A_{n-1}(q)$ if $q \in U^+(p)$. Then $A_{n-1}(q) \cap A_{k+1}(p)$ is a k-space, for all $q \in U^+(p)$. Let L be a k-space of accumulation of $A_{n-1}(q) \cap A_{k+1}(p)$ as $q \to p^+$. Since the lines $P_0P_1, \ldots, P_kP_{k+1}$ lie in $A_{k+1}(p)$, we have $P_0, \ldots, P_k \subset L$, by Theorem 6.1.2. Hence, $L = A_k(p)$.

6.2. The characteristic of A^*. Let p be a point of an arc A. Take P_i, S^+, S^-, E_i^+, E_i^- as before for Theorem 5.5.1.

THEOREM 6.2.1. *$\sigma_k(p) = 0$ if and only if $E_{k+1}^+ = E_{k+1}^-$, $0 \leqq k \leqq n-1$.*

Proof. Let $L = P_0 \ldots \hat{P}_{k+1} \ldots P_n$ (the hat over P_{k+1} indicates that P_{k+1} is to be omitted). Since $\delta(p, L) = k$, $\sigma_k(p) = 0$ if and only if L supports A at p. Let $H_\infty = P_1 \ldots P_n$. L supports A at p if and only if S^+ and S^- lie on the same side of L. This is the case if and only if the edges of S^+ and S^- along P_0P_{k+1} are the same, i.e., if and only if $E_{k+1}^+ = E_{k+1}^-$.

Suppose further that A is a dually differentiable arc. By Theorem 6.1.1, there exists $U^+(p)$ such that $P_i \not\subset A_{n-1}(q)$, for all $q \in U^+(p)$, $0 \leqq i \leqq n$. Put $\sigma_k^+(p) = 0$ $(\sigma_k^+(p) = 1)$ if $A_{n-1}(q)$ meets (does not meet) E_{k+1}^+, for all $q \in U^+(p)$, $0 \leqq k \leqq n-1$. Also, put $\sigma_{-1}^+(p) = 0$ and define $\sigma_k^-(p)$, $-1 \leqq k \leqq n-1$, similarly.

In this and in later sections all congruences are to be considered taken modulo 2.

THEOREM 6.2.2. *At any point p of a dually differentiable arc*

$$\sigma_k^* \equiv \sigma_{n-k-2}^+ + \sigma_{n-k-2} + \sigma_{n-k-2}^- + \sigma_{n-1}^+ + \sigma_{n-1} + \sigma_{n-1}^-,$$

$0 \leqq k \leqq n-1$.

Proof. Put $Q_i = P_0 \ldots \hat{P}_{n-i} \ldots P_n$, $0 \leqq i \leqq n$. Then $A_{n-i-1}(p) \subset Q_i$ and $A_{n-i}(p) \not\subset Q_i$. Thus, the hyperplane Q_i is a point Q_i^* of $\mathscr{P}^{n*}$ with

$$Q_i^* \in A_i^*(p) \backslash A_{i-1}^*(p), \quad 0 \leqq i \leqq n.$$

Put

$$\varphi(q) = A^*(q)Q_1^* \ldots \hat{Q}_{k+1}^* \ldots Q_n^*.$$

Then $A_k^*(p) \subset Q_0^* \ldots \hat{Q}_{k+1}^* \ldots Q_n^*$ and $A_{k+1}^*(p) \not\subset Q_0^* \ldots \hat{Q}_{k+1}^* \ldots Q_n^*$. Thus, $\sigma_k^*(p) = 0$ if and only if $\varphi(p)$ supports A^* at p. This occurs if and only if

φ has characteristic (2) at p. In $\mathscr{P}^n$, the space $Q_1^* \ldots \hat{Q}_{k+1}^* \ldots Q_n^*$ is

$$\begin{aligned} &Q_1 \cap \ldots \cap \hat{Q}_{k+1} \cap \ldots Q_n \\ &\quad = P_0 \ldots \hat{P}_{n-1}P_n \cap \ldots \cap (P_0 \ldots \hat{P}_{n-k-1} \ldots P_n)^\wedge \cap \ldots \cap \hat{P}_0 \ldots P_n \\ &\quad = P_{n-k-1}P_n. \end{aligned}$$

Thus, $\varphi(q) = A_{n-1}(q) \cap P_{n-k-1}P_n$. Hence, $\sigma_k^*(p) = 0$ if and only if there is a $U'(p)$ such that $A_{n-1}(q)$ meets the same open segment of $P_{n-k-1}P_n$ for all $q \in U'(p)$.

Suppose that $-1 \leqq k \leqq n-2$. Take $U^+(p)$, $U^-(p)$ such that $P_i \not\subset A_{n-1}(q)$, for all $q \in U^+(p) \cup U^-(p)$, $0 \leqq i \leqq n$. Let $r \in U^+(p)$ and $s \in U^-(p)$. By Theorem 6.2.1 and the definition of σ_i^+ and σ_i^-, $A_{n-1}(r)$ meets the same segment of P_0P_{n-k-1} as $A_{n-1}(s)$ if and only if

$$\sigma_{n-k-2}^+ + \sigma_{n-k-2} + \sigma_{n-k-2}^- \equiv 0.$$

Considering the triangle with vertices P_0, P_{n-k-1}, P_n and also making use of the case $k = -1$, it follows that $A_{n-1}(q)$ meets the same open segment of $P_{n-k-1}P_n$, for all $q \in U^+(p) \cup U^-(p)$, if and only if

$$(\sigma_{n-k-2}^+ + \sigma_{n-k-2} + \sigma_{n-k-2}^-) + (\sigma_{n-1}^+ + \sigma_{n-1} + \sigma_{n-1}^-) \equiv 0.$$

Suppose $k = n-1$. $A_{n-1}(q)$ meets the same open segment of P_0P_n if and only if $\sigma_{n-1}^+ + \sigma_{n-1} + \sigma_{n-1}^- \equiv 0$.

THEOREM 6.2.3. *For any point p of a dually differentiable arc*

$$\alpha_k^* \equiv \alpha_{n-k-1} + \sigma_{n-k-2}^+ + \sigma_{n-k-2}^- + \sigma_{n-k-1}^+ + \sigma_{n-k-1}^-,$$

$0 \leqq k \leqq n-1$.

Proof. By Theorem 6.2.2 and the definition of the characteristic

$$\alpha_0^* + \ldots + \alpha_k^* \equiv \alpha_{n-1} + \ldots + \alpha_{n-k-1} + \sigma_{n-k-2}^+ + \sigma_{n-k-2}^- + \sigma_{n-1}^+ + \sigma_{n-1}^-,$$

$0 \leqq k \leqq n-1$. If $1 \leqq k \leqq n-1$, add this congruence to the corresponding congruence with $k-1$ instead of k.

THEOREM 6.2.4. *Let A be strongly finite. Then*

$$\alpha_k^*(p) = \alpha_{n-k-1}(p),$$

for all $p \in J$, $0 \leqq k \leqq n-1$.

Proof. By Theorem 6.1.3, A is dually differentiable. With the notation as in Theorem 5.5.1 and $k = n-1$, one has that $A_{n-1}(q)$ meets E_m^+, $1 \leqq m \leqq n$, for all $q \in U^+(p)$. Hence, $\sigma_i^+(p) = 0$, $0 \leqq i \leqq n-1$. By Theorem 6.2.3, the result follows.

6.3. The Scherk–Derry duality theorem.

THEOREM 6.3.1. *The dual A^* of an arc A of order n is also of order n.*

Proof. By Theorem 5.2.1, A is strongly finite and hence, by Theorem 6.1.3, it is dually differentiable. By Theorems 3.2.2 and 6.2.4, A^* is regular. By

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Theorem 3.4.2, $A_k^* = A_{n-k-1}$ is continuous, $0 \leqq k \leqq n-1$. By Theorem 3.1.1, $A_i(p)$, $-1 \leqq i \leqq n$, is a tower for (p, q) on A; hence $A_i^*(p) = A_{n-i-1}(p)$, $-1 \leqq i \leqq n$, is a tower for (p, q) on A^*. By Theorem 4.1, (p, q) is of order n on A^*, and Theorem 6.3.1 follows.

THEOREM 6.3.2. *The dual A^* of an elementary arc is elementary.*

6.4. Existence of an elementary singularity.

THEOREM 6.4.1. *Let p be a point of a strongly finite arc. Let*

$$P_i \in A_i(p) \backslash A_{i-1}(p), \qquad 0 \leqq i \leqq n.$$

The dual projection of A into the line P_kP_{k+1} has the characteristic $(\alpha_k(p))$ at p, $0 \leqq k \leqq n-1$.

Proof. By Theorems 6.1.3 and 6.2.4, A is dually differentiable and $\alpha_k^*(p) = \alpha_{n-k-1}(p)$, for all $p \in J$, $0 \leqq k \leqq n-1$. Put $Q_i = P_0 \ldots \hat{P}_{n-i} \ldots P_n$, $0 \leqq i \leqq n$. The hyperplane Q_i is a point Q_i^* of P^{n*} and the point

$$P_i = Q_0 \cap \ldots \cap \hat{Q}_{n-i} \cap \ldots \cap Q_n$$

is a hyperplane $P_i^* = Q_0^* \ldots \hat{Q}_{n-i}^* \ldots Q_n^*$ of P^{n*}. One has

$$\begin{aligned}(P_kP_{k+1})^* &= P_k^* \cap P_{k+1}^* \\ &= Q_0^* \ldots \hat{Q}_{n-k}^* \ldots Q_n^* \cap Q_0^* \ldots \hat{Q}_{n-k-1}^* \ldots Q_n^* \\ &= Q_0^* \ldots \hat{Q}_{n-k-1}^* \hat{Q}_{n-k}^* \ldots Q_n^*.\end{aligned}$$

We may project A^* from $(P_kP_{k+1})^*$ by successively projecting from $Q_0^*, \ldots, Q_{n-k-2}^*$ and then from $Q_n^*, \ldots, Q_{n-k-1}^*$. The result is an arc with the characteristic $(\alpha_{n-k-1}^*(p)) = (\alpha_k(p))$ at p.

THEOREM 6.4.2. *If a finite arc has a singularity, it has an elementary singularity.*

Proof. Let $n = 1$. We may assume that there is a point P_∞ such that $A(p) \neq P_\infty$, for all $p \in J$. By Theorem 4.1, every regular subarc of A is of order 1.

Suppose that A has a singularity but no elementary singularity. Then A has an inflection p_1. There is a neighbourhood (q_1, r_1) of p_1 such that $A(p) \neq A(p_1)$, for all $p \in (q_1, r_1)$, $p \neq p_1$. We may assume that $A(q_1) = A(r_1)$ and that $A(p) \neq A(q_1)$, for all p $\in (q_1, r_1)$. Since p_1 is not elementary, at least one of the intervals (q_1, p_1), (p_1, r_1), say X_1, contains an inflection. Let Y_1 be the other interval. Note that $X_1 \cap Y_1 = \emptyset$ and $A(X_1) = A(Y_1)$.

Repeat the above using X_1 instead of J. Thus, p_2, q_2, r_2, X_2 and Y_2 are defined. Continuing indefinitely, one obtains sequences X_i, Y_i, $i = 1, 2, \ldots$, such that $X_i \cap Y_i = \emptyset$, $A(X_i) = A(Y_i)$, and $\bar{X}_{i+1}$, $\bar{Y}_{i+1} \subset X_i$, $i = 1, 2, \ldots$. It follows that $\overline{A(X_{i+1})} = A(\bar{X}_{i+1}) \subset A(X_i)$, $i = 1, 2, \ldots$. Thus, there is a point $P \in \bigcap_i A(X_i)$. Hence, P meets each Y_i. But the Y_i are disjoint. Thus, P meets $[q_1, r_1]$ infinitely often, contradicting Theorem 1.5.1.

Thus, Theorem 6.4.2 is true for $n = 1$; assume that it is true for $n - 1$. First, suppose that A has at most inflections. By Theorem 5.4.3, A has an

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inflection p_1. Let L be a line such that $A_{n-2}(p_1) \cap L = \emptyset$. Since p_1 is finite, we may assume that $A_{n-2}(p) \cap L = \emptyset$, for all $p \in J$. Put

$$\varphi(p) = A_{n-1}(p) \cap L,$$

for all $p \in J$. By Theorems 5.4.1 and 6.4.1, φ is an arc in L with characteristic $(\alpha_{n-1}(p))$ at p.

Since φ has the singularity p_1, it has an elementary singularity p_2. p_2 is an isolated inflection of A. There is a $U^+(p_2)$ which is regular and has a tower. By Theorem 5.4.2, $U^+(p_2)$ is of order n. Hence, p_2 is an elementary inflection.

Next, assume only that A has a singularity p_3. Let P be a point with $P \not\subset A_{n-1}(p_3)$. Since p_3 is finite, we may assume that $P \not\subset A_{n-1}(p)$, for all $p \in J$. If $\bar{A} = A|P$ is ordinary, then A has at most inflections and the theorem follows. If $\bar{A}$ has an elementary singularity p_4, take $U^+(p_4)$ and $U^-(p_4)$ of order $n - 1$ on $\bar{A}$. On A, $U^+(p_4)$ and $U^-(p_4)$ have at most inflections. If either contains an inflection, Theorem 6.4.2 follows. If both are regular, p_4 is elementary as above. By Theorem 5.2.3, p_4 is non-regular on $\bar{A}$. Hence, p_4 is non-regular on A.

7. MULTIPLICITIES

We define the multiplicity with which an osculating k-space $A_k(p)$ meets an $(n - k - 1)$-space L. The use of multiplicities allows us to prove some congruences which hold for any elementary curve; cf. Theorems 7.4.1 and 7.4.2. In Theorem 7.5.9, we give a geometric interpretation of multiplicites. From this there follows a lower bound for the k-th rank number of an elementary point; cf. Theorem 7.5.10.

7.1. Definition of multiplicities.

THEOREM 7.1.1. *Let a tower $\{H_i\}$ of spaces and an $(n - k - 1)$-space L be given,* $0 \leqq k \leqq n - 1$. *Then there exist points $P_{\tau(0)}, \ldots, P_{\tau(n-k-1)}$, where*

$$0 \leqq \tau(0) < \ldots < \tau(n - k - 1) \leqq n,$$

such that $L = \vee_{j=0}^{n-k-1} P_{\tau(j)}$ and $P_{\tau(j)}$ is on $H_{\tau(j)}$ but not on

$$H_{\tau(j)-1}, \quad 0 \leqq j \leqq n - k - 1.$$

The numbers $\tau(0), \ldots, \tau(n - k - 1)$ are uniquely determined by the spaces H_i and L.

Proof. Suppose that $0 \leqq i \leqq n$ and $\dim (H_i \cap L) = d$. We show there exist points $P_{\tau(0)}, \ldots, P_{\tau(d)}$, where

$$0 \leqq \tau(0) < \ldots < \tau(d) \leqq i,$$

such that $H_i \cap L = \vee_{j=0}^{d} P_{\tau(j)}$ and $P_{\tau(j)}$ is on $H_{\tau(j)}$ but not on $H_{\tau(j)-1}$, $0 \leqq j \leqq d$.

This is true for $i = 0$; assume that it is true for $i - 1 < n$. If

$$H_{i-1} \cap L = H_i \cap L,$$

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the statement is clear. Thus, assume that $H_{i-1} \cap L \neq H_i \cap L$. Then $\dim (H_{i-1} \cap L) = d - 1$. Let $P_{\tau(0)}, \ldots, P_{\tau(d)}$ be points where

$$0 \leqq \tau(0) < \ldots < \tau(d-1) \leqq i - 1, \quad \tau(d) = i,$$

such that

$$H_{i-1} \cap L = \bigvee_{j=0}^{d-1} P_{\tau(j)}, P_{\tau(j)} \subset H_i \cap L$$

and $P_{\tau(j)}$ is on $H_{\tau(j)}$ but not on $H_{\tau(j)-1}$, $0 \leqq j \leqq d$. Then $H_i \cap L = \bigvee_{j=0}^{d} P_{\tau(j)}$ and the statement follows.

If the $\tau(j)$ were not unique, there would exist more than $n - k$ independent points which span L, which is a contradiction.

Let p be a point of an arc and let L be an $(n - k - 1)$-space, $0 \leqq k \leqq n - 1$. Putting $H_i = A_i(p)$, $-1 \leqq i \leqq n$, Theorem 7.1.1 implies there exist points $P_{\tau(0)}, \ldots, P_{\tau(n-k-1)}$, where $0 \leqq \tau(0) < \ldots < \tau(n - k - 1) \leqq n$, such that $L = \bigvee_{j=0}^{n-k-1} P_{\tau(j)}$ and $\pi(P_{\tau(j)}, p) = \tau(j) - 1$, $0 \leqq j \leqq n - k - 1$. We say that L has the *type* $(\tau(0), \ldots, \tau(n - k - 1))$ relative to p. Thus, every space L has a uniquely determined type relative to p.

THEOREM 7.1.2. *Suppose that $n \geqq 2$. Let P be a point of an $(n - k - 1)$-space L. Suppose that P is on $A_{\tau(r)}(p)$ but not on $A_{\tau(r)-1}(p)$, where $0 \leqq r \leqq n - k - 1$. Then the type of L relative to p on $\bar{A} = A|P$ is given by*

$$\bar{\tau}(i) = \begin{cases} \tau(i) & \text{if } \ 0 \leqq i < r \\ \tau(i+1) - 1 & \text{if } \ r < i + 1 \leqq n - k - 1. \end{cases}$$

Proof. Choose points $P_{\tau(i)}$, $0 \leqq i \leqq n - k - 1$, as in Theorem 7.1.1 with $H_i = A_i(p)$, $-1 \leqq i \leqq n$. We may take $P_{\tau(r)} = P$. Then

$$L = (PP_{\tau(0)}) \ldots (PP_{\tau(r-1)})(PP_{\tau(r+1)}) \ldots (PP_{\tau(n-k-1)}),$$

and the statement follows.

Let p be a point of an arc and let L be an $(n - k - 1)$-space, $0 \leqq k \leqq n - 1$. The *multiplicity* with which L meets A_k is defined as

$$\mu(p, L) = \sum_{i=0}^{n-k-1} \sum_{j=\tau(i)}^{i+k} \alpha_j(p),$$

where $(\tau(0), \ldots, \tau(n - k - 1))$ is the type of L. Thus, if L is a hyperplane, then

$$\mu(p, L) = \sum_{i=0}^{\delta(p,L)} \alpha_i(p) = \beta_{\delta(p,L)}(p).$$

THEOREM 7.1.3. *Suppose that $n \geqq 2$. Let P be a point of an $(n - k - 1)$-space L. For $\bar{A} = A|P$, one has*

$$\bar{\mu}(p, L) \equiv \mu(p, L) - \sum_{i=\pi(P,p)+1}^{k} \alpha_i(p),$$

$0 \leqq k \leqq n - 2$.

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Proof. Put $\pi = \pi(P, p)$. By Theorem 1.4.3,

$$\sum_{i=i_1}^{i_2} \bar{\alpha}_i(p) \equiv \begin{cases} \sum_{i=i_1+1}^{i_2+1} \alpha_i(p) & \text{if } \pi < i_1 \\ \sum_{i=i_1}^{i_2+1} \alpha_i(p) & \text{if } i_1 \leqq \pi \leqq i_2 \\ \sum_{i=i_1}^{i_2} \alpha_i(p) & \text{if } i_2 < \pi. \end{cases}$$

Choose points $P_{\tau(i)}$, $0 \leqq i \leqq n - k - 1$, as in Theorem 7.1.1 with $H_i = A_i(p)$, $-1 \leqq i \leqq n$, such that $P_{\tau(r)} = P$, where $0 \leqq r \leqq n - k - 1$. Then $\pi = \tau(r) - 1$. By Theorem 7.1.2,

$$\begin{aligned} \bar{\mu}(p, L) &= \sum_{i=0}^{n-k-2} \sum_{j=\bar{\tau}(i)}^{i+k} \bar{\alpha}_j(p) \\ &= \sum_{i=0}^{r-1} \sum_{j=\tau(i)}^{i+k} \bar{\alpha}_j(p) + \sum_{i=r+1}^{n-k-1} \sum_{j=\tau(i)-1}^{i+k-1} \bar{\alpha}_j(p). \end{aligned}$$

Case 1. $\pi < \tau(0)$. Since $\pi = \tau(r) - 1$, $r = 0$. Hence,

$$\begin{aligned} \bar{\mu}(p, L) &= \sum_{i=1}^{n-k-1} \sum_{j=\tau(i)-1}^{i+k-1} \bar{\alpha}_j(p) \\ &\equiv \sum_{i=1}^{n-k-1} \sum_{j=\tau(i)}^{i+k} \alpha_j(p) \\ &= \mu(p, L) - \sum_{j=\tau(0)}^{k} \alpha_j(p). \end{aligned}$$

Case 2. $\tau(0) \leqq \pi \leqq k$. Since $\tau(r - 1) \leqq \pi < \tau(r + 1) - 1$, we have

$$\begin{aligned} \bar{\mu}(p, L) &\equiv \sum_{i=0}^{r-1} \sum_{j=\tau(i)}^{i+k+1} \alpha_j(p) + \sum_{i=r+1}^{n-k-1} \sum_{j=\tau(i)}^{i+k} \alpha_j(p) \\ &= \mu(p, L) - \sum_{j=\tau(r)}^{r+k} \alpha_j(p) + \sum_{j=k+1}^{r+k} \alpha_j(p) \\ &= \mu(p, L) - \sum_{j=\tau(r)}^{k} \alpha_j(p). \end{aligned}$$

Case 3. $k < \pi$. Since $\tau(r - 1) \leqq \pi < \tau(r + 1) - 1$, we obtain

$$\begin{aligned} \bar{\mu}(p, L) &= \sum_{i=0}^{\pi-k-1} \sum_{j=\tau(i)}^{i+k} \bar{\alpha}_j(p) + \sum_{i=\pi-k}^{r-1} \sum_{j=\tau(i)}^{i+k} \bar{\alpha}_j(p) + \sum_{i=r+1}^{n-k-1} \sum_{j=\tau(i)-1}^{i+k-1} \bar{\alpha}_j(p) \\ &\equiv \sum_{i=0}^{\pi-k-1} \sum_{j=\tau(i)}^{i+k} \alpha_j(p) + \sum_{i=\pi-k}^{r-1} \sum_{j=\tau(i)}^{i+k+1} \alpha_j(p) + \sum_{i=r+1}^{n-k-1} \sum_{j=\tau(i)}^{i+k} \alpha_j(p) \\ &= \mu(p, L) - \sum_{j=\tau(r)}^{r+k} \alpha_j(p) + \sum_{j=\pi+1}^{r+k} \alpha_j(p) \\ &= \mu(p, L). \end{aligned}$$

7.2. Multiplicities for A^*.

THEOREM 7.2.1. *Let A be dually differentiable. Let* $(\tau(0), \ldots, \tau(n - k - 1))$

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be the type of an $(n-k-1)$-space L relative to p. Then the type of L relative to p on A^* is given by

$$\tau^*(i) = \begin{cases} i & \text{if } 0 \leqq i < n-\tau(n-k-1) \\ i+1 & \text{if } n-\tau(n-k-1) < i+1 < n-\tau(n-k-2) \\ \cdot & \\ \cdot & \\ \cdot & \\ i-n-k & \text{if } n-\tau(0) < i+n-k \leqq n. \end{cases}$$

Proof. Choose points $P_i \in A_i(p)\backslash A_{i-1}(p)$, $0 \leqq i \leqq n$, such that $P_{\tau(j)} \subset L$, $0 \leqq j \leqq n-k-1$. Put $Q_i = P_0 \ldots \hat{P}_{n-i} \ldots P_n$, $0 \leqq i \leqq n$. The hyperplane Q_i is a point Q_i^* of $\mathscr{P}^{n*}$. The point $P_i = Q_0 \cap \ldots \cap \hat{Q}_{n-i} \cap \ldots \cap Q_n$ is a hyperplane $P_i^* = Q_0^* \ldots \hat{Q}_{n-i}^* \ldots Q_n^*$ of $\mathscr{P}^{n*}$. The $(n-k-1)$-space L is a k-space L^* of $\mathscr{P}^{n*}$.

$$\begin{aligned} L^* &= P^*_{\tau(0)} \cap \ldots \cap P^*_{\tau(n-k-1)} \\ &= Q_0^* \ldots \hat{Q}_{n-\tau(0)}^* \ldots Q_n^* \cap \ldots \cap Q_0^* \ldots \hat{Q}_{n-\tau(n-k-1)}^* \ldots Q_n^* \\ &= Q_0^* \ldots \hat{Q}_{n-\tau(n-k-1)}^* \ldots \hat{Q}_{n-\tau(0)}^* \ldots Q_n^*. \end{aligned}$$

THEOREM 7.2.2. *Let A be a strongly finite arc. Then*

$$\mu^*(p, L) = \mu(p, L),$$

for every $(n-k-1)$-space of $\mathscr{P}^n$, $0 \leqq k \leqq n-1$.

Proof. By Theorems 6.2.4 and 7.2.1,

$$\begin{aligned} \mu^*(p, L) &= \sum_{i=0}^{k} \sum_{j=\tau^*(i)}^{i+n-k-1} \alpha_j^*(p) \\ &= \sum_{i=0}^{n-\tau(n-k-1)-1} \sum_{j=i}^{i+n-k-1} \alpha_j^*(p) + \sum_{i=n-\tau(n-k-1)+1}^{n-\tau(n-k-2)-1} \sum_{j=i}^{i+n-k-2} \alpha_j^*(p) \\ &\quad + \ldots + \sum_{i=n-\tau(1)+1}^{n-\tau(0)-1} \sum_{j=i}^{i} \alpha_j^*(p) + 0 \\ &= \sum_{i=0}^{n-\tau(n-k-1)-1} \sum_{j=k-i}^{n-i-1} \alpha_j(p) + \sum_{i=n-\tau(n-k-1)+1}^{n-\tau(n-k-2)-1} \sum_{j=k-i+1}^{n-i-1} \alpha_j(p) \\ &\quad + \ldots + \sum_{i=n-\tau(1)+1}^{n-\tau(0)-1} \sum_{j=n-i-1}^{n-i-1} \alpha_j(p) \\ &= \sum_{j=k}^{n-1} \alpha_j(p) + \ldots + \sum_{j=k-(n-\tau(n-k-1))+1}^{\tau(n-k-1)} \alpha_j(p) \\ &\quad + \sum_{j=k-(n-\tau(n-k-1))}^{\tau(n-k-1)-2} \alpha_j(p) + \ldots + \sum_{j=k-(n-\tau(n-k-2))+2}^{\tau(n-k-2)} \alpha_j(p) \\ &\quad \cdot \qquad\qquad\qquad\qquad \cdot \\ &\quad \cdot \qquad\qquad\qquad\qquad \cdot \\ &\quad \cdot \qquad\qquad\qquad\qquad \cdot \\ &\quad + \sum_{j=\tau(1)-2}^{\tau(1)-2} \alpha_j(p) + \ldots + \sum_{j=\tau(0)}^{\tau(0)} \alpha_j(p) \\ &= \mu(p, L). \end{aligned}$$

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The last equality follows since the sum of the $(m + 1)$-st terms is

$$\sum_{j=\tau(m)}^{m+k} \alpha_j(p), \quad 0 \leqq m \leqq n - k - 1.$$

7.3. Reduced rank numbers. Let K be an oriented circle. A mapping C: $K \to \mathscr{P}_0{}^n$ is directly differentiable at $p \in K$ if the restriction of C to a neighbourhood of p is directly differentiable at p. C is a *curve* if it is directly differentiable at each of its points.

THEOREM 7.3.1. *Let C be a curve. Then*

$$\sum_{p \in K} \mu(p, L_1) \equiv \sum_{p \in K} \mu(p, L_2),$$

for any two hyperplanes L_1 and L_2.

Proof. By Theorem 1.5.1, both sides are finite.

Suppose that $n = 1$ and $L_1 \neq L_2$. Let S be one of the two open segments of $\mathscr{P}_0{}^1$ determined by L_1 and L_2. If $C(p) = L_i$, let $m_i(p)$ be the number of one-sided neighbourhoods of p which are mapped into S. Thus, $0 \leqq m_i(p) \leqq 2$. Put

$$m_i = \sum_{C(p)=L_i} m_i(p).$$

Then

$$m_i \equiv \sum_{p \in K} \mu(p, L_i).$$

But $m_1 + m_2$ is twice the number of intervals (p, q) mapped into S with $\{C(p), C(q)\} \subset \{L_1, L_2\}$. Hence, $m_1 \equiv m_2$.

Assume that the theorem is true for $n - 1$. Let L be a hyperplane and let P be a point on L not on C. We project from P and use Theorem 1.4.3.

Case 1. $P \not\subset C_{\delta(p,L)}(p)$. Then $\bar{\delta}(p, L) = \delta(p, L)$ and $\delta(p, L) + 1 \leqq \pi(P, p)$. Hence,

$$\bar{\mu}(p, L) = \bar{\beta}_{\bar{\delta}} = \bar{\beta}_{\delta} = \beta_{\delta} = \mu(p, L).$$

Case 2. $P \subset C_{\delta(q,L)}(p)$. Then $\bar{\delta}(p, L) = \delta(p, L) - 1$ and $0 \leqq \pi(P, p) < \delta(p, L)$. Hence,

$$\begin{aligned} \bar{\mu}(p, L) &= \sum_{i=0}^{\bar{\delta}(p,L)} \bar{\alpha}_i(p) \\ &\equiv \sum_{i=0}^{\pi(P,p)-1} \alpha_i(p) + \alpha_{\pi(P,p)}(p) + \alpha_{\pi(P,p)+1}(p) + \sum_{i=\pi(P,p)+1}^{\delta(p,L)-1} \alpha_{i+1}(p) \\ &= \sum_{i=0}^{\delta(p,L)} \alpha_i(p) \\ &= \mu(p, L). \end{aligned}$$

Except possibly when $n = 2$, there is a point $P \subset L_1 \cap L_2$ not on C. Then the induction hypothesis and the above cases apply. If $n = 2$ and

$L_1 \cap L_2$ is on C, let P_i be a point on L_i not on C, $i = 1, 2$. Put $L = P_1P_2$. Then

$$\sum \mu(p, L_1) \equiv \sum \mu(p, L) \equiv \sum \mu(p, L_2).$$

THEOREM 7.3.2. *Let C be a strongly finite curve. Then*

$$\sum_{p\in K} \mu(p, L_1) \equiv \sum_{p\in K} \mu(p, L_2),$$

for any two $(n-k-1)$-spaces L_1 and L_2, $0 \leqq k \leqq n-1$.

Proof. Since C is a finite curve, both sides are finite. The theorem is true for $n = 1$, by Theorem 7.3.1; assume that it is true for $n-1$.

First, assume that $0 \leqq k \leqq n-2$. We may assume that L_1, L_2 have a point P in common. Then the induction hypothesis and Theorem 7.1.3 apply.

If $k = n-1$, then Theorems 7.2.2 and 7.3.1 apply.

Let C be a curve such that

$$\sum_{p\in K} \mu(p, L_1) = \sum_{p\in K} \mu(p, L_2),$$

for any two $(n-k-1)$-spaces L_1 and L_2, $0 \leqq k \leqq n-1$. *The reduced* k-th *rank number* ρ_k of C is defined to be 0 or 1 such that

$$\rho_k \equiv \sum_{p\in K} \mu(p, L),$$

where L is an $(n-k-1)$-space, $0 \leqq k \leqq n-1$. Also, put $\rho_{-1} = \rho_n = 0$. By Theorem 7.3.2, the reduced k-th rank number of a strongly finite curve is always defined.

THEOREM 7.3.3. *Let C be a strongly finite curve. If $n \geqq 2$ and one projects C from a point P, then*

$$\bar{\rho}_k \equiv \rho_k - \sum_{p\in K} \sum_{i=\pi(P,p)+1}^{k} \alpha_i(p),$$

$-1 \leqq k \leqq n-1$.

Proof. For $k = -1$, this is clear. For $0 \leqq k \leqq n-2$, use Theorem 7.1.3. If $k = n-1$ then the type of P relative to p is $\tau = \pi(P, p) + 1$. Hence,

$$\mu(p, P) = \sum_{j=\tau}^{n-1} \alpha_j(p) = \sum_{j=\pi(P,p)+1}^{n-1} \alpha_j(p).$$

THEOREM 7.3.4. *Let C be a strongly finite curve. The reduced* k-th *rank number ρ_k^* of C^* is defined and*

$$\rho_k^* = \rho_{n-k-1},$$

$-1 \leqq k \leqq n$.

Proof. Use Theorems 7.2.2 and 7.3.2.

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7.4. Congruences for elementary curves.

THEOREM 7.4.1. *Let C be an elementary curve. Then*

$$\sum_{p\in K} (\alpha_k(p) - 1) \equiv \rho_{k-1} + \rho_{k+1},$$

$0 \leqq k \leqq n - 1.$

Proof. Suppose that $n = 1$. Since C is elementary, each of the h singularities is an inflection. Since C changes direction an even number of times, h is even. Thus, $h = \sum_{p\in K} (\alpha_0(p) - 1)$ is even.

Assume that the theorem is true for $n - 1$.

Case 1. $0 \leqq k \leqq n - 2$. Projecting from a point P, we obtain

$$\begin{aligned}\sum_{p\in K} (\bar{\alpha}_k(p) - 1) &\equiv \sum_{k<\pi(P,p)} (\alpha_k(p) - 1) + \sum_{k=\pi(P,p)} (\alpha_k(p) + \alpha_{k+1}(p) - 1)\\ &\quad + \sum_{\pi(P,p)<k} (\alpha_{k+1}(p) - 1)\\ &= \sum_{p\in K} (\alpha_k(p) - 1) + \sum_{k=\pi(P,p)} \alpha_{k+1}(p)\\ &\quad + \sum_{\pi(P,p)<k} (\alpha_{k+1}(p) - \alpha_k(p)).\end{aligned}$$

By Theorem 7.3.3,

$$\begin{aligned}\bar{\rho}_{k-1} + \bar{\rho}_{k+1} &\equiv \rho_{k-1} + \rho_{k+1} - \sum_{p\in K} \sum_{i=\pi(P,p)+1}^{k-1} \alpha_i(p) - \sum_{p\in K} \sum_{i=\pi(P,p)+1}^{k+1} \alpha_i(p)\\ &\equiv \rho_{k-1} + \rho_{k+1} + \sum_{\pi(P,p)<k} (\alpha_k(p) + \alpha_{k+1}(p)) + \sum_{\pi(P,p)=k} \alpha_{k+1}(p).\end{aligned}$$

Case 2. $k = n - 1$. By Theorem 7.3.4,

$$\begin{aligned}\sum_{p\in K} (\alpha_{n-1}(p) - 1) &= \sum_{p\in K} (\alpha_0^*(p) - 1)\\ &\equiv \rho_{-1}^* + \rho_1^*\\ &= \rho_n + \rho_{n-2}.\end{aligned}$$

THEOREM 7.4.2. *Let C be an elementary curve. Then*

$$\sum_{p\in K} \sum_{i=0}^{n-1} (\alpha_i(p) - 1) \equiv \rho_0 + \rho_{n-1},$$

$$\sum_{p\in K} \sum_{i=0}^{n-1} (n - i)(\alpha_i(p) - 1) \equiv (n + 1)\rho_0.$$

Proof. Use Theorem 7.4.1. For the significance of the left hand sides compare with Theorems 8.9.2 and 8.10.1.

In [**7**], it is proven that the k-th rank number of a curve of order n is bounded and it is conjectured that the k-th rank number is $(k + 1)(n - k)$. The following shows that the k-th rank number, with multiplicites taken into account, is at least congruent to $(k + 1)(n - k)$ modulo 2:

THEOREM 7.4.3. *Let C be a curve of order n. Then*

$$\rho_k \equiv (k + 1)(n - k).$$

Proof. By Theorem 3.2.2, each point of C is regular. By Theorem 7.4.1, $\rho_{k-1} + \rho_{k+1} \equiv 0$, if $0 \leqq k \leqq n - 1$. From $\rho_n + \rho_{n-2} \equiv \rho_{n-2} + \rho_{n-4} \equiv \ldots$ and $\rho_{n-1} + \rho_{n-3} \equiv \rho_{n-3} + \rho_{n-5} \equiv \ldots$ it follows that

$$\rho_k \equiv \begin{cases} 0 \text{ } if \text{ } k \equiv n \\ n \text{ } if \text{ } k \equiv n - 1. \end{cases}$$

In both cases, $\rho_k \equiv (k + 1)(n - k)$.

THEOREM 7.4.4. *Let C be an elementary curve with tower. Then $\rho_k = 0$, $-1 \leqq k \leqq n$, and $\sum_{p \in K} (\alpha_k(p) - 1)$ is even, $0 \leqq k \leqq n - 1$.*

Proof. $\rho_k \equiv \sum_{p \in K} \mu(p, H_{n-k-1}) = 0$. Thus, Theorem 7.4.4 follows from Theorem 7.4.1.

7.5. Interpretation of multiplicities.

THEOREM 7.5.1. *Let A be an ordinary arc and let H be an $(n - k)$-space such that $A_{k-1}(q) \cap H = \emptyset$, for all $q \in J$, $0 \leqq k \leqq n - 1$. Then $\bar{A}(q) = A_k(q) \cap H$ is an ordinary arc in H.*

Proof. Since $A_{k-1}(q)H$ has dimension n, $A_{n-k}^*(q) \cap H^* = \emptyset$. By Theorem 6.3.1, A_0^* is an ordinary arc. The projection $A_0^*|H^*$ of A_0^* from the $(k - 1)$-space H^* is ordinary, by successive applications of Theorem 5.2.4. In $\mathscr{P}^n$, $A_0^*(q)H^*$ is a hyperplane $A_{n-1}(q) \cap H$ of H. By Theorem 6.3.1 applied to the dual of the projective space H, $\bar{A}(q) = A_k(q) \cap H$ is an ordinary arc of H.

THEOREM 7.5.2. *Let p be a point of an arc A of order n and let L be an $(n - k - 1)$-space, $0 \leqq k \leqq n - 1$. If $\mu(p, L) = 1$, then there is a neighbourhood $U(L)$ of L such that, for every $M \in U(L)$, there is a $q \in J$ with $\mu(q, M) = 1$.*

Proof. Choose points $P_i \in A_i(p)\backslash A_{i-1}(p)$, $0 \leqq i \leqq n$, such that

$$L = P_k P_{k+2} \ldots P_n.$$

Let H_∞ be a hyperplane with $P_0, \ldots, P_k \not\subset H_\infty$ and $P_{k+1}, \ldots, P_n \subset H_\infty$. The equations

7.5.3 $$A_{k-1}(q) \cap P_{k+1}M = \emptyset$$

7.5.4 $$A_k(q) \cap P_{k+1}(M \cap H_\infty) = \emptyset$$

7.5.5 $$A_{k+1}(q) \cap (M \cap H_\infty) = \emptyset$$

hold for $q = p$, $M = L$. There exist neighbourhoods $U(p)$ and $U(L)$ such that 7.5.3, 7.5.4, and 7.5.5 hold, for all $q \in U(p)$, $M \in U(L)$. By 7.5.5, $P_{k+1} \not\subset M$ and $M \not\subset H_\infty$, for all $M \in U(L)$.

By Theorems 7.5.1 and 7.5.3, p is an ordinary point of the arc

$$\bar{A}(q) = A_k(q) \cap P_{k+1}L.$$

Since $\bar{A}_1(p) = P_k P_{k+1}$, L cuts $\bar{A}$ at p. Take $r, s \in U(p)$ with $r < p < s$ such that $\bar{A}(r)$ and $\bar{A}(s)$ are separated in $P_{k+1}L$ by L and $P_{k+1}(L \cap H_\infty)$. We may assume that $A_k(r) \cap P_{k+1}M$ and $A_k(s) \cap P_{k+1}M$ are separated in $P_{k+1}M$ by M and $P_{k+1}(M \cap H_\infty)$, for all $M \in U(L)$.

Let $M \in U(L)$. By 7.5.3, $A_k(q) \cap P_{k+1}M$ is a point of $P_{k+1}M$ for all $q \in U(p)$; by 7.5.4, this point does not lie on $P_{k+1}(M \cap H_\infty)$. Hence, $A_k(q) \cap M \neq \emptyset$, for some $q \in (r, s)$. By 7.5.3, $A_{k-1}(q) \cap M = \emptyset$. If $A_{k+1}(q) \cap M$ were a line, the infinite point on this line would contradict 7.5.5. Thus, there is no point of M in $A_{k+1}(q) \backslash A_k(q)$. Hence, $\mu(q, M) = 1$.

THEOREM 7.5.6. *Let p be an elementary point and let L be an $(n-k-1)$-space of type $(\tau(0), \ldots, \tau(n-k-1))$, $0 \leqq k \leqq n-1$. Suppose that there is a j such that either $\tau(j) + 1 < \tau(j+1)$, $0 \leqq j \leqq n-k-2$, or $\tau(j) < n$, $j = n-k-1$. Then any neighbourhood $U(L)$ of L contains a space M of type*

$$(\tau(0), \ldots, \tau(j-1), \tau(j)+1, \tau(j+1), \ldots, \tau(n-k-1)),$$

for which there exist $\alpha_{\tau(j)}(p)$ ordinary points q with $\mu(q, M) = 1$.

Proof. We may assume that each point $q \neq p$ is ordinary. Choose

$$P_i \in A_i(p) \backslash A_{i-1}(p), \qquad 0 \leqq i \leqq n,$$

such that $L = P_{\tau(0)} \ldots P_{\tau(n-k-1)}$. Put $G = P_{\tau(0)} \ldots P_{\tau(j-1)} P_{\tau(j+1)} \ldots P_{\tau(n-k-1)}$ and $H = P_{\tau(j)+1}L$. Then $G \subset L \subset H$. By Theorem 5.2.1, we may assume

7.5.7 $$A_{k-1}(q) \cap H = \emptyset,$$

7.5.8 $$A_{k+1}(q) \cap G = \emptyset,$$

for all $q \neq p$.

By Theorem 5.2.2, p is an elementary point of $\bar{A} = A|G$. Since $H = (P_{\tau(j)}G)(P_{\tau(j)+1}G)$, one has $\bar{H} = \bar{P}_{\tau(j)-j}\bar{P}_{\tau(j)-j+1}$. Let φ be the dual projection of $\bar{A}$ in $\bar{H}$. By Theorem 6.1.2, $\varphi(p) = \bar{P}_{\tau(j)-j}$; by Theorems 5.2.1 and 6.4.1, φ has the characteristic $(\bar{\alpha}_{\tau(j)-j}(p))$ at p. Thus, $\varphi(p) = L$; since $\tau(j) + 1 < \tau(j+1)$, φ has the characteristic $(\alpha_{\tau(j)}(p))$ at p.

If $q \neq p$, then

$$\begin{aligned} \varphi(q) &= \bar{A}_k(q) \cap \bar{H} \\ &= A_k(q)G \cap H \\ &= (A_k(q) \cap H)G. \end{aligned}$$

Thus, φ may be regarded as the projection of an arc in H from G. Hence, any neighbourhood $U(L)$ of $L = \varphi(p)$ contains a space $M \neq L$ with $G \subset M \subset H$ for which there exist $\alpha_{\tau(j)}(p)$ ordinary points q with $\varphi(q) = M$.

We show that $\mu(q, M) = 1$. By 7.5.7, $A_{k-1}(q) \cap M = \emptyset$. Since $\varphi(q) = M$, $A_k(q) \cap M \neq \emptyset$. The space $A_{k+1}(q) \cap M$ cannot be a line, for, otherwise, it would meet the hyperplane G of M in a point, contradicting 7.5.8. Thus, M has the type $(k, k + 2, \ldots, n)$ relative to q. By Theorem 3.2.2, q is regular. Thus, $\mu(q, M) = 1$.

Since $M \neq L$ and $L = P_{\tau(j)}G$, $P_{\tau(j)} \not\subset M$. Thus, the line $P_{\tau(j)}P_{\tau(j)+1}$ meets the hyperplane M of H in a point $P \in A_{\tau(j)+1}(p)\backslash A_{\tau(j)}(p)$. Since $G \subset M$, M has the required type.

THEOREM 7.5.9. *Let p be an elementary point and let L be an $(n - k - 1)$-space $0 \leqq k \leqq n - 1$. Any neighbourhood $U(L)$ of L contains a space M for which there exist $\mu(p, L)$ distinct ordinary points q with $\mu(q, M) = 1$.*

Proof. By successive applications of Theorem 7.5.6, one is able to construct the $\mu(p, L)$ ordinary points. By Theorem 7.5.2, no points obtained in preceding steps need be lost when new points are gained.

THEOREM 7.5.10. *The k-th rank of an elementary point p is at least*

$$\sum_{i=0}^{n-k-1} \sum_{j=i}^{i+k} \alpha_j(p).$$

Proof. By Theorem 7.5.9, with $L = A_{n-k-1}(p)$, there is an $(n - k - 1)$-space meeting $\mu(p, A_{n-k-1}(p))$ osculating k-spaces. But

$$\mu(p, A_{n-k-1}(p)) = \sum_{i=0}^{n-k-1} \sum_{j=i}^{i+k} \alpha_j(p).$$

8. BARNER ARCS

In [**1**], strongly convex arcs in $\mathscr{P}^n$ were defined. In the plane, the condition that an arc be strongly convex is roughly that through each point p of the arc there pass a line $B(p)$ which depends continuously on p and meets the arc only at p. Using analytic methods, Barner proved that strongly convex arcs satisfy an inequality similar to that of Theorem 8.6.1. Later, Haupt studied similar arcs which he called arcs without $(n - 2, k)$-secants in the strong sense. His work is completely synthetic and is outlined in [**7**].

The Barner arcs which we define may seem to have little in common with Barner's or Haupt's arcs; cf. 8.1.1. Nevertheless, we are able to prove (cf. Theorem 8.6.1), the analog of Barner's theorem. Perhaps the most significant way in which our arcs differ from Barner's or Haupt's is that they can have other points than regular points and inflections; cf. Theorem 8.4.1.

It is after our study of Barner arcs that we discovered arcs with tower. Such arcs satisfy a Barner-type inequality; cf. Theorem 8.7.1. By the early introduction, in § 4, of arcs with tower this theory has been considerably simplified. In Theorem 8.8.1, the inequality for arcs with tower is used to establish Denk's theorem.

8.1. Existence of Barner arcs. An arc is a *Barner arc* if there exists a continuous mapping $B: J^{n-1} \to \mathscr{P}^n{}_{n-1}$ such that

8.1.1 $$\delta(p, B(x)) = \beta_{\gamma(p,x)}(p) - 1,$$

for all $p \in J$, $x \in J^{n-1}$.

THEOREM 8.1.2. *Let A be an arc with tower which has at most inflections. Assume that A_k is continuous, $0 \leqq k \leqq n - 2$. Then A is a Barner arc.*

Proof. For $n = 1$, any arc whose image is not $\mathscr{P}_0{}^1$ is a Barner arc. Therefore, assume that $n \geqq 2$. Let $\{H_i\}$ be a tower for A. $\bar{A} = A|H_0$ is a regular arc with tower $\{H_0, \ldots, H_n\}$. $\bar{A}_k(p) = A_k(p)H_0$ is continuous, $0 \leqq k \leqq n - 2$. By Theorem 4.1, $\bar{A}$ is of order $n - 1$. Put $B(x) = \bar{A}^{n-2}(x)$. By Theorem 3.4.1, B is continuous. By Theorems 2.3.2 and 3.1.1, $\delta(p, B(x)) = \bar{\delta}(p, \bar{A}^{n-2}(x)) = \gamma(p,x)$ Since $\gamma(p, x) \leqq n - 2$ and p is at most an inflection,

$$\beta_{\gamma(p,x)}(p) - 1 = \sum_{0}^{\gamma(p,x)} 1 - 1 = \gamma(p, x).$$

Thus, A is a Barner arc.

For the remainder of § 8, we shall assume that A is a Barner arc.

8.2. Projection of Barner arcs.

THEOREM 8.2.1. *$A(p) \subset B(p_1, \ldots, p_{n-1})$ if and only if $p \in \{p_1, \ldots, p_{n-1}\}$.*

THEOREM 8.2.2. *$\delta(p, B(p, \ldots, p)) = \beta_{n-2}(p) - 1$, for all $p \in J$.*

THEOREM 8.2.3. *$A_{n-2}(p) \subset B(p, \ldots, p)$, for all $p \in J$.*

THEOREM 8.2.4. *If $p \neq q$, then $A(p) \not\subset A_{n-2}(q)$.*

Proof. By Theorem 8.2.1, $A(p) \not\subset B(q, \ldots, q)$. By Theorem 8.2.3, $A_{n-2}(q) \subset B(q, \ldots, q)$.

THEOREM 8.2.5. *If $n \geqq 2$, A is simple, i.e., if $p \neq q$, then $A(p) \neq A(q)$.*

THEOREM 8.2.6. *If $p \neq q$ and $A(p) \subset A_{n-1}(q)$, then $\beta_{n-2}(q) = n - 1$.*

Proof. Since $A(p) \subset A_{n-1}(q)$, $B(q, \ldots, q) \neq A_{n-1}(q)$, by Theorem 8.2.1. Thus, $\delta(q, B(q, \ldots, q)) \leqq n - 2$. By Theorem 8.2.3, $\delta(q, B(q, \ldots, q)) = n-2$. The statement now follows from Theorem 8.2.2.

THEOREM 8.2.7. *Suppose that $n \geqq 2$, $q \in J$. If A has at most inflections, then so has $A|q$.*

Proof. By Theorem 1.4.3, q is at most an inflection of $A|q$. If $p \neq q$, p is at most an inflection of $A|q$ by Theorems 1.4.3 and 8.2.4.

THEOREM 8.2.8. *Suppose that $n \geqq 2$, $q \in J$. If $\alpha_0(q) = 1$, then $A|q$ is a Barner arc. If $\alpha_0(q) = 2$, then the restriction of $A|q$ to either component of $J\backslash\{q\}$ is a Barner arc.*

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Proof. Given $y = (p_1, \ldots, p_{n-2})$, put $x = (q, p_1, \ldots, p_{n-2})$ and $\bar{B}(y) = B(x)$. $\bar{B}$ is continuous.

Suppose that $p = q$, $\alpha_0(q) = 1$. Then $\bar{\delta}(p, \bar{B}(y)) = \delta(p, B(x)) - 1$. By Theorem 1.4.3, $\bar{\beta}_k(p) = \beta_{k+1}(p) - \beta_0(p)$, for $-1 \leqq k \leqq n-2$. Since $\beta_0(p) = 1$

$$\bar{\beta}_{\gamma(p,y)}(p) = \beta_{\gamma(p,y)+1}(p) - 1 = \beta_{\gamma(p,x)}(p) - 1.$$

Thus, by 8.1.1,

8.2.9 $$\bar{\delta}(p, \bar{B}(y)) = \bar{\beta}_{\gamma(p,y)}(p) - 1.$$

Next, suppose that $p \neq q$, $A(q) \subset A_{n-1}(p)$. By Theorem 8.2.4,

$$A(q) \not\subset A_{n-2}(p);$$

hence, $\pi(q, p) = n - 2$. By Theorem 8.2.6, $\beta_{n-2}(p) = n - 1$; hence,

$$\delta(p, B(x)) = \beta_{\gamma(p,x)}(p) - 1 < \beta_{n-2}(p) - 1 = n - 2.$$

By Lemma 1.4.1,

8.2.10 $$\bar{\delta}(p, \bar{B}(y)) = \delta(p, B(x)).$$

If $p \neq q$ and $A(q) \not\subset A_{n-1}(p)$, then $\pi(q, p) = n - 1$ and 8.2.10 again holds.

If $p \neq q$, then $\gamma(p, y) = \gamma(p, x) < n - 2 \leqq \pi(q, p)$, so by Theorem 1.4.3,

$$\bar{\beta}_{\gamma(p,y)}(p) = \beta_{\gamma(p,y)}(p) = \beta_{\gamma(p,x)}(p).$$

By 8.1.1 and 8.2.10, it follows that 8.11 also hold if $p \neq q$.

8.3. Independence properties. Suppose that A is a Barner arc in $\mathscr{P}^4$ and p, q are distinct points with characteristic (2, 1, 1, 1). Then

$$A_1(p)A_1(q) \subset B(p, q, r),$$

for all $r \in J$. If $A_1(p) \cap A_1(q) = \emptyset$, then the inclusion is improper and the arc lies in the hyperplane $A_1(p)A_1(q)$, contradicting Theorem 1.2.1. Thus, A has a dependent 2-secant, namely $A_1(p)A_1(q)$.

THEOREM 8.3.1. *An $(n - 2)$-space L can meet a Barner arc in at most $n - 1$ distinct points.*

Proof. Suppose that there are distinct points $p_1, \ldots, p_n$ such that $A(p_i) \subset L$, $1 \leqq i \leqq n$. Then for some j, $A(p_j) \subset \vee_{i \neq j} A(p_i)$. Put $x = (p_1, \ldots, \hat{p}_j, \ldots, p_n)$. Then $A(p_i) \subset B(x)$, for all $i \neq j$. Thus, $A(p_j) \subset B(x)$, contradicting Theorem 8.2.1.

THEOREM 8.3.2. *Suppose that $n \geqq 3$, $1 \leqq k \leqq n - 2$. If $\beta_{k-2}(p) = k - 1$, for all $p \in J$, then A is k-independent.*

Proof. By Theorem 8.3.1, a line can meet A in at most 2 points. Hence, by Theorem 8.2.4, A is 1-independent and Theorem 8.3.2 is true if $k = 1$.

Assume that it is true for $k-1$, where $2 \leqq k \leqq n-2$. Let L be a k-secant of A and let $A(q) \subset L$. Then $\alpha_0(q) = 1$; by Theorem 8.2.8, $A|q$ is a Barner arc. By Theorem 1.4.3, $\bar{\beta}_{k-3}(q) = \beta_{k-2}(q) - \beta_0(q) = k-2$. If $p \neq q$, then $k-3 < n-2 \leqq \pi(q,p)$ by Theorem 8.2.4; hence, $\bar{\beta}_{k-3}(p) = \beta_{k-3}(p) = k-2$. Thus, $A|q$ is $(k-1)$-independent; in particular, L is independent on $A|q$. Thus,

$$k-1 = \sum_{p \in J} (\bar{\delta}(p,L)+1) - 1.$$

Now $\bar{\delta}(q,L) = \delta(q,L) - 1$. If $p \neq q$, then $A(q) \not\subset A_{n-2}(p)$, by Theorem 8.2.4; since $A(q) \subset L$ and $\dim L \leqq n-2$, $\delta(p,L) \leqq n-3 < \pi(q,p)$. Hence, if $p \neq q$, $\bar{\delta}(p,L) = \delta(p,L)$. Thus, L is an independent secant of A.

THEOREM 8.3.3. *If A has at most inflections it is $(n-2)$-independent.*

8.4. The characteristic.

THEOREM 8.4.1. *If $p \in J$, then $\alpha_i(p) = 2$, for at most one i, $0 \leqq i \leqq n-1$.*

Proof. For $n = 1$, this is obvious.

For $n = 2$, suppose that $\alpha_0(p) = \alpha_1(p) = 2$. Take $U'(p)$ and a line H_∞ such that $A(p) \not\subset H_\infty$ and

$$A(q) \not\subset A_1(p),\ A(q) \not\subset H_\infty, \text{ for all } q \in U'(p).$$

If one projects from p, $\bar{\alpha}_0(p) = \alpha_1(p) = 2$. Thus, there exist points $p_1, p_2 \in U'(p)$ and a line L such that $p_1 < p < p_2$ and $A(p_1), A(p), A(p_2) \subset L$. Since $\sigma_1(p) \equiv \alpha_0(p) + \alpha_1(p) = 4 \pmod 2$, $A(p_1), A(p_2)$ lie on the same side of $A_1(p)$. If, say, $A(p_1)$ lies between $A(p)$ and $A(p_2)$ on L, then there is a $q \in [p, p_2]$ with $A(q) \subset B(p_1)$, which is a contradiction.

Assume that the theorem is true for $n-1$, and suppose that for some p, $\alpha_i(p) = \alpha_j(p) = 2$, where $i < j$. Since $\delta(p, B(p, \ldots, p)) \leqq n-1$, $\beta_{n-2}(p) \leqq n$ by Theorem 8.2.2. Thus, in our case, $\beta_{n-2}(p) = n$ and $j = n-1$. Also, $i = 0$, for, otherwise, projection from p gives a contradiction. Projecting from $A_{n-2}(p)$ and using $\alpha_{n-1}(p) = 2$, one obtains points p_1, p_2 with $p_1 < p < p_2$ such that $A(p_1) \subset A_{n-2}(p)A(p_2)$. But this is impossible since

$$A_{n-2}(p)A(p_2) = B(p, \ldots, p, p_2),$$

by $\alpha_0(p) = 2$.

8.5. Ordinary subarcs of Barner arcs.

THEOREM 8.5.1. *If (p, q) is ordinary, then $A(p) \not\subset A_{n-1}(q)$.*

Proof. The theorem is true for $n = 1$, by Theorem 4.1; assume that it is true for $n-1$. If $\alpha_0(q) = 2$, then $A_{n-1}(q) = B(q, \ldots, q)$ and $A(p) \not\subset A_{n-1}(q)$. Hence, we may assume that $\alpha_0(q) = 1$.

Suppose that $A(p) \subset A_{n-1}(q)$. Then $\bar{A} = A|q$ is a Barner arc with $\bar{A}(p) \subset \bar{A}_{n-2}(q)$. By the induction assumption, (p, q) is not ordinary on $\bar{A}$. By

Theorem 5.2.4, there exists $p_1 \in (p, q)$ such that $A(q) \subset A_{n-1}(p_1)$. We now project the interval (p_1, q) on A from p_1 obtaining $q_1 \in (p_1, q)$ such that $A(p_1) \subset A_{n-1}(q_1)$.

Consider the set S of intervals (r, s) of (p_1, q_1) such that $A(r) \subset A_{n-1}(s)$. Repeating the above argument, $S \neq \emptyset$. Let s_0 be the infimum of the $s \in (p_1, q_1)$ for which there exists an r with $(r, s) \in S$. Since p_1 is ordinary, $p_1 < s_0$ by Theorem 3.1.1. Let r_i, s_i be such that $(r_i, s_i) \in S$, r_i converges, say, to r_0 and $s_i \to s_0$. Since s_0 is ordinary, $r_0 < s_0$ and $A(r_0) \subset A_{n-1}(s_0)$. Repeating the argument of the preceding paragraph we obtain $(r, s) \in S$ with $s < s_0$, contradicting the definition of s_0.

THEOREM 8.5.2. *If (p, q) is ordinary, then $[p, q]$ is of order n; thus, an ordinary Barner arc is of order n.*

Proof. The theorem is true for $n = 1$, by Theorem 4.1; assume that it is true for $n - 1$. Suppose that $p_1, \ldots, p_{n+1}$ lie in a hyperplane, where

$$p \leqq p_1 < \ldots < p_{n+1} \leqq q.$$

Since $p_2 \in (p, q)$, $\alpha_0(p_2) = 1$, by Theorem 3.2.2 and $A|p_2$ is a Barner arc. By Theorem 3.1.2, p_2 is ordinary on $A|p_2$. If $r \in (p, q)$, $r \neq p_2$, then

$$A(p_2) \not\subset A_{n-1}(r),$$

by Theorem 8.5.1; hence, r is ordinary on $A|p_2$, by Theorem 5.2.4. By the induction assumption, $[p, q]$ is of order $n - 1$ on $A|p_2$. This is a contradiction since $p_1, p_3, \ldots, p_{n+1}$ lie in a hyperplane in $\mathscr{P}^{n-1}(A(p_2))$

THEOREM 8.5.3. *If $p < q < r$ and (q, r) is ordinary, then*

$$A(p) \not\subset A_{n-1}(q) \cap A_{n-1}(r).$$

Proof. The theorem is true for $n = 1$; assume that $n \geqq 2$. By Theorems 8.5.2 and 3.4.1, A_{n-2} and A_{n-1} are continuous on $[q, r]$.

Suppose that $A(p) \subset A_{n-1}(q) \cap A_{n-1}(r)$, $p < q < r$. By Theorem 6.3.1, there is an $s_0 \in (q, r)$ such that $A(p) \not\subset A_{n-1}(s_0)$. Thus, there exist q_0, r_0 such that $q \leqq q_0 < s_0 < r_0 \leqq r$,

8.5.4 $$A(p) \not\subset A_{n-1}(s),$$

for all $s \in (q_0, r_0)$ and

8.5.5 $$A(p) \subset A_{n-1}(q_0) \cap A_{n-1}(r_0).$$

By 8.5.4 and Theorem 5.2.4, (q_0, r_0) is ordinary on $A|p$.

By Theorem 8.3.1, $A(p)$, $A(q_0)$, $A(r_0)$ span a plane M. Put $L_1 = A(p)A(q_0)$, $L_2 = A(p)A(r_0)$. If $s \in (q_0, r_0]$, then $A_{n-2}(s) \cap L_1 = \emptyset$; otherwise, projection from p yields a contradiction of Theorem 8.5.1. Thus, $A_{n-2}(s) \cap M$ is a point for all $s \in [q_0, r_0]$ and does not lie on L_1 or L_2, if $s \in (q_0, r_0)$. Since it depends continuously on s, there is a line L such that $A(p) \subset L \subset M$ and $A_{n-2}(s) \cap L = \emptyset$, for all $s \in [q_0, r_0]$. Put

$$\varphi(s) = A_{n-1}(s) \cap L, \quad \psi(s) = B(s, \ldots, s) \cap L,$$

for all $s \in [q_0, r_0]$. φ and ψ are continuous. By Theorem 7.5.1, φ is monotone. By 8.5.5, $\varphi(s)$ moves from $A(p)$ to $A(p)$ when s moves from q_0 to r_0. If $s \in (q_0, r_0)$, s is regular, by Theorem 3.2.2 and so $\varphi(s) \neq \psi(s)$, by Theorem 8.2.2. Also $\psi(s) \neq A(p)$, for all $s \in [q_0, r_0]$. Such φ and ψ cannot exist.

8.6. Barner's theorem. We define the *multiplicity* of a point p as

$$\nu(p) = \sum_{i=0}^{n-1} (n - i)(\alpha_i(p) - 1).$$

THEOREM 8.6.1. *Let A be an elementary Barner arc. Then*

$$\sum_{p \in J} \mu(p, L) \leqq \sum_{p \in J} \nu(p) + n,$$

for every hyperplane L.

Proof. Suppose that the theorem is true for $\sum_{p \in J} \mu(p, L)$ finite. Then

$$\sum_{p_1 < p < p_2} \mu(p, L) \leqq \sum_{p_1 < p < p_2} \nu(p) + n$$
$$\leqq \sum_{p \in J} \nu(p) + n.$$

If $\sum \mu(p, L)$ is infinite, then so is $\sum_{p \in J} \nu(p)$ and the theorem is true. We may, therefore, assume that $\sum_{p \in J} \mu(p, L)$ is finite. We may also assume that $\sum_{p \in J} \nu(p)$ is finite. Then A has only finitely many non-regular points.

Suppose that $n = 1$. If $A(p) = L$, for k points p, and h of these points are inflections, then $\sum_{p \in J} \mu(p, L) = h + k$. Since A is a Barner arc, each of the $k - 1$ open intervals determined by these k points contains an inflection. Thus,

$$k - 1 + h \leqq \sum_{p \in J} \nu(p).$$

Assume that the theorem is true for $n - 1$ and let L be a hyperplane.

Case 1. There is a point q with $\alpha_0(q) = 1$ and $A(q) \subset L$.
Put $\bar{A} = A|q$.

LEMMA 8.6.2.

$$\sum_{p \in J} \bar{\mu}(p, L) \leqq \sum_{p \in J} \bar{\nu}(p) + n - 1.$$

Proof. Use Theorems 5.2.2 and 8.2.8 and the induction hypothesis.

LEMMA 8.6.3. *Let X be the set of inflections $r \neq q$ of A with $A_{n-1}(r) = L$. Then*

$$\sum_{p \in J} \bar{\mu}(p, L) = \sum_{p \in J} \mu(p, L) - 2|X| - 1.$$

Proof. Write μ, β_k, δ instead of $\mu(p, L)$, $\beta_k(p)$, $\delta(p, L)$. Note that X is finite.

Case (i). $p = q$. By Lemma 1.4.1, $\bar{\delta} = \delta - 1$. Thus, by Theorem 1.4.3, $\bar{\mu} = \bar{\beta}_{\bar{\delta}} = \bar{\beta}_{\delta-1} = \beta_\delta - \beta_0 = \beta_\delta - 1 = \mu - 1$, since $\alpha_0(q) = 1$.

Case (ii). $\pi(q, p) = n - 2$, $\delta < n - 2$. One has $\bar{\delta} = \delta$; so $\bar{\mu} = \bar{\beta}_{\bar{\delta}} = \bar{\beta}_\delta = \beta_\delta = \mu$.

Case (iii). $\pi(q, p) = n - 2$, $\delta = n - 1$. One has $\bar{\delta} = \delta - 1$. Also,

$$\bar{\mu} = \bar{\beta}_{\bar{\delta}} = \sum_{i=0}^{n-2} \bar{\alpha}_i(p)$$

and

$$\mu = \beta_\delta = \sum_{i=0}^{n-1} \alpha_i(p).$$

If p is regular $\bar{\mu} = \mu$; if p is an inflection $\bar{\mu} = \mu - 2$; by Theorem 8.2.6, only these possibilities can occur.

Case (iv). $\pi(q, p) = n - 1$. Here, $\delta < n - 1$ and $\bar{\delta} = \delta$. Thus,

$$\bar{\mu} = \bar{\beta}_{\bar{\delta}} = \bar{\beta}_\delta = \beta_\delta = \mu.$$

LEMMA 8.6.4. *Let Y be the set of regular points $r \neq q$ of A with $A(q) \subset A_{n-1}(r)$; let Z be the set of non-regular points $r \neq q$. Then*

$$\sum_{p \in J} \bar{\nu}(p) = \sum_{p \in J} \nu(p) + |Y| - |Z|.$$

Proof. We are assuming that Z is finite. By Theorems 5.2.3 and 8.5.3, Y is finite.

Case (i). $p = q$. Then

$$\begin{aligned}\bar{\nu} &= \sum_{i=0}^{n-2} (n - i - 1)(\bar{\alpha}_i(p) - 1)\\ &= \sum_{i=0}^{n-2} (n - i - 1)(\alpha_{i+1}(p) - 1)\\ &= \sum_{i=1}^{n-1} (n - i)(\alpha_i(p) - 1) = \nu,\end{aligned}$$

since $\alpha_0(q) = 1$.

Case (ii). $\pi(q, p) = n - 2$. If p is regular, then $\nu = 0$, $\bar{\nu} = 1$. If p is an inflection, then $\nu = 1$, $\bar{\nu} = 0$.

Case (iii). $\pi(q, p) = n - 1$.

$$\begin{aligned}\bar{\nu} &= \sum_{i=0}^{n-2} (n - i - 1)(\bar{\alpha}_i(p) - 1)\\ &= \sum_{i=0}^{n-1} (n - i)(\alpha_i(p) - 1) - \sum_{i=0}^{n-1} (\alpha_i(p) - 1)\\ &= \nu - \sum_{i=0}^{n-1} (\alpha_i(p) - 1).\end{aligned}$$

Thus, $\bar{\nu} = \nu$ if p is regular; otherwise, $\bar{\nu} = \nu - 1$, by Theorem 8.4.1.

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LEMMA 8.6.5. *Let X, Y, Z be as in Lemmas* 8.6.3 *and* 8.6.4. *Then*

$$2|X| + |Y| - |Z| \leqq 0.$$

Proof. The set $X \cup Y \cup \{q\}$ determines $|X| + |Y|$ open intervals. By Theorems 8.5.1 and 8.5.3, none of these intervals is ordinary; by Theorem 5.2.3, each of these intervals contains a non-regular point. Hence,

$$|X| + |Y| \leqq |Z \backslash X| = |Z| - |X|.$$

By Lemmas 8.6.2–8.6.5,

$$\begin{aligned}\sum \mu(p, L) &= \sum \bar{\mu}(p, L) + 2|X| + 1\\ &\leqq \sum \bar{\nu}(p) + (n-1) + 2|X| + 1\\ &= \sum \nu(p) + |Y| - |Z| + 2|X| + n\\ &\leqq \sum \nu(p) + n.\end{aligned}$$

This concludes the proof of Theorem 8.6.1 in case 1.

Case 2. $\alpha_0(q) = 2$, for all q with $A(q) \subset L$, and there are at least two such points. By Theorem 8.2.6, $\delta(q, L) \leqq n - 2$, for all q with $A(q) \subset$ L. By Theorem 8.4.1, $\mu(q, L) \leqq n = \nu(q)$, for such q; hence,

$$\sum_{p \in J} \mu(p, L) \leqq \sum_{p \in J} \nu(p) \leqq \sum_{p \in J} \nu(p) + n.$$

Case 3. There is only one point q with $A(q) \subset L$ and this point satisfies $\alpha_0(q) = 2$. Then $\sum \mu(p, L) \leqq n + 1$. Let $M = A_{n-2}(q)A(r)$, $r \neq q$. Then $n + 1 \leqq \sum \mu(p, M)$, so $\sum \mu(p, L) \leqq \sum \mu(p, M)$ and Barner's theorem follows.

THEOREM 8.6.6. *The inequality of Theorem* 8.6.1 *holds if A is a Barner arc with at most countably many singularities.*

Proof. Suppose that A is regular. By Theorem 3.2.1, the set S of singularities of A is closed. By Theorems 5.2.3 and 8.5.2, S contains no isolated points. Since a non-empty perfect subset of J is uncountable, $S = \emptyset$; cf. [**8**]. By Theorem 8.5.2, A is of order n.

Suppose that A is not regular. We may assume that $\sum_{p \in J} \nu(p)$ is finite. Then the non-regular points decompose A into finitely many regular arcs. Thus, A is elementary and Theorem 8.6.1 applies.

8.7. Main theorem for arcs with tower.

THEOREM 8.7.1. *Let A be an elementary arc with tower. Then*

$$\sum_{p \in J} \mu(p, L) \leqq \sum_{p \in J} \sum_{i=0}^{n-1} (\alpha_i(p) - 1) + n,$$

for every hyperplane L.

Proof. As in Theorem 8.6.1, we may assume that both sides are finite.

For $n = 1$, Theorems 8.6.1 and 8.7.1 are the same. We may, therefore, assume that Theorem 8.7.1 is true for $n - 1$. Let $\{H_i\}$ be a tower for A.

Suppose that $H_0 \subset L$. By Theorem 5.2.2, $\bar{A} = A|H_0$ is an elementary arc with tower $\{H_0, \ldots, H_n\}$. Hence,

$$8.7.2 \qquad \sum_{p \in J} \bar{\mu}(p, L) \leqq \sum_{p \in J} \sum_{i=0}^{n-2} (\bar{\alpha}_i(p) - 1) + n - 1.$$

Since $H_0 \not\subset A_{n-1}(p)$ we have $\delta(p, L) \leqq n - 2$ and $\delta(p, L) < \pi(H_0, p)$. By Theorem 1.4.3, $\bar{\mu} = \bar{\beta}_{\bar{\delta}} = \bar{\beta}_\delta = \beta_\delta = \mu$ and $\bar{\alpha}_i(p) = \alpha_i(p)$, $0 \leqq i \leqq n - 2$. Thus, Theorem 8.7.1 follows from 8.7.2.

Next, assume that $H_0 \not\subset L$. Put $P = L \cap H_1$. As before, 8.7.2 holds for $\bar{A} = A|P$.

LEMMA 8.7.3. *Let X be the set of all points p such that $P \subset A_{n-1}(p)$. Then*

$$\sum_{p \in X} \alpha_{n-1}(p) \leqq \sum_{p \in J} (\alpha_{n-1}(p) - 1) + 1.$$

Proof. By Theorems 5.2.1 and 6.4.1, $\varphi(p) = A_{n-1}(p) \cap H_1$ is an arc with tower in H_1 with characteristic $(\alpha_{n-1}(p))$ at p. Thus, Lemma 8.7.3 follows from Theorem 8.7.1 for $n = 1$.

Suppose that $\delta(p, L) \leqq n - 2$. Since $A_{n-2}(p) \cap H_1 = \emptyset$, we have

$$n - 2 \leqq \pi(P, p).$$

If $\delta(p, L) = n - 2$, then $P \not\subset A_{n-1}(p)$ and $\pi(P, p) = n - 1$. Thus,

$$\delta(p, L) < \pi(P, p).$$

By Lemma 1.4.1 and Theorem 1.4.3, $\bar{\mu} = \bar{\beta}_{\bar{\delta}} = \bar{\beta}_\delta = \beta_\delta = \mu$.

Suppose that $\delta(p, L) = n - 1$. Then

$$\begin{aligned} \bar{\mu}(p, L) &= \sum_{i=0}^{n-2} \bar{\alpha}_i(p) \\ &= \mu(p, L) - (\alpha_{n-2}(p) + \alpha_{n-1}(p)) + \bar{\alpha}_{n-2}(p). \end{aligned}$$

Combining these cases and using the fact that $p \in X$ if $\delta(p, L) = n - 1$, one has

$$\begin{aligned} \sum_{p \in J} \mu(p, L) &= \sum_{p \in J} \bar{\mu}(p, L) + \sum_{\delta(p, L) = n-1} (\alpha_{n-2}(p) + \alpha_{n-1}(p) - \bar{\alpha}_{n-2}(p)) \\ &\leqq \sum_{p \in J} \bar{\mu}(p, L) + \sum_{p \in X} (\alpha_{n-2}(p) + \alpha_{n-1}(p) - \bar{\alpha}_{n-2}(p)). \end{aligned}$$

Also,

$$\begin{aligned} \sum_{p \in J} \sum_{i=0}^{n-2} (\bar{\alpha}_i(p) - 1) = \sum_{p \in J} \sum_{i=0}^{n-3} (\alpha_i(p) - 1) &+ \sum_{p \in X} (\bar{\alpha}_{n-2}(p) - 1) \\ &+ \sum_{p \notin X} (\alpha_{n-2}(p) - 1). \end{aligned}$$

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By 8.7.2,

$$\sum_{p \in J} \mu(p, L) \leqq \sum_{p \in J} \sum_{i=0}^{n-2} (\alpha_i(p) - 1) + \sum_{p \in X} \alpha_{n-1}(p) + n - 1.$$

Now Theorem 8.7.1 follows from Lemma 8.7.3.

8.8. Denk's theorem.

THEOREM 8.8.1. *The order with or without multiplicities taken into account, of an elementary point p is $\sum_{i=0}^{n-1} \alpha_i(p)$.*

Proof. Let ord(p) (Ord (p)) denote the order of p without (with) multiplicities taken into account. One has ord $(p) \leqq$ Ord (p). By Theorem 7.5.10 with $k = 0$,

$$\sum_{i=0}^{n-1} \alpha_i(p) \leqq \operatorname{ord}(p).$$

By Theorem 3.4.2, A_k is continuous at p, $0 \leqq k \leqq n - 1$. Thus, some neighbourhood $U(p)$ of p is an elementary arc with tower and contains only regular points except possibly p. By Theorem 8.7.1,

$$\begin{aligned} \operatorname{Ord}(p) &\leqq \sum_{i=0}^{n-1} (\alpha_i(p) - 1) + n \\ &= \sum_{i=0}^{n-1} \alpha_i(p). \end{aligned}$$

8.9. Barner curves. Barner curves are defined similarly to Barner arcs.

THEOREM 8.9.1. *Barner curves exist in every dimension.*

Proof. For $n = 1$, any curve whose image is not $\mathscr{P}_0{}^1$ is a Barner curve. Therefore, assume that $n \geqq 2$. Let C be a curve with at most inflections for which C_k is continuous, $0 \leqq k \leqq n - 1$. Assume that there is a point p such that the restriction A of C to $K \backslash \{p\}$ is an arc with tower $\{H_i\}$ with $H_0 \not\subset C_{n-1}(p)$. As in the proof of Theorem 8.1.2, $\bar{A} = A|H_0$ is of order $n - 1$. By Theorem 3.1.1, $\bar{C} = C|H_0$ is $(n - 2)$-independent. The remainder of the proof is as for Theorem 8.1.2.

THEOREM 8.9.2. *Let C be an elementary Barner curve. Then*

$$\sum_{p \in K} \mu(p, L) \leqq \sum_{p \in K} \nu(p),$$

for every hyperplane L. Both sides are congruent to $n + 1$ (mod 2).

Proof. The proof is the same as for Theorem 8.6.1 with slight modifications. In particular, the inequality of Lemma 8.6.5 can be improved to

$$2|X| + |Y| - |Z| + 1 \leqq 0.$$

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By definition, $\rho_0 \equiv \sum_{p\in K} \mu(p, L)$. Choose $L = B(q, \ldots, q)$. Then

$$\rho_0 \equiv \mu(q, B(q, \ldots, q)) = \sum_{i=0}^{\delta(q,L)} \alpha_i(q).$$

If q is at most an inflection, then $\delta(q, L) = n - 2$ and this sum is $n - 1$; otherwise, $\delta(q, L) = n - 1$ and, by Theorem 8.4.1, the sum is $n + 1$. In any case, $\rho_0 \equiv n + 1$. By Theorem 7.4.2,

$$\sum_{p\in K} \nu(p) \equiv (n+1)\rho_0 \equiv n + 1.$$

8.10. Curves with tower. It is clear that curves with tower exist in every dimension. For example, if A is an arc with tower and $p < q$, then the curve obtained by going from p to q and back to p again is a curve with tower.

THEOREM 8.10.1. *Let C be an elementary curve with tower. Then*

$$\sum_{p\in K} \mu(p, L) \leqq \sum_{p\in K} \sum_{i=0}^{n-1} (\alpha_i(p) - 1).$$

Both sides are even.

Proof. The proof is the same as that of Theorem 8.7.1 with slight modifications. Use Theorem 7.4.4.

8.11. Index of a curve. The *index* of a curve is the minimum number of points which can lie in a hyperplane. The points are not counted with multiplicity.

THEOREM 8.11.1. *Suppose that there is a hyperplane L which meets C in only one point. Then the index of C is* 0 *or* 1, *according as L supports or cuts C at p.*

Proof. The index is either 0 or 1.

Assume that L supports C at p. If $n = 1$, it is clear the index of C is 0. If $n \geqq 2$, take an $(n - 2)$-space $M \subset L$ with $C(p) \not\subset M$. Projection from M shows that C has index 0.

If L cuts C at p, then the supposition that there is a hyperplane H_∞ not meeting C leads to a contradiction. Hence, C has index 1.

THEOREM 8.11.2. *The index of a curve of order n (of a Barner curve) is* 0 *or* 1, *according as n is even or odd (odd or even). A curve with tower has index* 0.

Proof. If C is a curve of order n then, by Theorem 3.1.1, $C_{n-1}(p)$ meets C only in p. By Theorem 3.2.2, p is regular; thus, $\sigma_{n-1}(p) \equiv n$. Hence, $C_{n-1}(p)$ supports or cuts C according as n is even or odd and Theorem 8.11.1 applies.

If C is a Barner curve and $L = B(q, \ldots, q)$ then, from the proof of Theorem 8.9.2, $\sigma_{\delta(q,L)} \equiv n + 1$, and Theorem 8.11.1 applies.

If C is a curve with tower $\{H_i\}$, then H_{n-1} does not meet C.

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9. REGULAR ARCS

If one were able to show that every regular Barner arc is of order n, then the inequality of Theorem 8.6.1 would hold for all Barner arcs. For then the assumption that $\sum_p \nu(p)$ is finite implies that the arc is elementary and Theorem 8.6.1 applies. Unfortunately it is not known whether or not every regular Barner arc is of order n.

A large part of this section consists of the proof of Theorems 9.1.1 and 9.1.2. From Theorem 9.1.1, it follows that the inequality of Theorem 8.6.1 holds for any Barner arc for which A_{n-2} is continuous. Haupt's work on order-homogeneous arcs is related to Theorem 9.1.2; cf. [**7**]. From Theorem 7.5.10 and 9.1.2, it follows that the k-th rank of any arc is at least $(n - k)(k + 1)$; cf. [**13**].

One may ask whether or not a regular $(n - 2)$-independent arc is necessarily ordinary. In Theorem 9.2.3, we show that this is the case if $n = 2$.

9.1. The existence of an ordinary point.

THEOREM 9.1.1. *If A is a regular Barner arc and A_{n-2} is continuous, then A is of order n.*

THEOREM 9.1.2. *Every arc has an ordinary point.*

THEOREM 9.1.3. *If A is a Barner arc with at most inflections and A_{n-1} is continuous, then either A is of order n or A has an elementary singularity.*

Proof of Theorems 9.1.1, 9.1.2, *and* 9.1.3 *for* $n = 1$. Use Theorems 6.4.2 and 4.1.

Assume that Theorems 9.1.1, 9.1.2, and 9.1.3 are true for $n - 1$.

LEMMA 9.1.4. *Let A be a Barner arc with at most inflections in $\mathscr{P}^m$. If A_{m-1} is continuous at p, then so is A_{m-2}.*

Proof. By Theorem 8.2.2, $\delta(q, B(q, \ldots, q)) = m - 2$, for all $q \in J$. Hence, $A_{m-2}(q) = A_{m-1}(q) \cap B(q, \ldots, q)$, for all $q \in J$.

LEMMA 9.1.5. *Let A be an arc with at most inflections. Assume that A_{n-2} is continuous. Then there is a point at which A_{n-1} is continuous.*

Proof. We may assume that there is a line L such that $A_{n-2}(p) \cap L = \emptyset$, for all $p \in J$. Let $\{P_i| \, i = 1, 2, \ldots\}$ be a set of points of L which is dense in L. By Theorem 9.1.2$(n - 1)$, there exists a sequence X_i, $i = 1, 2, \ldots$ of open intervals such that X_i has order $n - 1$ on $A|P_i$ and $\bar{X}_{i+1} \subset X_i$, $i = 1, 2, \ldots$. Take $p \in \bigcap_{i=1}^{\infty} X_i$ and put $P = A_{n-1}(p) \cap L$.

To show A_{n-1} that is continuous at p it is sufficient to show

$$\lim_{q \to p} A_{n-1}(q) \cap L = P.$$

Let $U(P)$ be a neighbourhood of P on L, say one with the end points P_i, P_j, where $i < j$. Take $q, r \in X_j$ with $q < p < r$. Since X_j is of order $n - 1$ on

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$A|P_j$, $A(s) \not\subset A_{n-2}(t)$, for all $s, t \in X_j$, $s \neq t$. We may define a continuous path in $\mathscr{P}^n{}_{n-1}$ by:

$$A_{n-1}(p) \xrightarrow{\varphi_1} A_{n-2}(p)A(r) \xrightarrow{\varphi_2} A_{n-2}(q)A(r) \xrightarrow{\varphi_3} A_{n-1}(q)$$

where

$$\begin{aligned} \varphi_1(s) &= A_{n-2}(p)A(s) \quad \text{if} \quad s \in (p, r) \\ \varphi_2(s) &= A_{n-2}(s)A(r) \quad \text{if} \quad s \in (q, p) \\ \varphi_3(s) &= A_{n-2}(q)A(s) \quad \text{if} \quad s \in (q, r). \end{aligned}$$

Since X_j is of order $n - 1$ on $A|P_i$, $A|P_j$, no hyperplane of this path, except possibly $A_{n-1}(q)$, contains P_i or P_j. Hence, $A_{n-1}(q) \cap L \in (U(P))^-$, where $(U(P))^-$ is the closure of $U(P)$. Similarly, $A_{n-1}(r) \cap L \in (U(P))^-$. Since q and r are arbitrary points of X_j with $q < p < r$, the continuity of A_{n-1} at p follows.

LEMMA 9.1.6. *Every arc A contains a Barner arc with at most inflections on which A_{n-2} is continuous.*

Proof. We may assume that there is a hyperplane H_∞ not meeting A. Let $P_1, \ldots, P_n$ be independent points of H_∞. By Theorem 9.1.2$(n - 1)$, there are subarcs $X_1, \ldots, X_n$ such that X_i has order $n - 1$ on $A|P_i$, $1 \leqq i \leqq n$, and $X_n \subset \ldots \subset X_1$. X_n has at most inflections on A; for, if $p \in X_n$, there is an i such that $P_i \not\subset A_{n-1}(p)$. By Theorem 3.2.2, p is regular on $A|P_i$. By Theorem 1.4.3, p is either a regular point or an inflection on A.

If $p \in X_n$, then $P_i \not\subset A_{n-2}(p)$, for all i, $1 \leqq i \leqq n$. For, otherwise, by Theorem 1.4.3, and $P_i \neq A(p)$, one has $\alpha_j(p) = 2$, for some j, $0 \leqq j \leqq n - 2$, contradicting the above argument. Hence, $A_{n-2}(p)P_i$ is the osculating $(n - 2)$-space of $A|P_i$, for all $p \in X_n$, $1 \leqq i \leqq n$. Since

$$A_{n-2}(p) = \bigcap_{i=1}^{n} A_{n-2}(p)P_i,$$

for all $p \in X_n$, A_{n-2} is continuous on X_n; cf. Theorem 3.4.1.

By Lemma 9.1.5, there is a point $p \in X_n$ at which A_{n-1} is continuous. Take $P \not\subset A_{n-1}(p)$ and $U(p) \subset X_n$ such that $P \not\subset A_{n-1}(q)$ for all $q \in U(p)$. By Theorem 9.1.2$(n - 1)$, there is a subarc $X \subset U(p)$ which is of order $n - 1$ on $\bar{A} = A|P$. Put $B(x) = \bar{A}^{n-2}(x)$. As in the proof of Theorem 8.1.2, X is a Barner arc.

Assumption. From now until the end of the proof of 9.3(n), we assume that A is a Barner arc with at most inflections and that A_{n-2} is continuous. This is possible by Lemmas 9.1.4 and 9.1.6.

If H_∞ is a hyperplane and P and Q are distinct points not on H_∞, then $L_\infty(L_f)$ will denote the open segment of the line $L = PQ$ which does (does not) contain $L \cap H_\infty$.

LEMMA 9.1.7. *Suppose that (p, q) is of order n and H_∞ is a hyperplane not* *meeting* $[p, q]$. *Put* $L = A(p)A(q)$. *If* n *is odd (even), then* $A_{n-1}(r) \cap L \in L_f(L_\infty)$, *for all* $r \in (p, q)$.

Proof. By Theorem 3.1.1, $A_{n-1}(r)$ meets $[p, q]$ only at r. If n is odd (even), then $A_{n-1}(r)$ cuts (supports) A at r; cf. Theorem 8.11.2. Thus, $A(p)$ and $A(q)$ lie in different open half spaces (the same open half space) determined by $A_{n-1}(r)$ and H_∞.

LEMMA 9.1.8. *Suppose that* $p < q < r$. *Let* H_∞ *be a hyperplane not meeting* $[p, r]$. *Put* $L = A(p)A(q)$. *If* (q, r) *is of order* n *and* $A(p) \subset A_{n-1}(r)$, *then* $A_{n-1}(s)$ *meets* L_f, *for all* $s \in (q, r)$.

Proof. Since Lemma 9.1.8 is true for $n = 1$, we may assume that $n \geqq 2$. By Theorem 8.2.5, $A(p) \neq A(q)$; hence, L is a line.

By Theorem 8.5.3, $A(p) \not\subset A_{n-1}(s)$, for all $s \in (q, r)$. Hence, by Theorem 5.2.4, each $s \in (q, r)$ is ordinary on $A|p$ and (q, r) is of order $n - 1$ on $A|p$, by Theorem 8.5.2. Using this and Theorem 8.5.1, we obtain

$$A_{n-2}(s) \cap L \neq \emptyset,$$

if $s \in (q, r]$, and

$$A(p) \not\subset A^{n-1}(s, \ldots, s, t),$$

if $s, t \in (q, r)$, $s \neq t$. By Theorem 8.5.3, this is true even when $s = t$. By Theorem 3.1.1,

$$A(q) \not\subset A^{n-1}(s, \ldots, s, t),$$

if $s, t \in (q, r)$. By the continuity of A^{n-1} on (q, r), $A^{n-1}(s, \ldots, s, t)$ meets either L_f, for all $s, t \in (q, r)$, or L_∞, for all $s, t \in (q, r)$.

Since $A_{n-2}(r) \cap L = \emptyset$,

$$\varphi(t) = A^{n-1}(r, \ldots, r, t) \cap L$$

is a point of L, for all $t \in [q, r]$. Since $[p, q]$ lies in one of the open half spaces determined by $B(r, \ldots, r)$ and H_∞, $B(r, \ldots, r) \cap L \in L_\infty$. Now

$$\varphi(t) \neq B(r, \ldots, r) \cap L,$$

for all $t \in (q, r)$. Since $\varphi(t)$ moves continuously from $A(q)$ to $A(p)$ as t moves from q to r, there is a $t_0 \in (q, r)$ such that $\varphi(t_0) \in L_f$. Thus,

$$A_{n-2}(r)A(t_0) \cap L \in L_f.$$

By the continuity of A_{n-2} on $[q, r]$, there is an $s_0 \in (q, r)$ such that $A^{n-1}(s_0, \ldots, s_0, t_0)$ meets L_f. Thus, $A^{n-1}(s, \ldots, s, t)$ meets L_f for all $s, t \in (q, r)$.

LEMMA 9.1.9. *Suppose that* A *is regular and there exist* $p < q$, q *ordinary, such that* $A(p) \subset A_{n-1}(q)$. *Then there exist* r, s, *with* s *singular, such that* $A_{n-1}(s)$ *cuts* A *at* r.

Proof. Take s as small as possible such that (s, q) is ordinary. By Theorem 8.5.1, $p < s < q$ and s is singular. By Theorem 8.5.2, (s, q) is of order n. Let H_∞ be a hyperplane which does not meet $[p, q]$ and put $P = A(s)A(q)$, $S = A(p)A(q)$, $Q = A(p)A(s)$. For any $t \in (s, q)$, $A_{n-1}(t)$ meets $P_f(P_\infty)$ if n is odd (even), by Lemma 9.1.7; $A_{n-1}(t)$ meets Q_f, by Lemma 9.1.8; thus, $A_{n-1}(t)$ meets $S_\infty(S_f)$ if n is odd (even). Since $A(p) \not\subset A_{n-1}(s)$, by Theorem 8.5.3, and $A(q) \not\subset A_{n-1}(s)$, by Theorem 8.5.1, it follows that $A_{n-1}(s)$ meets $S_\infty(S_f)$ if n is odd (even). But, s being regular, $A_{n-1}(s)$ cuts (supports) A at s if n is odd (even). Since $A_{n-1}(s)$ does not meet (s, q), there is an $r \in (p, s)$ such that $A_{n-1}(s)$ cuts A at r.

LEMMA 9.1.10. *If A is not of order n, then there exist p, q, with $p < q$, such that* $\delta(p, A_{n-1}(q)) = 0$.

Proof. Some hyperplane contains points $p, p_1, \ldots, p_n$, where

$$p < p_1 < \ldots < p_n.$$

Consider $\bar{A} = A|p$. By Theorem 8.2.4, $\bar{A}_{n-2}(q) = A_{n-2}(q)A(p)$, for all $q \in (p, p_n)$; thus, $\bar{A}_{n-2}$ is continuous on (p, p_n). By Lemma 9.1.4, $\bar{A}_{n-3}$ is continuous on (p, p_n). By Theorem 8.2.7, (p, p_n) contains at most inflections of $\bar{A}$. Now (p, p_n) contains an inflection of $\bar{A}$; otherwise, by Theorems 9.1.1$(n-1)$ and 8.5.2, $[p, p_n]$ would be of order $n-1$ on $\bar{A}$, contradicting the fact that $p_1, \ldots, p_n$ lie in a hyperplane.

If there is only one inflection q of $\bar{A}$ in (p, p_n), then (p, q) is of order $n-1$ on $\bar{A}$, by Theorem 9.1.1$(n-1)$, and $\bar{A}(p) \not\subset \bar{A}_{n-2}(q)$, by Theorem 8.5.1. Since q is an inflection of $\bar{A}$, q is regular on A and $A(p) \subset A_{n-1}(q)$. Since $\bar{A}(p) \not\subset \bar{A}_{n-2}(q)$, $A_1(p) \not\subset A_{n-1}(q)$ and Lemma 9.1.10 holds.

Suppose that $q_1 < q_2$ are inflections of $\bar{A}$ in (p, p_n). By Theorem 9.1.2$(n-1)$, there is an ordinary point q_3 of $\bar{A}$ in (q_1, q_2). Take q_4, q_5 such that

$$q_1 \leqq q_4 < q_3 < q_5 \leqq q_2,$$

(q_4, q_5) is regular on $\bar{A}$, and every neighbourhood of q_4 and q_5 contains inflections of $\bar{A}$. By Theorem 9.1.1$(n-1)$, (q_4, q_5) is of order $n-1$ on $\bar{A}$. If $\bar{A}(p) \subset \bar{A}_{n-2}(q)$, for all inflections q of $\bar{A}$ in (p, p_n), then $\bar{A}(p) \subset \bar{A}_{n-2}(q_4) \cap \bar{A}_{n-2}(q_5)$, contradicting Theorem 8.5.3. Hence, $\bar{A}(p) \not\subset \bar{A}_{n-2}(q)$, for some inflection q of $\bar{A}$ in (p, p_n), and Lemma 9.1.10 follows, as in the preceding paragraph.

LEMMA 9.1.11. *If A is regular but not of order n, there exist p, q, r, with $p < q < r$, q singular, such that $A(p)A_{n-2}(q)A(r)$ is a hyperplane which cuts A at p.*

Proof. By Lemma 9.1.10, there exist s, q, with $s < q$, such that $\delta(s, A_{n-1}(q)) = 0$. Since $\alpha_0(s) = 1$, $A_{n-1}(q)$ cuts A at s. By Lemma 9.1.9, we may assume that q is singular. By projection from $A_{n-2}(q)$, there is an r with $q < r$ such that $A_{n-2}(q)A(r)$ cuts A at a point $p < q$.

Proof of Theorem 9.1.1(n). Suppose that A is not of order n. Take p_1, q_1, r_1 with the properties of p, q, r in Lemma 9.1.11. Let X_1 be a neighbourhood of q_1

such that $r_1 \notin X_1$ and that for all $q \in X_1$, $A_{n-2}(q)A(r_1)$ meets A in a point $p \notin X_1, p < q_1$. Since q_1 is a singularity, we may repeat this argument using X_1 instead of J and obtain p_2, q_2, r_2 and X_2. Continuing, one obtains p_i, q_i, r_i, X_i, $i = 1, 2, \ldots$, such that $\bar{X}_{i+1} \subset X_i, r_i \notin X_i, q_i < r_i$, and $A_{n-2}(q)A(r_{i+1})$ meets $X_i \backslash X_{i+1}$ in a point $p < q_{i+1}$, whenever $q \in X_{i+1}, i = 1, 2, \ldots$. We may assume that $\bigcap_{i=1}^{\infty} X_i = \{q\}$. Then $q < r_i$, $i = 1, 2, \ldots$. Also, $A_{n-2}(q)A(r_{i+1})$ meets X_i in a point $p < q$, $i = 1, 2, \ldots$. Since q is regular on A, q is regular on $\bar{A} = A|A_{n-2}(q)$. Thus, there exist $U^+(q), U^-(q)$ with $\bar{A}(U^+(q)) \cap \bar{A}(U^-(q)) = \emptyset$. Taking $X_i \subset U^-(q) \cup \{q\} \cup U^+(q)$, we obtain a contradiction.

LEMMA 9.1.12. *Suppose that A_{n-1} is continuous at each point of a non-empty set W. Then there is a subarc X which contains a point of W such that, if $p, q \in X$, $p \neq q$, $p \in W$, then $A_{n-1}(p)$ does not cut A at q.*

Proof. Suppose that for every subarc X with $X \cap W \neq \emptyset$ there exist $p, q \in X$, $p \neq q$, $p \in W$ such that $A_{n-1}(p)$ cuts A at q. Take $p_1, q_1 \in J$, $p_1 \neq q_1, p_1 \in W$ such that $A_{n-1}(p_1)$ cuts A at q_1. Since $p_1 \in W$, there are disjoint neighbourhoods X_1, Y_1 of p_1, q_1, respectively, such that if $p \in X_1$, then $A_{n-1}(p)$ meets Y_1. Since $p_1 \in X_1$, we may repeat this construction replacing J by X_1. This yields two intervals X_2, Y_2 of X_1. Thus, $\bar{X}_2$, $\bar{Y}_2 \subset X_1$. Continuing, one obtains sequences X_i, Y_i, $i = 1, 2, \ldots$, with $X_i \cap Y_i = \emptyset$ and $\bar{X}_{i+1}$, $\bar{Y}_{i+1} \subset X_i$, such that if $p \in X_i$, then $A_{n-1}(p)$ meets Y_i. Since $\bar{X}_{i+1} \subset X_i$, there is a point $p \in \bigcap_{i=1}^{\infty} X_i$. Thus, $A_{n-1}(p)$ meets every Y_i. Since the Y_i are disjoint, $A_{n-1}(p)$ meets the compact set $\bar{X}_1$ infinitely often, contradicting Theorem 1.5.1.

Notation. The following notation will remain fixed: p_0 is a point at which A_{n-1} is continuous; cf. Lemma 9.1.5; (p_1, p_2) is a neighbourhood of p_0 and H_∞ is a hyperplane such that H_∞ does not meet $[p_1, p_2]$, $A_{n-1}(p_0)$ meets $[p_1, p_2]$ only in p_0 and $A(p_1) \not\subset A_{n-1}(p)$, for all $p \in (p_0, p_2)$; $L = A(p_1)A(p_0)$, a line by Theorem 8.2.5, $L_\infty(L_f)$ is the open segment of L with the end points $A(p_1)$, $A(p_0)$ which meets (does not meet) H_∞.

LEMMA 9.1.13. *(p_0, p_2) has order $n - 1$ on $A|p_1$; $A_{n-2}(p) \cap L = \emptyset$, for all $p \in (p_0, p_2)$.*

Proof. (p_0, p_2) is regular on $\bar{A} = A|p_1$. By Theorem 8.2.4,

$$\bar{A}_{n-2}(p) = A_{n-2}(p)A(p_1),$$

for all $p \in (p_0, p_2)$. Hence, $\bar{A}_{n-2}$ is continuous on (p_0, p_2). As $\bar{A}_{n-3}$ is continuous on (p_0, p_2) by Lemma 9.1.4, the first statement follows by Theorem 9.1.1 $(n-1)$ and the second by projection from p_1 and Theorem 8.2.4.

LEMMA 9.1.14. *Suppose that p_0 is an inflection. If there is a $p_3 \in (p_0, p_2)$ such that (p_0, p_3) has order $n - 1$ on $A|p_0$, then for each $p \in (p_0, p_3)$, either $A_{n-1}(p)$ meets L_f or $A(p_0) \subset A_{n-1}(p)$.*

Proof. By Theorem 3.4.1 applied to $A|p_0$, $A(p_0)A_{n-2}(p)$ tends to $A_{n-1}(p_0)$ as $p \to p_0^+$. If n is odd (even), p_1, p_3 lie on the same side (opposite sides) of $A_{n-1}(p_0)$. Take $p_4 \in (p_0, p_3)$ such that if n is odd (even), p_1, p_3 lie on the same side (opposite sides) of $A(p_0)A_{n-2}(p_4)$. By Theorem 3.2.2, p_4 is regular on $A|p_0$. Hence, by projection from p_0 and Theorem 1.4.3, $A(p_0)A_{n-2}(p_4)$ supports (cuts) A in p_4 if n is odd (even). Since $A(p_0)A_{n-2}(p_4)$ meets (p_0, p_3) only in p_4, p_1 and (p_0, p_4) lie on the same side of $A(p_0)A_{n-2}(p_4)$, for n odd or even. Now by projection from $A_{n-2}(p_4)$, there is a $p_5 \in (p_0, p_4)$ such that $A(p_5)A_{n-2}(p_4)$ meets L_f.

Since (p_0, p_3) has order $n - 1$ on $A|p_1$, by Lemma 9.1.13, and on $A|p_0$, by hypothesis, it follows, by Theorem 3.1.1, that neither $A(p_1)$ nor $A(p_0)$ lie on $A(p)A_{n-2}(q)$, if $p_0 < p < q < p_3$. Thus, $A(p)A_{n-2}(q)$ meets L_f, if

$$p_0 < p < q < p_3,$$

and Lemma 9.1.14 follows.

Proof of Theorem 9.1.2(n). Let W be the set of all points at which A_{n-1} is continuous. By Lemma 9.1.5, $W \neq \emptyset$. Choose X according to Lemma 9.1.12; $p_0 \in X \cap W$. We may assume that $X = J$. Then if $p \in W$, $A_{n-1}(p)$ does not cut A at any point except possibly p.

Case 1. p_0 is regular. Then p_1, p_2 lie on the same side (opposite sides) of $A_{n-1}(p_0)$, if n is even (odd). Take $p_3 \in (p_0, p_2)$ such that p_1, p_2 lie on the same side (on opposite sides) of $A_{n-1}(p)$, for all $p \in (p_0, p_3)$. Thus, $A_{n-1}(p)$ supports (cuts) A at p, for all $p \in (p_0, p_3) \cap W$. Hence, each point $p \in (p_0, p_3) \cap W$ is regular.

Let q be a point of (p_0, p_3). By Lemma 9.1.5, there are points

$$q_i \in (p_0, p_3) \cap W, \qquad i = 1, 2, \ldots$$

such that $q_i \to q$. Let M be any hyperplane of accumulation of the sequence $A_{n-1}(q_i)$. Let $(r_1, r_2) \subset (p_0, p_3)$ be a neighbourhood of q such that $A_{n-1}(q)$ meets $[r_1, r_2]$ only in q. There is an i with $q_i \in (r_1, r_2)$ such that r_1, r_2 lie on the same side of $A_{n-1}(q_i)$ if and only if they lie on the same side on M. Since q_i is regular, this is the case if and only if n is even. Hence, M supports (cuts) A at q, if n is even (odd). Since $A_{n-2}(q) \subset M$, q is regular. By Theorem 9.1.1(n), (p_0, p_3) is of order n.

Case 2. p_0 is an inflection and there exists $p_3 \in (p_0, p_2)$ such that (p_0, p_3) has order $n - 1$ on $A|p_0$. By Lemma 9.1.5, there is a point $p \in (p_0, p_3) \cap W$. If $A_{n-1}(p)$ meets L_f, then $A_{n-1}(p)$ cuts A at a point of (p_1, p_0) which is a contradiction. Hence, by Lemma 9.1.14, $A(p_0) \subset A_{n-1}(p)$. If $A_1(p_0) \not\subset A_{n-1}(p)$, $A_{n-1}(p)$ cuts A at p_0, which is a contradiction. If $A_1(p_0) \subset A_{n-1}(p)$, projection from p_0 yields a contradiction. Hence, Case 2 cannot occur.

Case 3. p_0 is an inflection and no $U^+(p_0)$ has order $n - 1$ on $A|p_0$. By Theorem 9.1.3($n - 1$) applied to $A|p_0$, there exist points p_3, p_4, p_5 such that $p_0 < p_4 < p_3 < p_5 < p_2$, p_3 is an inflection on $A|p_0$ and (p_4, p_3), (p_3, p_5) are of order $n - 1$ on $A|p_0$. Hence, p_3 is regular on A and $A(p_0) \subset A_{n-1}(p_3)$. Since p_3 is regular on $A|A_{n-2}(p_3)$, there is a $p_6 \in (p_4, p_3) \cup (p_3. p_5)$ such that $A(p_6)A_{n-2}(p_3)$ meets L_f.

Suppose that $p_6 \in (p_4, p_3)$. Projection from p_1, p_0 shows

$$A(p_1), A(p_0) \not\subset A(p)A_{n-2}(q),$$

if $p_4 < p < q \leqq p_3$. Thus, $A(p)A_{n-2}(q)$ meets L_f, for these p, q and $A_{n-1}(r)$ meets $L_f \cup \{A(p_0)\}$, for all $r \in (p_4, p_3)$. Take $r_1, r_2 \in (p_4, p_3) \cap W$, $r_1 \neq r_2$. The arguments of Case 2 show a contradiction occurs unless

$$A_1(p_0) \subset A_{n-1}(r_1) \cap A_{n-1}(r_2).$$

But this gives a contradiction of Theorem 8.5.3, by projection from p_0. Similarly, $p_6 \in (p_3, p_5)$ gives a contradiction. Thus, Case 3 cannot occur.

Proof of Theorem 9.1.3(n). Let W be the set of inflections of A. By Theorem 9.1.1(n), we may assume that $W \neq \emptyset$. As in the proof of Theorem 9.1.2(n), we may assume that the subarc X of Lemma 9.1.12, is J. Let $p_0 \in W$.

Case 1. There exists $p_3 \in (p_0, p_2)$ such that (p_0, p_3) has order $n - 1$ on $A|p_0$. If there is a $p \in (p_0, p_3) \cap W$, Lemma 9.1.14, and an argument as in Case 2 above give a contradiction. Thus, (p_0, p_3) is regular; by Theorem 9.1.1(n), it is of order n.

Case 2. No $U^+(p_0)$ has order $n - 1$ on $A|p_0$. By Theorem 9.1.1(n), every $U^+(p_0)$ contains an inflection on $A|p_0$. By Theorem 9.1.3($n - 1$), there exist p_3, p_4, p_5 such that $p_0 < p_4 < p_3 < p_5 < p_2$; p_3 is an inflection on $A|p_0$; (p_4, p_3), (p_3, p_5) are of order $n - 1$ on $A|p_0$; and every neighbourhood of p_4 and p_5 contains inflections of $A|p_0$. If p is an inflection of $A|p_0$ and $p \neq p_0$, then p is regular on A and $A(p_0) \subset A_{n-1}(p)$. By the continuity of A_{n-1},

$$A(p_0) \subset A_{n-1}(p_4) \cap A_{n-1}(p_5).$$

Since p_3 is regular on A and $A(p_0) \subset A_{n-1}(p_3)$, it follows by projection from $A_{n-2}(p_3)$ that there is a $p_6 \in (p_4, p_3) \cup (p_3, p_5)$ such that $A(p_6)A_{n-2}(p_3)$ meets L_f. Suppose that $p_6 \in (p_4, p_3)$. Then, as in Case 3 above, for each $p \in (p_4, p_3)$, $A_{n-1}(p)$ meets $L_f \cup \{A(p_0)\}$. Theorem 8.5.3, (p_4, p_3) is not ordinary; by Theorem 9.1.1(n), there is an $r_1 \in (p_4, p_3) \cap W$. Since $A_{n-1}(r_1)$ meets $L_f \cup \{A(p_0)\}$, $A_1(p_0) \subset A_{n-1}(r_1)$, as in Case 2 above. Repeating this argument, one obtains $r_2 \in (r_1, p_3) \cap W$. Again $A_1(p_0) \subset A_{n-1}(r_2)$. This contradicts Theorem 8.5.3 applied to (r_1, r_2) on $A|p_0$. Similarly, $p_6 \in (p_3, p_5)$ gives a contradiction. Hence, Case 2 cannot occur.

We conclude that p_0 has a right neighbourhood of order n. Symmetrically p_0 has a left neighbourhood of order n. Thus p_0 is an elementary inflection.

THEOREM 9.1.15. *Let A be a regular Barner arc. If A_{n-1} is continuous at p, then p is ordinary.*

Proof. This is true for $n = 1$; assume that it is true for $n - 1$. Take p_1, p_2 with $A(p_i) \not\subset A_{n-1}(p)$, $i = 1, 2$, and $A_{n-2}(p)A(p_1) \neq A_{n-2}(p)A(p_2)$. Take

$U_1(p)$ such that $A(p_i) \not\subset A_{n-1}(q)$, for all $q \in U_1(p)$, $i = 1, 2$. Put $A^{(i)} = A|p_i$. $U_1(p)$ is regular on $A^{(i)}$. By Lemma 9.1.4, A_{n-2} is continuous at p; by Theorem 8.2.4, $A_{n-2}{}^{(i)}$ is continuous at p. Hence, p is an ordinary point of $A^{(i)}$; in particular, there is a $U_2(p) \subset U_1(p)$ on which $A_{n-2}{}^{(i)}$ is continuous. Take $U(p) \subset U_2(p)$ such that $A_{n-2}(q) = \bigcap_{i=1}^{2} A_{n-2}(q)A(p_i)$, for all $q \in U(p)$. Then A_{n-2} is continuous on $U(p)$; by Theorem 9.1.1, $U(p)$ is of order n.

9.2. Regular simple arcs in $\mathscr{P}^2$.

LEMMA 9.2.1. *Let (p, q) be an ordinary simple subarc of an arc A in $\mathscr{P}^2$. Suppose that $\{H_i\}$ is a tower of spaces such that H_1 does not meet A. If the lines $L = A(p)H_0$ and $M = A(q)H_0$ are distinct and do not cut (p, q), then (p, q) is of order* 2.

Proof. First, assume that p and q are ordinary. By Theorem 5.2.1, there are only finitely many points $r \in (p, q)$ with $H_0 \subset A_1(r)$, say $r_1 < \ldots < r_k$. Put $r_0 = p$, $r_{k+1} = q$ and $L_i = H_0A(r_i)$, $0 \leqq i \leqq k+1$. Thus, $L_i = A_1(r_i)$, $1 \leqq i \leqq k$. By Theorem 4.1, each of the intervals (r_{i-1}, r_i), $1 \leqq i \leqq k+1$, is of order 2. Thus, if $k = 0$, (p, q) is of order 2.

Suppose that $k \geqq 1$. We think of the lines through H_0, other than H_1, as being vertical and order them so that $L < M$. We may assume that (p, r_1) lies above the line $A(p)A(r_1)$. Since r_1 is regular (r_1, r_2) lies below the line $A(r_1)A(r_2)$ and $L \leqq L_2 < L_1 \leqq M$. Thus, $k \geqq 2$. Since r_2 is regular (r_2, r_3) lies above the line $A(r_2)A(r_3)$. Since (p, q) is simple, (r_2, r_3) lies in the region determined by L_2 and (p, r_2). Thus, $L_2 < L_3 < L_1$ and $k \geqq 3$. Continuing, it follows that k is arbitrarily large, which is a contradiction.

Suppose that p is not necessarily ordinary. Take $r \in (p, q)$ such that the lines L and $A(r)H_0$ are distinct and do not meet (p, r). For any $p_1 \in (p, r)$, there is a $p_2 \in (p, p_1)$ such that $A(p_2)H_0$ does not meet (p_2, r). By the preceding paragraph, (p_2, r) is of order 2. Thus, $(p_1, r) \subset (p_2, r)$ is of order 2. Since p_1 is arbitrary, (p, r) is of order 2. Similarly, q has a left neighbourhood of order 2 and Lemma 9.2.1 follows.

LEMMA 9.2.2. *Suppose that a regular simple arc in $\mathscr{P}^2$ has a singularity p_0. Let $\{H_i\}$ be a tower such that H_1 does not meet A. Then there exist points p_1, q_1, r_1 such that p_1 is a singularity, $p_1 \notin [q_1, r_1]$, and $A(p_1)H_0$ lies between $A(q_1)H_0$ and $A(r_1)H_0$.*

Proof. There is a point p such that $H_0 \subset A_1(p)$; for, otherwise, A is a Barner arc, by Theorem 8.1.2, and of order 2, by Theorem 9.1.1.

Case 1. $H_0 \subset A_1(p)$ for some singularity p.

Let (q, r) be a neighbourhood of p such that $A(q)H_0 = A(r)H_0 = L$, say. We may assume that $A_1(p)$ and L are distinct and do not meet (q, p) or (p, r). One of these intervals contains a singularity p_1; for, otherwise, they are of order 2, by Lemma 9.2.1, and p is ordinary, by Theorem 5.2.3. If $p_1 \in (q, p)$ say, choose $q_1 = p$, $r_1 = r$.

Case 2. $H_0 \subset A_1(p)$ only for ordinary points p.

Let (p, q) be a neighbourhood of p_0 such that the lines $A(p)H_0$ and $A(q)H_0$ are distinct and do not meet (p, q). Let $r \in (p, q)$ be such that $H_0 \subset A_1(r)$ and $A_1(r)$ does not meet (p, r). Take $s \in (r, q)$ such that $A(s) \subset A_1(r)$ and $A_1(r)$ does not meet (r, s). Take $t \in (r, s)$ such that $H_0 \subset A_1(t)$ and $A_1(t)$ does not meet (r, t) or (t, s). Take $u \in (p, r)$ such that $A(u) \subset A_1(t)$ and $A_1(t)$ does not meet (u, r). Then $u < r < t < s$. By Lemma 9.2.1, (u, s) contains a singularity p_1. Since $H_0 \subset A_1(r) \cap A_1(t)$, p_1 is neither r nor t. If $p_1 \in (u, r)$ choose $q_1 = t$, $r_1 = s$. If $p_1 \in (r, t)$ or (t, s), similar choices for q_1 and r_1 are possible.

THEOREM 9.2.3. *Every regular simple arc A in* $\mathscr{P}^2$ *is ordinary.*

Proof. Suppose that A has a singularity p_0. Let $\{H_i\}$ be a tower such that H_1 does not meet some $U_0 = U(p_0)$. Take $p_1, q_1, r_1 \in U_0$, as in Lemma 9.2.2. Let $U_1 = U(p_1)$ be such that $\bar{U}_1 \subset U_0$, $U_1 \cap [q_1, r_1] = \emptyset$ and $A(p)H_0$ lies between $A(q_1)H_0$ and $A(r_1)H_0$, for all $p \in U_1$.

Repeating this construction, one obtains p_i, q_i, r_i, U_i such that $p_i \in U_i$, $\bar{U}_i \subset U_{i-1}$, $q_i \in U_{i-1}$, $r_i \in U_{i-1}$, $U_i \cap [q_i, r_i] = \emptyset$, and $A(p)H_0$ lies between $A(q_i)H_0$ and $A(r_i)H_0$, for all $p \in U_i$. Take $p \in \bigcap_{i=0}^{\infty} U_i$. Then $A(p)H_0$ meets each of the disjoint intervals $[q_i, r_i]$, contradicting Theorem 1.5.1.

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