8
A stochastic process for string decay

8.1 Introduction

In Chapter 7 we considered the kinematics of string decay. At the same time we found and formulated a set of constraints stemming from causality, confinement and Lorentz covariance which are necessary for a consistent description of the decay process.

The intention of this chapter is to show that *there is only one stochastical process for string decay which is consistent with the requirements derived in Chapter* 7 and it contains essentially two parameters. The discussion is based upon results obtained in [19].

Once again only semi-classical physical arguments as well as probability concepts will be used during the discussion. We begin by listing the basic concepts which were derived in Chapter 7. They must all be incorporated into the stochastical process for which we are looking.

A The process of string breakup corresponds to the production of a set of yoyo-states with given masses. Each yoyo-hadron is composed of a q-particle and a $\overline{q}$-particle stemming from adjacent vertices (i.e. string breakup points) together with the string piece between them.

B1 Each pair from a vertex is massless (*local energy-momentum conservation*); the particles start to move apart after their production, due to the force from the string field.

B2 There is no interaction between the q and $\bar{q}$ of such a vertex after their production, i.e. the string force field ends on the endpoint charges (this implies *confinement*).

C The separation of the vertices is spacelike with adjacent vertices, in particular, on hyperbolas determined by the yoyo-hadron masses (this implies *causality conditions*).

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D1 All vertices therefore are of the same dynamical status. There should not be a different treatment of any one of these decay situations.

D2 Each vertex corresponds to the partitioning of the final state into one left-moving and one right-moving set of final-state yoyo-hadrons.

E Each vertex pair contains the internal (flavor) quantum numbers of the vacuum (*local conservation of internal quantum numbers*).

With regard to the ordering and the variables we have found:

F A convenient ordering of the process is *rank-ordering*. Two hadrons of adjacent rank share a $q\overline{q}$-pair produced at a vertex and therefore (according to property E above) contain the corresponding internal quantum numbers (e.g. flavors and antiflavors). Rank-ordering corresponds to an ordering along either the positive or the negative lightcone. *The process should be independent of which lightcone we use.*

G Rank ordering also implies that the process can be described as a set of steps from one vertex to the next. The steps correspond to choosing a partitioning of the energy-momentum of the original $q\bar{q}$-pair p_{+0}, p_{-0} (which at the time of the breakup goes into field energy and is then given back to the produced particles). This implies *total energy-momentum conservation.*

H1 A convenient *Lorentz-invariant* set of variables are the scaled lightcone energy-momentum fractions $p_{\pm j}/p_{\pm 0}$, with $p_{\pm j}$ the positive or negative lightcone energy-momentum of the rank-j yoyo-hadron. The $p_{\pm j}$ are carried by the q- or $\bar{q}$-particle, respectively, at the time when they meet during the yoyo-cycle.

H2 The steps referred to under property G above correspond to the space-time interval during which the particles have obtained that energy-momentum, i.e. $\Delta x_{\pm j} = p_{\pm j}/\kappa$ where κ is the string tension.

It is necessary to introduce a further assumption, which later we will show to be consistent with the results.

J Even when the energy of the original pair becomes very large the proper times of the vertices stay finite.

At the end of the chapter we will bring up a different approach, the Artru-Menessier model, [26], which was further extended and improved by Bowler, [32] (it is therefore known as the AMB model). This model contains many similarities to the Lund model fragmentation formulas

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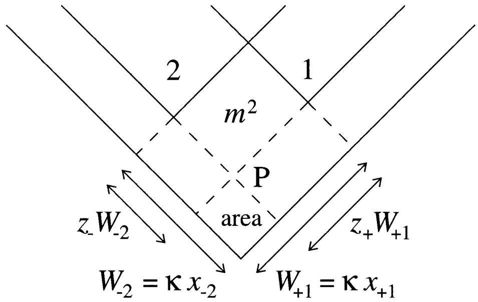

Fig. 8.1. The production of a hadron with mass m between two adjacent vertices 1 and 2 (the notation is explained in the text).

and it was conceived many years before we started our work. It was not until Artru, [24], pointed out to us that our considerations of hadrons produced in a linear potential (i.e. the yoyo string modes [14]) were similar to his results that we realised that these states actually correspond to some particular modes of the MRS. The two models, the Lund model and the AMB model, nevertheless contain major differences which we will briefly consider at the end of this chapter.

8.2 The unique breakup distribution for a single hadron

If the squared mass $s = p_{+0}p_{-0}$ of the original $q\bar{q}$-pair is very large then there will be many yoyo-hadrons produced, i.e. the process will contain many steps. A hadron produced at the centre will be little affected by the original pair and will be essentially independent of the many steps and production points that occur before its own production (or 'after'). We are introducing the idea that the process leads to a *steady-state fragmentation behaviour*. Property J, above, means that the density of hadrons will stay finite in the centre, as we will see further on.

1 The distributions H and f

We now consider two adjacent vertices at the space-time points 1 and 2, a hadron of mass m being produced in between (see Fig. 8.1).

We may describe this process as the result of taking many steps along the positive lightcone to reach vertex 1 and then one further step to reach vertex 2, thereby producing the hadron m. Another way would be to consider vertex 2 as the result of many steps along the negative lightcone, the production of m being one further step from 2 to 1.

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In the first description the positive-lightcone energy-momentum remaining before the hadron m is produced is given by $W_{+1} = \kappa x_{+1}$. Similarly in the second description the negative-lightcone energy-momentum remaining is given by the corresponding 2-component $W_{-2} = \kappa x_{-2}$. We are now going to make use of assumption J above and conclude that there is a finite probability of arriving at vertex 1 after many steps:

$$H'(1)dx_{+1}dx_{-1} \equiv H(\Gamma_1)d\Gamma_1 dy_1 \tag{8.1}$$

In this expression we have introduced hyperbolic coordinates Γ_1, y_1 instead of the lightcone variables for the vertex 1:

$$\Gamma_1 = \kappa^2 x_{+1}x_{-1}, \quad y_1 = \frac{1}{2}\log\left(\frac{x_{+1}}{x_{-1}}\right) \tag{8.2}$$

Owing to Lorentz invariance the distribution H can depend only upon Γ_1, the only Lorentz invariant available. From its definition it is obvious that Γ_1 is essentially equal to the squared proper time of vertex 1, $\kappa^2 x_{+1}x_{-1} = \kappa^2(t_1^2 - x_1^2)$ (cf. Chapter 2).

There is, of course, a corresponding probability of reaching vertex 2 after many steps along the negative lightcone:

$$H(\Gamma_2)d\Gamma_2 dy_2 \tag{8.3}$$

Given that we have arrived at vertex 1 the production of the hadron corresponds to taking a step to 2, with probability

$$f(z_+)dz_+ \tag{8.4}$$

of taking a fraction z_+ of the remaining energy-momentum W_{+1} defined above. Note that z_+ is defined by a scaling with W_{+1} instead of with the original energy-momentum p_{+0}. This is a convenient quantity to use at this point, *its range* $1 > z_+ > 0$ *being independent of the other variables.*

The joint probability of being at vertex 1 and of producing the hadron is then given by the product of the two probabilities in Eqs. (8.1), (8.4). The hadron is the result of the last in a long row of steps along the positive axis. On the other hand it may also be considered as the result of the final step of many along the negative axis. Then the joint probability is

$$H(\Gamma_2)d\Gamma_2 dy_2 f(z_-)dz_- \tag{8.5}$$

where z_- is likewise scaled with respect to the energy-momentum remainder, in this case W_{-2}.

We are now going to equate these two probabilities. Surprisingly enough we will then be able to prove that there is a single (two-parameter) solution for H and f. (To be more precise there will, in principle, be $n_f + 1$ parameters if there are n_f different $q\overline{q}$-flavors).

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2 The derivation of the distributions

We start by noting that the two quantities $dy_{1,2}$ can evidently be taken to be equal and that there is a set of relations between the remaining variables $\Gamma_{1,2}$ and $z_\pm$. From Fig. 8.1 we obtain the relations

$$\begin{aligned} \Gamma_1 &= (1-z_-)W_{-2}W_{+1} \\ \Gamma_2 &= W_{-2}(1-z_+)W_{+1} \\ m^2 &= (z_-W_{-2})(z_+W_{+1}) \end{aligned} \tag{8.6}$$

Thus there are only two independent variables in the problem (assuming m^2 as fixed), which we may take as e.g. $z_\pm$. We obtain immediately

$$\Gamma_1 = \frac{m^2(1-z_-)}{z_+z_-}, \quad \Gamma_2 = \frac{m^2(1-z_+)}{z_+z_-} \tag{8.7}$$
$$d\Gamma_1\frac{dz_+}{z_+} = d\Gamma_2\frac{dz_-}{z_-}$$

Therefore the requirement of equality introduced at the end of the last subsection reduces to

$$H(\Gamma_1(z_+,z_-))z_+f(z_+) = H(\Gamma_2(z_+,z_-))z_-f(z_-) \tag{8.8}$$

where the $z_\pm$-dependence has been explicitly written out.

Taking the logarithm of this equation we obtain with $h(\Gamma) = \log H(\Gamma)$ and $g(z) = \log(zf(z))$

$$h(\Gamma_1) + g(z_+) = h(\Gamma_2) + g(z_-) \tag{8.9}$$

If this expression is differentiated first with respect to z_+ and then with respect to z_- (keeping the other one fixed, i.e. using partial differentiation) then *all the g-dependence vanishes*. We will be left with only the variations in h. The result is

$$\frac{dh(\Gamma_1)}{d\Gamma_1} + \Gamma_1\frac{d^2h(\Gamma_1)}{d\Gamma_1^2} = \frac{dh(\Gamma_2)}{d\Gamma_2} + \Gamma_2\frac{d^2h(\Gamma_2)}{d\Gamma_2^2} \tag{8.10}$$

To obtain this result a $z_\pm$-dependent expression has been divided out from both sides. Further the chain rule for differentiation has been used:

$$\frac{\partial h(\Gamma_1)}{\partial z_+} = \frac{dh(\Gamma_1)}{d\Gamma_1}\frac{\partial \Gamma_1}{\partial z_+} = \frac{dh(\Gamma_1)}{d\Gamma_1}\left[-\frac{m^2(1-z_-)}{z_-z_+^2}\right] \tag{8.11}$$

An important property of the differential equation in Eq. (8.10) is that *the left-hand side only depends on* Γ_1 *and the right-hand side only on* Γ_2. The two Γ-variables are just as independent of each other as the two $z_\pm$-variables. The $z_\pm$ can of course be expressed in terms of the Γ's by the equations above. Since the Γ's are taken as independently varying quantities then *the only way that the equation can be fulfilled is if both sides are equal to the same constant, to be called* $-b$.

Then the differential equation for h is

$$\frac{d}{d\Gamma}\left(\Gamma\frac{dh}{d\Gamma}\right) = -b \tag{8.12}$$

which implies

$$h(\Gamma) = -b\Gamma + a\log\Gamma + \log C \tag{8.13}$$

In this way we obtain for $H(\Gamma)$ (neglecting the indices 1 and 2 as the equation works equally well for both)

$$H(\Gamma) = C\Gamma^a \exp(-b\Gamma) \tag{8.14}$$

The parameters b, a and C are all constants of integration. While b (which has the dimension of an inverse squared mass) must be the same for all the vertices the (dimensionless) constants a and C may have different values. They may e.g. depend upon the flavor quantum numbers of the pair produced at a particular vertex. The constant C plays the role of a normalisation constant for the distribution H. We will later show the significance of a and b.

If we introduce the results for h into the original equation for h and g, Eq. (8.9), it is possible to derive an expression for the original distribution $f(z)$. This can be arranged so that all the dependence on z_+ is on one side of the equation and all the z_--dependence on the other:

$$\begin{aligned} g_{12}(z_+) + \frac{bm^2}{z_+} - a_1\log\left(\frac{m^2}{z_+}\right) - a_2\log\left(\frac{1-z_+}{z_+}\right) + \log C_1 \\ = g_{21}(z_-) + \frac{bm^2}{z_-} - a_2\log\left(\frac{m^2}{z_-}\right) - a_1\log\left(\frac{1-z_-}{z_-}\right) + \log C_2 \end{aligned} \tag{8.15}$$

Then we use the same argument based upon independence to deduce that both sides must be equal to the same constant. The result for f is

$$f(z) = N\frac{1}{z}(1-z)^a \exp\left(-\frac{bm^2}{z}\right) \tag{8.16}$$

if there is only a single value of the a-parameter for all vertices. The quantity N is again a normalisation constant. When there are different values a_α, a_β at two adjacent vertices then we obtain, with a labelling such that the produced hadron stems from a step from vertex α to vertex β,

$$f_{\alpha\beta} = N_{\alpha\beta}\frac{1}{z}z^{a_\alpha}\left(\frac{1-z}{z}\right)^{a_\beta} \exp\left(\frac{bm^2}{z}\right) \tag{8.17}$$

From Eq. (8.15) we conclude that the normalisation constants N_{12} and N_{21} are related to the normalisation of the distributions H_j, $j = 1, 2$, by a

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common factor N_c:

$$N_{12} = \frac{N_c}{C_1 m^{2a_1}}, \quad N_{21} = \frac{N_c}{C_2 m^{2a_2}} \tag{8.18}$$

The combined probability of being at vertex 1 and of taking the step z_+ towards vertex 2, thereby producing the hadron m, is (for equal values of a):

$$\begin{aligned} &CN\,[\Gamma(1-z_+)]^a \exp\left[-b\left(\Gamma + \frac{m^2}{z_+}\right)\right] d\Gamma \frac{dz_+}{z_+} \\ &= CN\left[m^2 \frac{(1-z_-)(1-z_+)}{z_- z_+}\right]^a \exp\left(-\frac{bm^2}{z_+ z_-}\right) \frac{m^2 dz_+ dz_-}{(z_+ z_-)^2} \end{aligned} \tag{8.19}$$

From the second line we find that the distribution is the same if we decide to go 'in the opposite direction', i.e. express the distribution in terms of the variables relevant for the negative lightcone description. We leave it to the reader to derive the corresponding relations for the case when a and C are different at neighboring vertices. In particular it is useful to note that the product CNm^{2a} becomes N_c as defined in Eq. (8.18).

Phenomenologically it has not up to now been necessary in the Lund model to use different a-values to describe the data from the experiments. We will present an idea of Bowler, [32], in connection with the discussion of heavy flavor fragmentation in Chapter 13 which fits very nicely into the Lund model scenario and would require a different a-value for the first-rank hadron in the fragmentation of a heavy quark jet.

If we should, nevertheless, require to use several a-values then it would be necessary to normalise the distributions H_j (j being an index corresponding to different flavor values) to the relative occurrence of the different flavors in phase space and to choose the normalisation(s) of the distributions f_{jk} in a similar way. We will come back to these normalisations in a later chapter.

Thus, using a remarkably simple assumption, we have obtained a very precise result for the string-breaking process. For the Lund model to work it is essential that the expressions we have obtained really do fit the experimental data.

It is, however, necessary, before we can compare with data, to extend the model. We need to remember that the hadronic momenta are measured in a three-dimensional world: therefore the model must be extended outside $1+1$ dimensions. We also need to prescribe a way of normalising our distributions in the case where we would like to describe several different flavors and different hadrons (and one should not forget that we should also be able to account for baryon-antibaryon production!).

Before doing all these things in later chapters we will provide an interpretation of the results we have obtained.

3 The interpretation of the distributions H and f

We will start with the combined expressions occurring in the exponentials of the distributions in Eq. (8.19). For the case when we have arrived at vertex 1 and take a step z_+ we obtain the negative exponential of

$$b\left(\Gamma_1 + \frac{m^2}{z_+}\right) \tag{8.20}$$

From Fig. 8.1 we find that the sum multiplying the parameter b is the area which is spanned below the first meeting point of the two constituents (the q_1-particle from vertex 1 and the $\overline{q}_2$-particle from vertex 2) of the hadron; it is evidently common to the two situations because it can just as well be described as follows (if we are at the vertex 2 and take step z_-):

$$\Gamma_2 + \frac{m^2}{z_-} \equiv \frac{m^2}{z_+ z_-} \equiv W_{+1} W_{-2} \tag{8.21}$$

We leave it to the reader to prove the equality of the expressions in Eqs. (8.20) and (8.21).

Thus *the exponential suppression is related to the size of an area characteristic of the production process.* We will come back to this property later on in Chapter 11 when we provide a quantum mechanical interpretation of the Lund fragmentation distributions.

For the remaining non-exponential factors obtained by multiplying f and H in the two cases we obtain (for different a_α, a_β)

$$\frac{dz_+ dz_-}{z_+^2 z_-^2} \left(\frac{1 - z_-}{z_-}\right)^{a_\alpha} \left(\frac{1 - z_+}{z_+}\right)^{a_\beta} \tag{8.22}$$

(besides some constant factors). This expression is evidently again symmetric between the two vertices and can also be interpreted as the size of certain areas. For the case when $a_\alpha = a_\beta$ we obtain the symmetrical area marked *area* in Fig. 8.1 as the common factor, i.e.

$$(area)^a \tag{8.23}$$

From this result we conclude (parameter a being positive) that there is a (power-)suppression if we take too large a step in the production process, i.e. when any one of the variables $z_\pm$ is chosen to be close to unity.

We will later see that the appearance of the parameter a stems from the requirement of not using up all the remaining energy-momentum. The reason is, of course, that we are implicitly assuming in all our considerations that we are far from the end or the beginning of the process. The distributions f and H are called *inclusive* distributions, i.e. they are characteristic of a single-production event independent of anything else that comes before or after. But there is, of course, a tacit assumption that there are other particles produced, over which we are summing.

8.3 The production of a finite-energy cluster of hadrons

We will in this section derive the distribution for a finite number of hadrons which are rank-ordered, for definiteness along the positive light-cone. From the resulting formulas all other possible situations can be deduced. Such a group of particles is often called a *cluster* or a *single jet*. Together they will have a finite mass, conventionally called $\sqrt{s}$.

The first-rank particle will then contain the flavor f_0 of the original q_0 together with the antiflavor $\bar{f}_1$ of the $\bar{q}_1$ produced at the first vertex. The second-rank particle will contain the flavor f_1 and antiflavor $\bar{f}_2$ of the q_1 from the first vertex and the $\bar{q}_2$ from the second, etc.

The probability of obtaining a first-rank meson with mass m_{01} and with a fractional lightcone component z_1 of the total energy-momentum p_{+0} of the original q_0 is according to Eq. (8.17)

$$f(z_1)dz_1 = N\frac{dz_1}{z_1}z_1^{a_0}\left(\frac{1-z_1}{z_1}\right)^{a_1}\exp\left(-\frac{bm_{01}^2}{z_1}\right) \tag{8.24}$$

In order to simplify the formulas we will from now on consider the case when all the a-values as well as the masses are the same. At the end of the derivation we will provide the formulas for the general case. We will also use the convention of writing z_{oj} for the lightcone energy-momentum fraction of the hadron of rank j, scaled with respect to the original quark's energy-momentum p_{+0}; we call z_{oj} the 'observable' fraction.

Thus the variable z_1 in Eq. (8.24) equals z_{o1} while for the second-rank hadron, which takes a fraction z_2 of the remaining energy-momentum, $(1 - z_{o1})p_{+0}$, we have

$$z_{o2} = z_2(1 - z_{o1}) \tag{8.25}$$

The variable z_2 is again distributed according to the function f (for equal a-values cf. Eq. (8.16)). Therefore the combined distribution for producing first- and second-rank hadrons with observable fractional lightcone components z_{o1} and z_{o2} is

$$\begin{aligned} f(z_1)dz_1 f(z_2)dz_2 &= f(z_{o1})dz_{o1} f\left(\frac{z_{o2}}{1-z_{o1}}\right)\frac{dz_{o2}}{1-z_{o1}} \\ &= \left(\frac{Ndz_{o1}}{z_{o1}}\right)\left(\frac{Ndz_{o2}}{z_{o2}}\right)(1-z_{o1})^a\left(1-\frac{z_{o2}}{1-z_{o1}}\right)^a \exp[-b(A_1+A_2)] \end{aligned} \tag{8.26}$$

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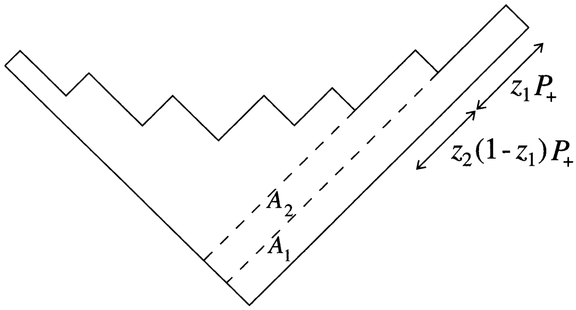

Fig. 8.2. The production of the first- and second-rank hadrons, with the areas in the exponent of Eq. (8.26) indicated.

The a-dependent factors obviously combine to give

$$(1 - z_{o1} - z_{o2})^a \tag{8.27}$$

and the fractional differentials can be reexpressed as follows:

$$\left(\frac{dz_{o1}}{z_{o1}}\right)\left(\frac{dz_{o2}}{z_{o2}}\right) = d^2p_{o1}d^2p_{o2}\delta^+(p_{o1}^2 - m^2)\delta^+(p_{o2}^2 - m^2) \tag{8.28}$$

Here $d^2p = dp_+dp_-$. We consequently introduce two new variables, in this case the negative-lightcone energy-momenta (note that $p_{+oj} = z_{oj}p_{+0}$). This is done by the introduction of two δ-distributions which fix their values. We have used the following properties of the δ-distribution, which was also used in Chapter 3 with the requirement that $C = D/B$:

$$dBdC\delta(BC - D) \to \frac{dB}{B} \tag{8.29}$$

The arrow implies that the left- and right-hand sides are equal if we actually perform the integral. We shall always use an equality sign even if we do not perform the integrals. The right-hand side of Eq. (8.28) explicitly exhibits the Lorentz invariance of the phase-space factors. The factor $A_1 + A_2$ in the exponential in Eq. (8.26) corresponds to the two regions indicated in Fig. 8.2 (the interpretation as an area size is given to the exponential factor in the fragmentation function in Eqs. (8.20), (8.21)).

From this result we may already guess what the result will be if we produce n particles with energy-momenta $\{p_{+oj}\} \equiv \{z_{oj}p_{+0}, p_{-oj}\}$:

$$dP(p_{o1}, \dots, p_{on}) \equiv \left(1 - \sum_{j=1}^{n} z_{oj}\right)^a \prod_{j=1}^{n} Nd^2p_{oj}\delta^+(p_{oj}^2 - m^2)\exp(-bA_j). \tag{8.30}$$

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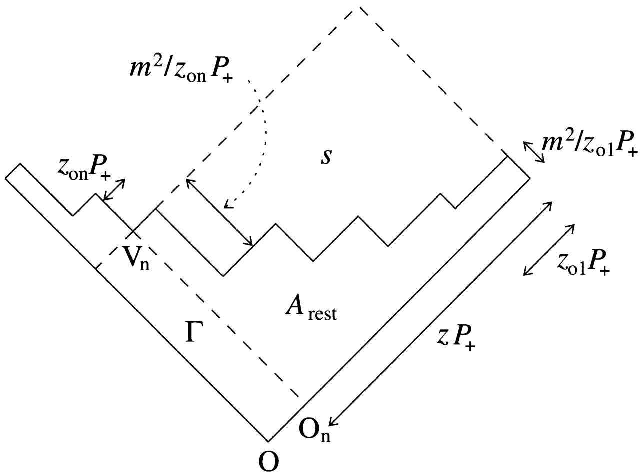

Fig. 8.3. An n-particle cluster with notation as explained in the text.

This formula is straightforward to prove; we will leave this to the reader.

The situation after n steps is depicted in Fig. 8.3. We note firstly that the total area in the exponent, $\sum_{j=1}^{n} A_j \equiv A_{tot}$, can be subdivided into two parts,

$$A_{tot} = A_{rest} + \Gamma \tag{8.31}$$

as shown in the figure. The quantity Γ then corresponds to the proper time (cf. Eq. (8.2)) of the 'last' vertex of the cluster.

Secondly we note that it seems as if the system of the n particles could have been produced just as well by the original q-particle and an antiparticle $\bar{q}_n$. This pair would then have started out at the point O_n in Fig. 8.3. We know in fact that the cluster is part of the system produced from the force field of the original $q\bar{q}$-pair which started at the point O and produced the pair $q_n\bar{q}_n$ at the vertex V_n. But we would not have that knowledge unless we had been able to observe some parts of the system outside the cluster!

The energy-momentum of the 'new' pair is then (W_{+n}, W_{-n}) where $W_{+n} = zp_{+0}$, $z \equiv \sum_{j=1}^{n} z_{oj}$ and

$$W_{-n} = \sum_{j=1}^{n} \frac{m_j^2}{z_{oj}p_{+0}} \tag{8.32}$$

The formulas for $W_{\pm n}$ are a somewhat complex way of writing the total

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energy-momenta of all the particles in the cluster: $W_{\pm n} = \sum_{j=1}^{n} p_{\pm oj}$. We conclude that the total squared mass of our n-particle system is

$$s = W_{+n}W_{-n} = \sum_{j=1}^{n} \frac{m^2 z}{z_{oj}} \tag{8.33}$$

The variable s is also the size of an area according to Fig. 8.3. Using this fact it is easy to convince oneself that the area Γ defined in Eq. (8.31) is given by

$$\Gamma = \frac{s(1-z)}{z} \tag{8.34}$$

which we again leave to the reader to prove.

Consequently all the interesting *external* properties of our n-particle system (i.e. its properties with regard to the original origin O) are given by the two Lorentz invariants s and z. It is useful to introduce these variables into the formulas and define (using Eq. (8.30))

$$dP(z, s; p_{o1}, \ldots, p_{on}) \equiv dz\delta\left(z - \sum_{j=1}^{n} z_{oj}\right) ds\delta\left(s - \sum_{j=1}^{n} \frac{m^2 z}{z_{oj}}\right) \times dP(p_{o1}, \ldots, p_{on}). \tag{8.35}$$

As $z > 0$ we may change the first δ-distribution as follows:

$$dz\delta\left(z - \sum_{j=1}^{n} z_{oj}\right) = \frac{dz}{z}\delta\left(1 - \sum_{j=1}^{n} \frac{z_{oj}}{z}\right) \tag{8.36}$$

Then the two new δ-distributions (i.e. the s-definition and the above reorganised z-definition) only depend upon the *internal* variables

$$u_j \equiv \frac{z_{oj}}{z} = \frac{p_{+oj}}{W_{+n}} \tag{8.37}$$

These would be the scaling variables if we consider the cluster as arising from the $q\bar{q}_n$-pair produced at the space-time point O_n in Fig. 8.3. We then obtain for the expression in Eq. (8.35)

$$dP(z, s; p_{o1}, \ldots, p_{on}) = ds\frac{dz}{z}(1-z)^a \exp(-b\Gamma)\delta\left(1 - \sum_{j=1}^{n} u_j\right)\delta\left(s - \sum_{j=1}^{n} \frac{m^2}{u_j}\right) \times \prod_{j=1}^{n} N d^2 p_{oj} \delta^+(p_{oj}^2 - m^2) \exp(-bA_{rest}) \tag{8.38}$$

By a further 'division trick' the two δ-distributions can be written as

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follows:

$$\delta\left(1-\sum_{j=1}^{n}u_j\right)\delta\left(s-\sum_{j=1}^{n}\frac{m^2}{u_j}\right)$$
$$=\delta\left(W_{+n}-W_{+n}\sum_{j=1}^{n}u_j\right)\delta\left(W_{-n}-\sum_{j=1}^{n}\frac{m^2}{u_jW_{+n}}\right)\equiv\delta^2(P_{rest}-\sum_{j=1}^{n}p_{oj}) \tag{8.39}$$

where the quantity P_{rest} is the energy-momentum of the n-particle system ($P_{rest}=(W_{+n},W_{-n})$). The superscript 2 on the δ at the end of Eq. (8.39) indicates that here we use (the lightcone-component version of) the two-dimensional energy-momentum conserving δ-distribution.

In this way we have been able to partition the formula for the production probability of an n-particle cluster with a given endpoint (squared) proper time Γ, Eq. (8.34), and a given total energy-momentum $W_{\pm n}$ with a squared mass $s=W_{+n}W_{-n}$, into two parts. These will be called, according to the notions introduced above, the *external part*

$$dP_{ext}=ds\frac{dz}{z}(1-z)^a\exp(-b\Gamma) \tag{8.40}$$

and the *internal part*:

$$dP_{int}=\prod_{j=1}^{n}Nd^2p_{oj}\delta^+(p_{oj}^2-m^2)\exp(-bA_{rest})\delta^2(\sum_{j=1}^{n}p_{oj}-P_{rest}) \tag{8.41}$$

The external part corresponds to the (non-normalised) probability that the cluster as defined above will occur, while the internal part in the same way corresponds to the probability that the cluster will decay into the particular channel considered, containing the given n particles.

The general result for an n-particle cluster which starts at a vertex with the parameter a_0 and ends at a vertex with a_n is, for the external part,

$$dP_{ext}=ds\frac{dz}{z}z^{a_0}\left(\frac{1-z}{z}\right)^{a_n}\exp(-b\Gamma) \tag{8.42}$$

The corresponding general formula for the internal part is

$$dP_{int}=\prod_{j=1}^{n}N_{j-1,j}\left(\frac{du_j}{u_j}\right)u_j^{a_{j-1}-a_j}$$
$$\times\exp(-bA_{rest})\delta\left(1-\sum_{j=1}^{n}u_j\right)\delta\left(s-\sum_{j=1}^{n}\frac{m_{j-1,j}^2}{u_j}\right) \tag{8.43}$$

where we have kept the scaling variable description. We leave it to the reader to derive Eqs. (8.42), (8.43).

The result in Eq. (8.41) is evidently completely symmetric with respect to the different particles and therefore it has an obvious left-right symmetry with respect to the lightcones. This property is not so obvious in Eq. (8.43). We note, however, that the negative-lightcone variables v_j corresponding to the u_j obviously should fulfil

$$v_j u_j = \frac{m_{j-1,j}^2}{s} \tag{8.44}$$

(these are the mass-shell conditions). Therefore a change from the variables u_j to v_j can be carried through in a straightforward manner in Eq. (8.43). We obtain for the terms in Eq. (8.43):

$$\begin{gathered}
\frac{du_j}{u_j} \to \frac{dv_j}{v_j} \\
N_{j-1,j} u_j^{a_{j-1}-a_j} \to N'_{j-1,j} v_j^{a_j - a_{j-1}} \\
\delta\left(1 - \sum_{j=1}^{n} u_j\right) \delta\left(s - \sum_{j=1}^{n} \frac{m_{j-1,j}^2}{u_j}\right) \to \delta\left(s - \sum_{j=1}^{n} \frac{m_{j-1,j}^2}{v_j}\right) \delta\left(1 - \sum_{j=1}^{n} v_j\right)
\end{gathered} \tag{8.45}$$

In the second line we have absorbed a $(j, j-1)$-dependent mass factor into the normalisation constant $N'_{j-1,j}$ and in the third line we have again made use of a 'division trick' for the two δ-distributions.

Obviously the result in Eq. (8.43) is, after these operations, the same in the u_j-language as in the v_j-language apart from the fact that we are now ordering the vertices as $j, j-1, \ldots$ along the negative lightcone.

In the following chapters we will investigate the internal- and external-part formulas in great detail and also exhibit several different interpretations from both quantum field theory and statistical mechanics.

8.4 The Artru-Menessier-Bowler model

We will now briefly consider a different approach to the decay of a high-mass string, the AMB model, [26], [32]. Here the idea is to take classical probability arguments, which also occur in the Lund model derivation as presented above, as far as they can go. There are two basic rules.

AMB1 There is a constant probability $\mathscr{P}$ per unit time and per unit length in the string's space-time history that it may break up by the production of $q\bar{q}$-pairs.

AMB2 The string cannot break up further in the forward lightcone with respect to an 'earlier' vertex.

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The procedure can be visualised as the game of stochastic dart-throwing on a target corresponding to the original string's space-time history. The landing of each dart then produces a possible vertex and one accepts those vertices which have no other vertex in their prehistory.

A continuous mass spectrum will then be obtained for the produced particles. There is then a third rule to interpret the result.

AMB3 Using AMB1 and AMB2 one obtains a *first generation of breakups* producing a first generation of yoyo-hadrons. These states are then considered as 'resonances' and will be allowed to decay again, independently, according to the same rules.

If we go back to Fig. 8.1 then we conclude that one will obtain (just as for a radiative decay) that the probability for an allowed vertex at a point (Γ_1, y_1) is

$$dP_{AMB}(1) = bd\Gamma_1 dy_1 \exp(-b\Gamma_1) \tag{8.46}$$

where $b = \mathscr{P}/\kappa^2$. We will consider this result in more detail below when we compare to the Lund model results.

Similarly there is a joint probability of having two primary neighboring AMB vertices at the two points 1 and 2 in Fig. 8.1. It is equal to

$$\begin{aligned} dP_{AMB}(12) &= dP_{AMB}(1)dP_{AMB}(1 \to 2) \\ dP_{AMB}(1 \to 2) &= b(W_{+1}dz_+)(W_{-2}dz_-)\exp\left[-b(W_{+1})(z_- W_{-2})\right] \end{aligned} \tag{8.47}$$

with $dP_{AMB}(1 \to 2)$ *the conditional probability* that given 1 we may also obtain 2. We are using the notation of Fig. 8.1 and the Eqs. (8.6). This time there is no mass-shell condition to constrain the location of the two vertices 1 and 2. Therefore we need all four (independent) quantities Γ_1, y_1, z_+, z_-. (Note that due to Lorentz covariance there is no dependence on the rapidity variable y_1 in the formulas.)

The probability distribution $dP_{AMB}(1 \to 2)$ contains the negative exponential of the region (cf. Eq. (8.6))

$$W_{+1}W_{-2}z_- = \frac{m^2}{z_+} \tag{8.48}$$

with m the mass produced between the adjacent vertices 1 and 2; together the exponentials of the two distributions $dP(1)dP(1 \to 2)$ contain the symmetrical surface $W_{+1}W_{-2}$ from Eqs. (8.20), (8.21). Therefore the joint distribution $dP(12)$ is symmetric with respect to vertices 1 and 2.

The distribution $dP_{AMB}(12)$ can be reformulated into a distribution in

z_+ and the mass m as

$$dP_{AMB}(12) = \frac{bdz_+ dm^2}{z_+} \exp\left(-\frac{bm^2}{z_+}\right) \tag{8.49}$$

From this expression it is then possible to obtain the distribution in the mass m by means of an integral over z_+:

$$\frac{dP}{dm^2} = \int_0^1 \frac{bdz_+}{z_+} \exp\left(-\frac{bm^2}{z_+}\right) = bE_1(bm^2) \tag{8.50}$$

where E_1 is the exponential integral of the first rank. This function is singular when $m^2 \to 0$, which means that there is a large probability that the string in the AMB model breaks up into very tiny pieces. It is then necessary to introduce a lower cutoff in the mass spectrum. Such a cutoff is difficult to introduce in a consistent way if one wants to keep to the classical probability concepts which are at the basis of the model. It is nevertheless possible to interpret the resulting spectrum in a way similar to the resonance spectrum suggested by Hagedorn, [76] (although Hagedorn obtained a linear dependence upon the masses in the exponent).

The results of the AMB model are evidently (apart from the continuous mass spectrum) similar to the results of the Lund model. It contains an iterative structure based upon an area suppression law. It is, however, not possible to obtain the Lund model relations by the use of the probability concepts in the AMB model.

To see this, suppose that we specialise the AMB model to particular masses, e.g. a single mass with a width δm^2 around m^2. This would mean that a new vertex would only be allowed in a band along the mass hyperbola corresponding to m. If we are at vertex V and we are looking for the next vertex H in that band we may subdivide the band into many small boxes (see Fig. 8.4) and call them $1, 2, \ldots, n \ldots$. The boxes have areas $(\delta a)_j$ and the probability of finding a vertex in such a box is equal to $b(\delta a)_j$.

Then the probability of not finding a vertex in the first n boxes will be

$$\prod_{j=1}^{n} [1 - b(\delta a)_j] \to \exp\left[-\int bd(\delta a)\right] \tag{8.51}$$

Here the right-hand expression is the limit found when we subdivide the band indefinitely, i.e. when $n \to \infty$. The expression for $d(\delta a)$ is $\delta m^2 dz_+/z_+$ i.e. the width times an infinitesimal angular segment along the hyperbola.

Therefore the probability of finding a vertex at the value z_+ without having found it for any larger value of z_+ (i.e. for any 'earlier' vertex,

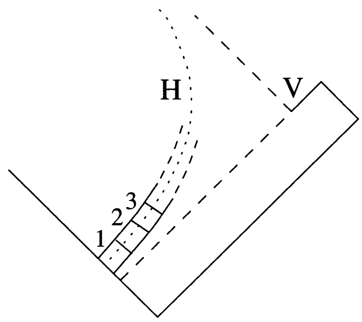

Fig. 8.4. The allowed region for finding the next vertex H, after the vertex produced at V, is a band around a hyperbola. This region can be subdivided into small boxes as discussed in the text.

closer to the origin, see Fig. 8.4) is

$$\frac{b\delta m^2 dz_+}{z_+} \exp\left(-b\delta m^2 \int_{z_+}^{1} \frac{dz'_+}{z'_+}\right) = b\delta m^2 dz_+ z_+^{b\delta m^2 - 1} \tag{8.52}$$

This corresponds to a power law in z_+, owing to the fact that we no longer have a two-dimensional surface on which to apply the probability rule.

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