GROUPS WITH EQUAL UNIFORMITIES

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If $G = (G, \tau)$ is a topological group with topology τ, then there is a smallest topology $\tau^* \supseteq \tau$ such that $G^* = (G, \tau^*)$ is a topological group with equal left and right uniformities (**1**). Bagley and Wu introduced this topology in (**1**), and studied the relationship between G and G^*. In this paper we prove some additional results concerning G^* and groups with equal uniformities in general. The structure of locally compact groups with equal uniformities has been studied extensively. If G is a locally compact connected group, then G has equal uniformities if and only if $G \cong V \times K$, where V is a vector group and K is a compact group (**5**). More generally, every locally compact group with equal uniformities has an open normal subgroup of the form $V \times K$ (**4**).

Definition. If $G = (G, \tau)$ is a topological group with topology τ, and $\mathscr{B}$ is a neighbourhood base at the identity e in G, then $G^* = (G, \tau^*)$, where a neighbourhood base at e for τ^* is $\{\bigcap_{t \in G} tVt^{-1}: V \in \mathscr{B}\}$. We let V^* denote $\bigcap_{t \in G} tVt^{-1}$ whenever V is a neighbourhood of e in G.

It is easy to see that a group G has equal uniformities if and only if V^* is a neighbourhood of e whenever V is a neighbourhood of e. Thus, it follows that G^* is a topological group with equal uniformities, and τ^* is the smallest topology containing τ for which this is so.

If G is a locally compact connected group or a connected Lie group, then G^* is a locally compact group or a Lie group, respectively; see (**1**). However, there are locally compact groups G such that G^* is not locally compact (**1**). The following theorem gives a specific example of how G and G^* are related. $\mathrm{GL}(n)$ is the group of non-singular $n \times n$ real matrices with the n^2 Euclidean topology.

THEOREM 1. *If $G = \mathrm{GL}(n)$, and V is a bounded neighbourhood of the identity, then $V^* = V \cap Z$, where Z is the centre of G.*

Proof. Recall that $Z = \{cI: c \neq 0\}$, where c is a real number and I is the identity matrix. We first prove the following result.

LEMMA. *Let AB be a well-defined matrix product, where B is fixed and the coordinates of A are allowed to vary. If B has a non-zero coordinate b_{ij}, then we can vary the coordinates of A to obtain an arbitrarily large number as the kj coordinate of the product AB (k is any number such that AB has a kj coordinate).*

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Proof. If c_{kj} is the kj coordinate of AB, then $c_{kj} = \sum_l a_{kl} b_{lj}$. By setting $a_{kl} = 0$ for $l \neq i$ and letting a_{ki} be suitably large, we obtain the desired result.

The proposition is true for $G = \mathrm{GL}(1)$. Using induction, suppose that it is true for $n - 1$, and consider $G = \mathrm{GL}(n)$. Let $x \in V^* = \bigcap_{t \in G} tVt^{-1}$. Then $txt^{-1} \in V$ for all $t \in G$. In particular, this is true for any t of the form

$$\left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & a_2 \end{array}\right],$$

where A_1 is an $(n-1) \times (n-1)$ matrix. Partitioning x the same way, we obtain

$$\begin{aligned} txt^{-1} &= \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & a_2 \end{array}\right]\left[\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right]\left[\begin{array}{c|c} A_1^{-1} & 0 \\ \hline 0 & a_2^{-1} \end{array}\right] \\ &= \left[\begin{array}{c|c} A_1 X_1 A_1^{-1} & A_1 X_2 a_2^{-1} \\ \hline a_2 X_3 A_1^{-1} & a_2 X_4 a_2^{-1} \end{array}\right]. \end{aligned}$$

By the induction hypothesis, since

$$\left\{\left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right]\right\}$$

is isomorphic to $\mathrm{GL}(n-1)$, it follows that $X_1 = c_1 I$. Now $a_2 \neq 0$, since t is non-singular. Thus, if X_2 has a non-zero coordinate, by letting A_1 vary, we can obtain an arbitrarily large coordinate in $A_1 X_2 a_2^{-1}$. Since V is bounded, $X_2 = 0$. Similarly, $X_3 = 0$. Thus,

$$x = \left[\begin{array}{c|c} c_1 I & 0 \\ \hline 0 & c_2 \end{array}\right],$$

where I is an $(n-1) \times (n-1)$ matrix. By a similar argument we show that

$$x = \left[\begin{array}{c|c} c_1 & 0 \\ \hline 0 & c_2 I \end{array}\right],$$

where I is an $(n-1) \times (n-1)$ matrix.

If we knew that $n > 2$, then we would have $c_1 = c_2$, and the proof would be complete. In order to complete the induction argument, we must present a separate proof that $c_1 = c_2$ in the case $n = 2$. Notice that

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} c_1 & bc^{-1}(c_2 - c_1) \\ 0 & c_2 \end{bmatrix} \quad \text{for all} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathrm{GL}(2).$$

If $c_1 \neq c_2$, then we can vary b to make $bc^{-1}(c_2 - c_1)$ large. Thus, $c_1 = c_2$. We have shown that $V^* \subset V \cap Z$. It is now obvious that $V^* = V \cap Z$.

Notice that this theorem tells us that if $G = \mathrm{GL}(n)$, then G^* is one-dimensional. The following statement undoubtedly follows directly. We prove it, however, as a corollary to our theorem.

COROLLARY. $\mathrm{GL}(n)$ *has no non-finite normal compact subgroups.*

Proof. Let N be a compact normal subgroup of $\mathrm{GL}(n)$. Then by (**1**, Lemma 1), the topologies of G and G^* agree on N. Thus, there is a neighbourhood of the identity in N which is contained in Z. However, $N \cap Z \subset \{\pm I\}$, and thus N is discrete, and hence finite.

More generally, we have the following two theorems which state that either $G = G^*$ or there is quite a difference between them.

THEOREM 2. *Either $G = G^*$, or there is a neighbourhood of e in G^* which is first category in G.*

Proof. If V is a closed symmetric neighbourhood of e in G, then V^* is closed in G, and hence V^* is almost open (**3**, p. 211). Now, if V^* is second category, then $(V^*)^2$ is open in G (**3**). Since $\{(V^*)^2$: V a closed symmetric neighbourhood of e in $G\}$ is a basis at e for the topology of G^*, it follows that $G = G^*$, or some such V^* is first category in G.

THEOREM 3. *If G is second category, and V is a compact neighbourhood of e, then the subgroup generated by V^* is either first category in G or open in G.*

Proof. $(V^*)^n$ is compact in G for each n, and therefore closed. Thus, the subgroup generated by V^* is a Borel set, hence almost open in G (**3**). Thus, the subgroup is either first category in G or open in G.

COROLLARY. *If G^* is locally compact, then the identity component of G^* is either first category in G or it is the identity component in G.*

Proof. The identity component of a locally compact group is the intersection of the open subgroups.

We now prove several results which establish sufficient conditions for a group to have equal uniformities.

THEOREM 4. *If H is a subgroup of G, and H has equal uniformities, then so does $\bar{H}$.*

Proof. H has a basis at e of neighbourhoods U such that $U = \bigcap_{t \in H} tUt^{-1}$. For any such U, it is not hard to show that $\bar{U}$ is a neighbourhood of e in $\bar{H}$, and $\bar{U} = \bigcap_{t \in H} t\bar{U}t^{-1}$. Thus, $\{\bar{U}\}$ will form a basis at e in $\bar{H}$, and the proof is complete.

THEOREM 5. *If $G = \prod_\alpha G_\alpha$, and each G_α has equal uniformities, then G has equal uniformities.*

Proof. For each α, let p_α: $G \to G_\alpha$ be the natural projection $p_\alpha(g) = g_\alpha$, where $g = (g_\alpha)$. If U_α is a neighbourhood of e in G_α such that $U_\alpha = \bigcap_{t \in G_\alpha} tU_\alpha t^{-1}$, then $p_\alpha(gp_\alpha^{-1}(U_\alpha)g^{-1}) = g_\alpha U_\alpha g_\alpha^{-1} = U_\alpha$ for all $g \in G$, and thus $p_\alpha^{-1}(U_\alpha) = \bigcap_{g \in G} gp_\alpha^{-1}(U_\alpha)g^{-1}$. Since sets of the form $p_\alpha^{-1}(U_\alpha)$, where $U_\alpha = \bigcap_{t \in G} tU_\alpha t^{-1}$, form a subbasis for the topology of G, it follows that G has equal uniformities.

COROLLARY. *If $\{G_\alpha\}$ is an inverse system of groups such that each G_α has equal uniformities, and G is the projective limit of this system, then G has equal uniformities.*

Proof. $G \subset \prod_\alpha G_\alpha$. Thus, by Theorem 5, G has equal uniformities.

The next theorem is closely related to the above corollary in the case G is locally compact.

THEOREM 6. *If G has arbitrarily small closed normal subgroups H such that G/H has equal uniformities, then G has equal uniformities.*

Proof. Let U be a neighbourhood of e in G and pick a neighbourhood V of e such that $V^2 \subset U$. Let $H \subset V$ be a closed normal subgroup such that G/H has equal uniformities. Then $\bigcap_{t\in G} tUt^{-1} \supset \bigcap_{t\in G} tV^2t^{-1} \supset \bigcap_{t\in G} tVHt^{-1}$. Since G/H has equal uniformities, this last set is a neighbourhood of e in G. Thus, G has equal uniformities.

If a group G has equal uniformities, and H is a normal subgroup, then both H and G/H have equal uniformities. The converse is not true in general. For example, even a semi-direct product of groups with equal uniformities may fail to have equal uniformities. Let G be the semi-direct product of the plane R^2 by the circle group C where the automorphism of R^2 induced by $e^{i\theta} \in C$ is a rotation of R^2 through an angle θ. G is a locally compact connected group with trivial centre. It follows that G cannot be isomorphic to the direct product of a vector group and a compact group, and thus G fails to have equal uniformities, even though R^2 is abelian and C is compact.

The following theorem is related to (**1**, Theorem 3 and its corollaries). R denotes the additive real numbers.

THEOREM 7. *If R is a closed normal subgroup of G and G/R is compact, then G has equal uniformities.*

Proof. We first prove a lemma.

LEMMA. *If G is locally compact, R a closed normal subgroup such that R has a compact neighbourhood U of e which is invariant under the inner automorphisms of G, then the centralizer $Z(R)$ of R in G has index 2 in G.*

Proof. Take $\alpha \in U$, $g \in G$, $\alpha > 0$, and consider the sequence $\alpha_n = |g\alpha_{n-1}g^{-1}|$. This is a monotone sequence, and we may assume that it is increasing. Thus, it converges to $\beta \in U$, $\beta > 0$. Now $|g\beta g^{-1}| = \lim |g\alpha_n g^{-1}| = \beta$, and therefore either $g\beta g^{-1} = \beta$ or $g\beta g^{-1} = -\beta$. It follows that $g\alpha g^{-1} = \alpha$ or $g\alpha g^{-1} = -\alpha$ for all $\alpha \in R$. Thus, we have proved our lemma.

Now suppose that G/R is compact. Then $G = U \cdot R$, where U is a compact neighbourhood of e. Let V be any neighbourhood of zero in R. Then $\bigcap_{t\in G} tVt^{-1} = \bigcap_{t\in U} tVt^{-1}$, which is a neighbourhood of zero, since U is compact. Thus, we can apply our lemma, and $Z(R)$ is an open subgroup of index 2. It follows easily that G has equal uniformities.

This last theorem generalizes (**2**, Proposition 12.2, part (v)).

THEOREM 8. *If G is locally compact, and G is mapped by a continuous isomorphism into a group G_1 with equal uniformities, then G has equal uniformities if either one of the following conditions holds:*
(1) *G is generated by a compact neighbourhood of e;*
(2) *G has a compact neighbourhood U of e such that $U^* = U$.*

Proof. Part (1). Let U be a compact neighbourhood of e which generates G. Let V be any open neighbourhood of e such that $V \subset U$. Now, if ϕ is the continuous isomorphism of G into G_1, then $\phi | U^3$ is a homeomorphism onto $\phi(U^3)$. Thus, $\phi(V)$ is open in $\phi(U^3)$. Therefore, there is a neighbourhood W of e in G_1 such that $W \cap \phi(U^3) = \phi(V)$. Let N be a neighbourhood of e in G_1 such that $N \subset W$ and $N = \bigcap_{t \in G_1} tNt^{-1}$. Finally, let $V_0 = V \cap \phi^{-1}(N)$. Thus, V_0 is a neighbourhood of e in G. If $x \in V_0$ and $t \in U$, then $txt^{-1} \in U^3$; hence, $\phi(txt^{-1}) \in \phi(U^3) \cap N \subset \phi(U^3) \cap W = \phi(V)$. Thus, $txt^{-1} \in V \cap \phi^{-1}(N) = V_0$. Since U generates G, it follows that for any $t \in G$, $x \in V_0$, that $txt^{-1} \in V_0$. Thus, $V_0 = V_0^*$, and G has equal uniformities.

Proof. Part (2). Let U be a compact neighbourhood of e such that $U = U^*$, and let V be any neighbourhood of e which is contained in U. The isomorphism ϕ of G into G_1 is a homeomorphism when restricted to U. Thus, we can pick a neighbourhood W of e in G_1 such that $W \cap \phi(U) \subset \phi(V)$ and $W = \bigcap_{t \in G_1} tWt^{-1}$. Then for any $t \in G$,

$$\phi(tVt^{-1}) = \phi(t)\phi(V)\phi(t)^{-1} \supset \phi(t)[W \cap \phi(U)]\phi(t)^{-1} = W \cap \phi(U).$$

Thus, $V^* \supset \phi^{-1}(W \cap \phi(U))$, and this is a neighbourhood of e in G. Thus, G has equal uniformities.

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