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ON FINITE PRODUCTS OF TOTALLY PERMUTABLE GROUPS

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In this paper the structure of finite groups which are the product of two totally permutable subgroups is studied. In fact we can obtain the $\mathcal{F}$-residual, where $\mathcal{F}$ is a formation, $\mathcal{H}$-projectors and $\mathcal{H}$-normalisers, where $\mathcal{H}$ is a saturated formation, of the group from the corresponding subgroups of the factor subgroups.

1. INTRODUCTION

All groups considered here are finite. Let G be a group. Following Maier [4], we say that G is the product of the totally permutable subgroups H and K if $G = HK$ and every subgroup of H permutes with every subgroup of K. Direct and central products are typical examples of such products. The structure of the groups which are the product of two totally permutable subgroups have been studied in [1] and [4]. Our purpose here is to carry these studies further. A well-known theorem of Doerk and Hawkes states that for a formation $\mathcal{F}$ of soluble groups the $\mathcal{F}$-residual respects the operation of forming direct products. This result has been generalised in [1]: if $G = HK$ is the product of the totally permutable subgroups H and K, then $G^{\mathcal{F}} = H^{\mathcal{F}}K^{\mathcal{F}}$. Here, $\mathcal{F}$ is a saturated formation of soluble groups containing the class $\mathcal{U}$ of all supersoluble groups. Recall that $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of G, that is, the smallest normal subgroup of G such that $G/G^{\mathcal{F}}$ belongs to $\mathcal{F}$.

Our first theorem extends this result to arbitrary formations of soluble groups containing $\mathcal{U}$.

THEOREM A. *Let $\mathcal{F}$ be a formation of soluble groups such that $\mathcal{U} \subseteq \mathcal{F}$. If $G = HK$ is the product of the totally permutable subgroups H and K, then $G^{\mathcal{F}} = H^{\mathcal{F}}K^{\mathcal{F}}$.*

We would like to mention that the corresponding result for saturated formations is used in the proof of Theorem A.

Recall that if $\mathcal{X}$ is a class of groups, a subgroup P of a group G is an $\mathcal{X}$-projector of G if for every normal subgroup N of G, the factor group PN/N is $\mathcal{X}$-maximal in G/N. It is known that if $\mathcal{X}$ is a Schunck class (in particular if $\mathcal{X}$ is a saturated formation), each group G has $\mathcal{X}$-projectors [3, III, 3.10]. It is quite clear that knowledge of the $\mathcal{X}$-projectors of a group usually reveals little about the $\mathcal{X}$-subgroup structure of the group.

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In fact there is no conection in general between the projectors of a group and those of a proper subgroup. Totally permutable subgroups are exceptions to this general rule when $\mathcal{X}$ is a saturated formation containing the formation of all supersoluble groups.

THEOREM B. *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{U} \subseteq \mathcal{F}$. Let $G = HK$ be the product of the totally permutable subgroups H and K. If A is an $\mathcal{F}$-projector of H and B is an $\mathcal{F}$-projector of K, then AB is an $\mathcal{F}$-projector of G.*

Our last theorem shows that Theorem B also holds for $\mathcal{F}$-normalisers in the soluble universe.

THEOREM C. *Let the soluble group $G = HK$ be the product of the totally permutable subgroups H and K. If $\mathcal{F}$ is a saturated formation such that $\mathcal{U} \subseteq \mathcal{F}$, A is an $\mathcal{F}$-normaliser of H and B is an $\mathcal{F}$-normaliser of K, then AB is an $\mathcal{F}$-normaliser of G.*

The above results become false if the formation $\mathcal{F}$ does not contain the class $\mathcal{U}$, as the formation of all nilpotent groups and the symmetric group of degree 3 show.

The notation is standard and can be found in [**3**]. This book is also the reference for the results concerning formations.

2. Preliminary results

In this section we collect some results which will be used in the proofs of our main theorems.

LEMMA 1. [**1**, Lemma 1] *Let the group $G = NB$ be the product of two subgroups N and B. Suppose that N is normal in G. Since B acts by conjugation on N, we can construct the semidirect product, $X = [N]B$, with respect to this action. Then the natural map $\alpha : X \longrightarrow G$ given by $(nb)\alpha = nb$, for every $n \in N$ and every $b \in B$, is an epimorphism, $\ker \alpha \cap N = 1$ and $\ker \alpha \leqslant C_X(N)$.*

LEMMA 2. [**1**, Lemmas 6 and 8] *Let $G = HK$ be the product of the totally permutable subgroups H and K.*

(a) $[H^{\mathcal{U}}, K^{\mathcal{U}}] = 1$.
(b) *If K is supersoluble, then K centralises $G^{\mathcal{U}} = H^{\mathcal{U}}$.*

As a consequence of Lemma 2 we can stipulate the following result.

COROLLARY. *Let $G = H_1H_2$ be the product of the totally permutable subgroups H_1 and H_2. Then H_i centralises $(H_{3-i})^{\mathcal{U}}$ for $i = 1, 2$.*

PROOF: Apply Lemma 2 taking into account that $H_i = H_i^{\mathcal{U}}U$, where U is an $\mathcal{U}$-projector of H_i (i=1,2). ∎

Lemma 3. *Let $G = HK$ be the product of the totally permutable subgroups H and K. Let $\mathcal{F}$ be a formation containing $\mathcal{U}$. Then $H^{\mathcal{F}}$ and $K^{\mathcal{F}}$ are normal subgroups of G.*

Proof: Notice that $H^{\mathcal{F}} \leqslant H^{\mathcal{U}}$. The Corollary 1 implies that $H^{\mathcal{F}}$ is centralised by K, and obviously $H^{\mathcal{F}}$ is normal in H. Consequently $H^{\mathcal{F}}$ is a normal subgroup of G, and the same is true for $K^{\mathcal{F}}$. ∎

Lemma 4. [1, Theorem] *Let $\mathcal{F}$ be a formation such that $\mathcal{U} \subseteq \mathcal{F}$. Suppose that the group $G = HK$ is the product of the totally permutable subgroups H and K. If H and K belong to $\mathcal{F}$, then G belongs to $\mathcal{F}$.*

Lemma 5. *Let $\mathcal{F}$ be a formation such that $\mathcal{U} \subseteq \mathcal{F}$. Assume that $G = HK$ is the product of the totally permutable subgroups H and K. If $G \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \in \mathcal{F}$.*

Proof: If H is supersoluble, then $H \in \mathcal{F}$ and we are done. Suppose that H is not supersoluble. Then $1 \neq H^{\mathcal{U}}$ and $H = H^{\mathcal{U}}U$, where U is a supersoluble projector of H. By Lemma 1, we have that H is an epimorphic image of the group $C = [H^{\mathcal{U}}]U$. We show that C belongs to $\mathcal{F}$. By Lemma 3, $H^{\mathcal{U}}$ is a normal subgroup of $G = H^{\mathcal{U}}(UK)$. Consequently, by Lemma 1, there exists an epimorphism $\alpha : X \longrightarrow G$ such that $\ker\alpha \cap H^{\mathcal{U}} = 1$, where $X = [H^{\mathcal{U}}](UK)$. In particular, $X/\ker\alpha \simeq G$ and so $X/\ker\alpha$ belongs to $\mathcal{F}$. On the other hand, $X/H^{\mathcal{U}}$ is isomorphic to UK. Since UK is the product of the totally permutable subgroups U and K, it follows that $UK \in \mathcal{F}$ by Lemma 4. This means that $X/H^{\mathcal{U}} \in \mathcal{F}$ and so $X \in R_0(\mathcal{F}) = \mathcal{F}$.

Let $\langle K^X \rangle$ be the normal closure of K in X. Since $H^{\mathcal{U}}$ centralises K by Corollary 1, it follows that $\langle K^X \rangle = \langle K^U \rangle \leqslant UK$ and then $\langle K^X \rangle \cap H^{\mathcal{U}} = 1$. Moreover $X/\langle K^X \rangle$ is isomorphic to $[H^{\mathcal{U}}]U/(\langle K^X \rangle \cap H^{\mathcal{U}}U)$. Hence $[H^{\mathcal{U}}]U/(\langle K^X \rangle \cap H^{\mathcal{U}}U)$ belongs to $\mathcal{F}$. Since $[H^{\mathcal{U}}]U/H^{\mathcal{U}}$ is supersoluble, we have that $[H^{\mathcal{U}}]U/(\langle K^X \rangle \cap H^{\mathcal{U}}U \cap H^{\mathcal{U}}) = [H^{\mathcal{U}}]U \in \mathcal{F}$. This implies that $C \in \mathcal{F}$ and then $H \in \mathcal{F}$. ∎

Lemma 6. *Let $\mathcal{F}$ be a formation containing $\mathcal{U}$ such that either $\mathcal{F}$ is saturated or $\mathcal{F} \subseteq \mathcal{S}$, the class of all soluble groups. Assume that $G = HK$ is the product of the totally permutable subgroups H and K. If $G \in \mathcal{F}$, then $H \in \mathcal{F}$ and $K \in \mathcal{F}$.*

Proof: For a saturated formation the result was obtained in [1]. Assume now that $\mathcal{F}$ is a formation of soluble groups such that $\mathcal{U} \subseteq \mathcal{F}$. We use induction on $|G| + |H| + |K|$. If K is supersoluble, then $K \in \mathcal{F}$ and the result follows from Lemma 5. Suppose that $K^{\mathcal{U}} \neq 1$. Since $\mathcal{U}$ is a saturated formation, we have that $K^{\mathcal{U}}$ is not contained in $\Phi(K)$, the Frattini subgroup of K. Since $G \in \mathcal{F}$, we have that G is soluble. In particular, K is soluble and then $F/K^{\mathcal{U}} \cap \Phi(K)$, the Fitting subgroup of $K^{\mathcal{U}}/K^{\mathcal{U}} \cap \Phi(K)$ is different from 1. Now $K^{\mathcal{U}} \cap \Phi(K) = F \cap \Phi(K)$ and $F/F \cap \Phi(K)$ is nilpotent. Since F is a subnormal subgroup of K, we have that F is nilpotent by

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[**2**, Theorem 3.7]. Notice that F is indeed subnormal in G because $K^{\mathcal{U}}$ is normal in G and F is normal in $K^{\mathcal{U}}$. Therefore $F \leqslant F(G) \cap K^{\mathcal{U}} = R$ and so R is a normal subgroup of G which is not contained in $\Phi(K)$. Let M be a maximal subgroup of K such that R is not contained in M. Then $K = RM$ and $G = R(MH)$. This implies that MH is a subgroup of G supplementing a nilpotent normal subgroup of G. By [**3**, IV,1.14], we have that $MH \in \mathcal{F}$. Now $J = MH$ is a product of the totally permutable subgroups M and H and $|J| + |M| + |H| < |G| + |H| + |K|$. By induction, $M \in \mathcal{F}$ and $H \in \mathcal{F}$. Therefore we have the following situation: $G = HK$ is the product of the totally permutable subgroups H and K, $G \in \mathcal{F}$ and $H \in \mathcal{F}$. By Lemma 5, $K \in \mathcal{F}$. This completes the proof of the Lemma. ☐

3. Proofs of the Theorems

Proof of Theorem A:

We argue by induction on $|G|$. First of all notice that

$$G/G^{\mathcal{F}} = (HG^{\mathcal{F}}/G^{\mathcal{F}})(KG^{\mathcal{F}}/G^{\mathcal{F}})$$

is the product of the totally permutable subgroups $HG^{\mathcal{F}}/G^{\mathcal{F}}$ and $KG^{\mathcal{F}}/G^{\mathcal{F}}$. Since $G/G^{\mathcal{F}}$ belongs to $\mathcal{F}$, it follows that $HG^{\mathcal{F}}/G^{\mathcal{F}} \in \mathcal{F}$ and $KG^{\mathcal{F}}/G^{\mathcal{F}} \in \mathcal{F}$ by Lemma 6. In particular, $H^{\mathcal{F}} \leqslant G^{\mathcal{F}}$ and $K^{\mathcal{F}} \leqslant G^{\mathcal{F}}$. On the other hand, by Lemma 3 we know that $H^{\mathcal{F}}$ and $K^{\mathcal{F}}$ are normal subgroups of G. Hence $H^{\mathcal{F}}K^{\mathcal{F}}$ is a normal subgroup of G contained in $G^{\mathcal{F}}$. Suppose that $H^{\mathcal{F}}K^{\mathcal{F}} = 1$. Then $H \in \mathcal{F}$ and $K \in \mathcal{F}$. By applying Lemma 4, we have that $G \in \mathcal{F}$ and we are done. So we may suppose that $H^{\mathcal{F}}K^{\mathcal{F}} \neq 1$. Let N be a minimal normal subgroup of G contained in $H^{\mathcal{F}}K^{\mathcal{F}}$. Since G/N is the product of the totally permutable subgroups HN/N and KN/N, we apply induction to conclude that $G^{\mathcal{F}}/N = (H^{\mathcal{F}}N/N)(K^{\mathcal{F}}N/N)$. Consequently $G^{\mathcal{F}} = (H^{\mathcal{F}}K^{\mathcal{F}})N = H^{\mathcal{F}}K^{\mathcal{F}}$ and the theorem is proved. ☐

Proof of Theorem B:

Assume that the theorem is false and let $G = HK$ be a counterexample of minimal order. Then $H \neq 1$ and $K \neq 1$. Applying [4], H or K contains a minimal normal subgroup of G. Suppose that H contains such a minimal normal subgroup, N say. By minimality of G, we have that $(AN/N)(BN/N) = (AB)N/N$ is an $\mathcal{F}$-projector of G/N. Let $C = (AB)N$, which is a subgroup of G. Suppose that C is a proper subgroup of G. Since C is the product of the totally permutable subgroups AN and B and A is an $\mathcal{F}$-projector of AN by [**3**, III, 3.14 and 3.18], it follows that AB is an $\mathcal{F}$-projector of C by minimality of G. Now we apply [**3**, III, 3.7] to conclude that AB is an $\mathcal{F}$-projector of G, a contradiction. Consequently $G = (AB)N$. By Lemma 4, $AB \in \mathcal{F}$. This implies that $G^{\mathcal{F}} \leqslant N$ and so either $G^{\mathcal{F}} = 1$ or $G^{\mathcal{F}} = N$. If $G^{\mathcal{F}} = 1$, we have that $A = H$ and $B = K$ by Lemma 6, a contradiction. Therefore $G^{\mathcal{F}} = N$.

Let U be an $\mathcal{F}$-maximal subgroup of G containing AB. Since $G = N(AB)$, it follows that U is an $\mathcal{F}$-projector of G by [**3**, III, 3.14 and 3.18]. Moreover $U = (A(N \cap U))B$ is the product of the totally permutable subgroups $A(N \cap U)$ and B. From $U \in \mathcal{F}$ and $B \in \mathcal{F}$, we have that $A(N \cap U) \in \mathcal{F}$ by Lemma 5. Since A is $\mathcal{F}$-maximal in H, we have that $N \cap U \leqslant A$ and $U = AB$ is an $\mathcal{F}$-projector of G, a contradiction. □

Proof of Theorem C:

We argue by induction on $|G| + |H| + |K|$. If $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $G \in \mathcal{F}$ by Lemma 4 and we are done. So we may assume that H does not belong to $\mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, it follows that $H^{\mathcal{F}}$ is not contained in $\Phi(H)$. So $1 \neq T/H^{\mathcal{F}} \cap \Phi(H) = F\big(H^{\mathcal{F}}/H^{\mathcal{F}} \cap \Phi(H)\big)$ because H is soluble. Now $H^{\mathcal{F}} \cap \Phi(H) = T \cap \Phi(H)$. By [**2**, Theorem 3.7], we have that T is a nilpotent subnormal subgroup of G. Let M be a maximal subgroup of H such that T is not contained in M. Then $H = TM = H^{\mathcal{F}}M = F(H)M$. Therefore M is an $\mathcal{F}$-critical maximal subgroup of H. By [**3**, V,3.7], every $\mathcal{F}$-normaliser of M is an $\mathcal{F}$-normaliser of G and by [**3**, V, 3.2], the $\mathcal{F}$-normalisers of H are conjugate. So we may assume that $A \leqslant M$. Moreover $G = T(MK) = F(G)(MK) = G^{\mathcal{F}}(MK)$ because $H^{\mathcal{F}} \leqslant G^{\mathcal{F}}$ by Lemma 6. Suppose that $G = MK$. Then $|G| + |M| + |K| < |G| + |H| + |K|$ and G is the product of the totally permutable subgroups M and K. By induction AB is an $\mathcal{F}$-normaliser of G and the theorem is proved. Therefore we may assume that MK is a proper subgroup of G. This means that MK is an $\mathcal{F}$-critical maximal subgroup of G. Again by [**3**, V,3.7] every $\mathcal{F}$-normaliser of MK is an $\mathcal{F}$-normaliser of G. Since by induction AB is an $\mathcal{F}$-normaliser of the totally permutable product MK, it follows that AB is an $\mathcal{F}$-normaliser of G. Therefore the theorem is proved. □

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