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Genuine representations of the metaplectic group

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Abstract. This paper determines the θ-correspondence for the dual pairs $(\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}))$ when $p+q=2n+1$. As a consequence, there is a natural bijection between genuine irreducible representations of the metaplectic group $\mathrm{Mp}(2n,\mathbb{R})$ and irreducible representations of $\mathrm{SO}(p,q)$ with $p+q=2n+1$.

0. Introduction

Consider the dual pairs $(\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}))$ with $p+q=2n+1$. Let $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ be the metaplectic group, and $\widetilde{\mathrm{O}}(p,q)$ the $\det^{1/2}$ cover of $\mathrm{O}(p,q)$ (we will be more precise in Section 1). For ψ a nontrivial additive character of $\mathbb{R}$, the oscillator representation $\omega(\psi)$ yields a bijection $\theta(\psi,p,q)$ between subsets of the irreducible representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ and those of $\widetilde{\mathrm{O}}(p,q)$ [5]. The representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ which arise are all genuine, i.e. do not factor to the linear group $\mathrm{Sp}(2n,\mathbb{R})$. The main result of this paper is an explicit description of this correspondence (Theorem 5.1).

Fix the discriminant $\delta=(-1)^q$ of the orthogonal space. An immediate corollary of Theorem 5.1 is a bijection, depending on ψ, between the set

$$\widetilde{\mathrm{Sp}}(2n,\mathbb{R})\widehat{\ }_{\text{genuine}}$$

of (equivalence classes of) genuine irreducible admissible representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ and the union

$$\bigcup_{\substack{p+q=2n+1\\ (-1)^q=\delta}} \mathrm{SO}(p,q)\widehat{\ }$$

of the irreducible admissible representations of the groups $\mathrm{SO}(p,q)$ (cf. Corollary 6.2 for details). This result confirms, in the real case, part of a conjecture of Kudla [8], which in turn is a generalization of a result of Waldspurger [22] in the case $n=1$.

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The bijection is one of similarity, rather than of duality, in that it takes small representations to small representations. For example it takes the trivial representations of $\mathrm{SO}(n+1,n)$ and $\mathrm{SO}(n,n+1)$ to the even halves of the oscillator representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$.

The metaplectic group is an example of a non-linear group, to which the machinery of the L-group does not apply. On the other hand it is of great importance in the theory of automorphic representations, so it is of interest to understand it in these terms. With this in mind notions such as L-packet, stability, etc. may be defined for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ by carrying over the corresponding notions from $\mathrm{SO}(p,q)$. Even in the easiest examples it is clear that care must be taken in making such extensions. For example the representations in an L-packet defined in this manner may fail to have the same central character, a phenomenon which is forbidden for linear groups (and also for the larger L-packets and Arthur-packets of [4]).

This bijection is natural in terms of the Langlands classification. The Cartan subgroups of $\mathrm{O}(p,q)$ are isomorphic to those of $\mathrm{Sp}(2n,\mathbb{R})$, and very roughly speaking the matching is given by the same characters. For example discrete series representations having the 'same' Harish–Chandra parameter correspond. This naturality is expressed in the commutative diagram of Proposition 6.1. The correspondence of K-types on the space of joint harmonics also has nice properties; each K-type for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is harmonic for precisely one choice of p,q with given discriminant. Furthermore lowest K-types in the sense of Vogan are always of lowest degree in the sense of Howe [5].

These properties are special to the range in which the two groups are roughly the same size. Similar properties also hold for the dual pairs $(\mathrm{O}(p,q),\mathrm{Sp}(2n,\mathbb{R}))$ with p,q *even* and $p+q=2n,2n+2$ [11]. In fact our approach is quite close to that of [11], with the additional complications arising from the presence of nontrivial covering groups.

1. Preliminaries

In this section we describe facts about the double covers, dual pairs and generalities about the metaplectic representation that we will need. The main reference for the double covers and the metaplectic representation are [10] and [15]. The setup is for any local field $\mathbb{F}$, but we concentrate on the case $\mathbb{F}=\mathbb{R}$. We omit the details of many straightforward calculations.

For any positive integer m we equip $W=\mathbb{R}^{2m}$ with the usual symplectic structure given by $J=\left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0\end{smallmatrix}\right)$, and standard basis $e_1,\dots,e_m,f_1,\dots,f_m$. Then $\mathrm{Sp}(2m,\mathbb{R})$ is the isometry group of this form, and the metaplectic cover $\widetilde{\mathrm{Sp}}(2m)$ is defined by the normalized cocycle $c(\ ,\ )$ of [15] or [10]. Thus

$$\widetilde{\mathrm{Sp}}(2m,\mathbb{R})=\mathrm{Sp}(2m,\mathbb{R})\times\mathbb{Z}/2\mathbb{Z},\qquad (g,\varepsilon)(g',\varepsilon')=(gg',\varepsilon\varepsilon' c(g,g')).$$

If ψ is a nontrivial (unitary) additive character of $\mathbb{R}$, let $\omega(\psi)$ be the Harish–Chandra module of the oscillator representation of $\widetilde{\mathrm{Sp}}(2m,\mathbb{R})$, ([15], Section 4), ([10], Part I). The character ψ may be written $\psi_a(x) = \mathrm{e}^{iax}$; up to isomorphism $\omega(\psi_a)$ only depends on the image of a in $\mathbb{R}^*/\mathbb{R}^{*^2}$.

Let V be a real $2n+1$-dimensional vector space equipped with a non-degenerate symmetric bilinear form $(\ ,\ )$ of signature (p,q), and basis $v_1,\ldots,v_p,v'_1,\ldots,v'_q$ for which the matrix of $(\ ,\ )$ is $\mathrm{diag}(I_p,\Leftrightarrow I_q)$. We let $\mathrm{O}(p,q)$ denote the isometry group of $(\ ,\ )$. Now $V\otimes W$ has a symplectic structure with standard basis $v_1\otimes e_1,\ldots,v'_q\otimes f_n$. The natural map $\alpha\colon \mathrm{O}(p,q)\times \mathrm{Sp}(2m,\mathbb{R})\to \mathrm{Sp}(2m(2n+1),\mathbb{R})$ makes $(\mathrm{O}(p,q),\mathrm{Sp}(2n,\mathbb{R}))$ into a dual pair. We write $p_0=[p/2]$ and $q_0=[q/2]$ for the ranks of $\mathrm{O}(p)$ and $\mathrm{O}(q)$.

The main result concerns the case $m=n$, but many secondary results hold with little or no restriction.

Let $\widetilde{\mathrm{GL}}(m)$ be the two-fold cover of $\mathrm{GL}(m)$ defined by the cocyle

$$c(g,h) = (\det(g),\det(h))_{\mathbb{R}}$$

where $(x,y)_{\mathbb{R}}$ is the Hilbert symbol [16] for $\mathbb{R}$. It is convenient to let $\widetilde{\mathrm{O}}(p,q)$ be the two-fold cover of $\mathrm{O}(p,q)$ defined by $c(g,h)=(\det(g),\det(h))^n_{\mathbb{R}}$ $(p+q=2n+1)$. This is split over $\mathrm{SO}(p,q)$ and splits over $\mathrm{O}(p,q)$ if and only if n is even. For later use we let $\widetilde{\mathrm{O}}(p,q)[k]$ be the cover defined by cocycle $c(g,h)^k$.

Now α lifts to a map $\tilde{\alpha}\colon \widetilde{\mathrm{O}}(p,q)\times\widetilde{\mathrm{Sp}}(2n,\mathbb{R})\to\widetilde{\mathrm{Sp}}(2n(2n+1),\mathbb{R})$. In particular for $g\in\mathrm{O}(p,q)$,

$$\tilde{\alpha}(g,\varepsilon) = (\alpha(g),\varepsilon) \tag{1.1a}$$

and for $g\in\mathrm{Sp}(2n,\mathbb{R})$

$$\tilde{\alpha}(g,\varepsilon) = (\alpha(g),\varepsilon^{2n+1}\Delta(g)) \tag{1.1b}$$

for a certain map $\Delta\colon \mathrm{Sp}(2n,\mathbb{R})\to\pm1$.

The image of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ in $\widetilde{\mathrm{Sp}}(2n(2n+1),\mathbb{R})$ is the inverse image of $\mathrm{Sp}(2n,\mathbb{R})$, and the image of the center of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is the center of $\widetilde{\mathrm{Sp}}(2n(2n+1),\mathbb{R})$. Our choice of covering $\widetilde{\mathrm{O}}(p,q)$ implies that the analogous statements hold for $\mathrm{O}(p,q)$. It also has the advantage that n odd and n even may be treated uniformly.

Given ψ, let

$$\chi(\psi)(x,\varepsilon) := \gamma(x,\tfrac{1}{2}\psi)\varepsilon, \tag{1.2a}$$

where

$$\gamma(a,\psi) := \frac{\gamma(a\psi)}{\gamma(\psi)} \tag{1.2b}$$

is the Weil index [15]. We have $\gamma(\psi_a)=\mathrm{e}^{(2i\pi/8)\operatorname{sgn} a}$ and

$$\chi(\psi_a)(x,\varepsilon) = \mathrm{sgn}(a)\,\mathrm{e}^{(2i\pi/8)(\operatorname{sgn} x-1)}\varepsilon. \tag{1.2c}$$

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This is a character of the $(\ ,\ )_{\mathbb{R}}$ double cover of $\mathbb{R}^*$, and so we can compose $\chi(\psi)$ with the determinant to get a character of $\widetilde{\mathrm{GL}}(m)$. We denote the ensuing genuine character by the same symbol, $\chi(\psi)(g,\varepsilon) := \chi(\psi)(\det(g),\varepsilon)$.

This satisfies

$$\overline{\chi(\psi)}(x,\varepsilon) = \mathrm{sgn}\,(x)\chi(\psi)(x,\varepsilon), \qquad \chi(\psi)(x,\varepsilon)^2 = \mathrm{sgn}(x). \tag{1.2d}$$

If V is an orthogonal space of dimension m and discriminant δ, we let (cf. [7], 2.5)

$$\chi(\psi,V)(g,\varepsilon) = \gamma(\det(g),\tfrac{1}{2}\psi)^{-m}(\delta,\det(g))_{\mathbb{R}}\varepsilon^m. \tag{1.2e}$$

In general, if $\widetilde{G}$ is a double cover of G and π is a representation of $\widetilde{G}$, we say π is of type k if $\pi(\varepsilon) = \varepsilon^k$ for ε in the kernel of the covering. With this convention $\chi(\psi,V)(g,\varepsilon)$ is a character of $\widetilde{\mathrm{GL}}(V)$ of type $m = \dim V$.

If the signature of V is p,q with $p+q$ odd, then

$$\chi(\psi,V) = \chi(\psi)^{-p+q}. \tag{1.2f}$$

This is the formula we will use most of the time. By (d) this may be thought of as $\mathrm{sgn}(\det)^{(-p+q/2)}$.

We fix a genuine character

$$\xi(\psi)(g,\varepsilon) = \begin{cases} \varepsilon & n \text{ even} \\ \chi(\psi)(\det(g),\varepsilon)^{-1} & n \text{ odd} \end{cases} \tag{1.2g}$$

of $\widetilde{\mathrm{O}}(p,q)$. The map $\pi \to \pi \otimes \xi$ defines a bijection between the irreducible representations of $\mathrm{O}(p,q)$ and the irreducible genuine representations of $\widetilde{\mathrm{O}}(p,q)$.

If ψ is fixed we drop it from the notation and write $\omega = \omega(\psi)$, $\chi = \chi(\psi)$, $\chi_V = \chi(\psi,V)$ and $\xi = \xi(\psi)$.

Pulling $\omega(\psi)$ back to $\widetilde{\mathrm{O}}(p,q) \times \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ via $\tilde{\alpha}$ we obtain the representation correspondence for this dual pair [5]. This is a correspondence between certain irreducible Harish–Chandra modules.

By (1.1) the representations of $\widetilde{\mathrm{O}}(p,q)$ and $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ in the image of the correspondence are genuine, i.e. of type 1. If π,π' are genuine irreducible representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ and $\widetilde{\mathrm{O}}(p,q)$ respectively which correspond, we write

$$\pi' = \theta(\psi,p,q)(\pi), \qquad \pi = \theta(\psi)(\pi'). \tag{1.3}$$

If ψ is fixed we write $\theta_{p,q} = \theta(\psi,p,q)$ and $\theta = \theta(\psi)$.

For an irreducible (admissible) representation π of a group G, we denote by π^* its contragredient. In particular, for ξ as in (1.2g), we have

$$\xi^*(g,\varepsilon) = \xi \otimes \mathrm{sgn}(\det g)^n. \tag{1.4a}$$

Combined with the results on p. 91 of [12], we find the following expressions for π^*. In case $G = \widetilde{\mathrm{Sp}}$, let $\tau = \mathrm{Ad}\,\kappa$, where $\kappa = \mathrm{diag}(I_m, \Leftrightarrow I_m)$ (this is an outer automorphism of G). Then

$$\pi^* = \begin{cases} \pi \otimes \mathrm{sgn}^n & \text{for } G = \widetilde{\mathrm{O}}(V), \\ \pi \circ \tau, & \text{for } G = \widetilde{\mathrm{Sp}}(W). \end{cases} \tag{1.4b}$$

From (1.2d) and the fact that $\omega(\overline{\psi}) \simeq \omega(\psi)^*$ we see

$$\theta(\overline{\psi}, p, q)(\pi) = \theta(\psi, p, q)(\pi^*) \otimes \mathrm{sgn}^n. \tag{1.4c}$$

Let V' denote the same space as V with form $Q' = \Leftrightarrow Q$, of signature (q, p). Let δ be the tautological identification of $\mathrm{O}(V)$ and $\mathrm{O}(V')$. Note that $\mathrm{O}(V') \cong \mathrm{O}(q, p)$, and so we can identify representations of $\mathrm{O}(p, q)$ and $\mathrm{O}(q, p)$ by choosing such an isomorphism; however the θ-correspondences are different. The next lemma gives the relationship between $\theta_{p,q}$ and $\theta_{q,p}$.

LEMMA 1.5. *For any irreducible representation* π *of* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$,

$$\theta(\psi, p, q)(\pi) = \theta(\psi, q, p)(\pi^*).$$

Proof. The map $\mathrm{Id} \otimes \kappa$ is an isomorphism between $V \otimes W$ and $V' \otimes W$ which interchanges $Q \cdot \langle\, ,\rangle$ and $\Leftrightarrow Q \cdot \langle\, ,\rangle$. Let Ψ be the ensuing isomorphism $\mathrm{Sp}(V \otimes W) \to \mathrm{Sp}(V' \otimes W)$. The diagram

$$\begin{tikzcd} \mathrm{O}(V) \times \mathrm{Sp}(W) \arrow[r, "\alpha"] \arrow[d, "\delta\otimes\tau"'] & \mathrm{Sp}(V \otimes W) \arrow[d, "\Psi"] \\ \mathrm{O}(V') \otimes \mathrm{Sp}(W) \arrow[r, "\alpha'"] & \mathrm{Sp}(V' \otimes W) \end{tikzcd}$$

is commutative and $\omega(\psi, V \otimes W) = \omega(\psi, V' \otimes W) \circ \Psi$. The lemma follows from the formula for π^*. $\square$

Note that (1.4c) and the Lemma give

$$\theta(\overline{\psi}, p, q)(\pi) = \theta(\psi, q, p)(\pi) \otimes \mathrm{sgn}^n. \tag{1.6}$$

Suppose W_1 and W_2 are symplectic spaces. Then $W_1 \oplus W_2$ inherits a natural symplectic structure and there is a canonical map $\widetilde{\mathrm{Sp}}(W_1) \otimes \widetilde{\mathrm{Sp}}(W_2) \to \widetilde{\mathrm{Sp}}(W_1 \oplus W_2)$. We will use this map in the special case $\widetilde{\mathrm{Sp}}(2m, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(2m, \mathbb{R}) \to \widetilde{\mathrm{Sp}}(4m, \mathbb{R})$. Similarly there is a canonical map $\widetilde{\mathrm{O}}(p, q) \times \widetilde{\mathrm{O}}(p', q') \to \widetilde{\mathrm{O}}(p+p', q+q')$.

LEMMA 1.7. *Let* $\omega_{n,p,q}$ *be the oscillator representation of* $\widetilde{\mathrm{Sp}}(2n(p+q), \mathbb{R})$ *restricted to the dual pair* $(\widetilde{\mathrm{Sp}}(2n), \widetilde{\mathrm{O}}(p, q))$.

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(1) $\omega_{n,p,q} \otimes \omega_{n,p,q} \simeq \omega_{2n,p,q} \otimes \mathrm{sgn}^n$ *as representations of* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \times \widetilde{\mathrm{O}}(p, q)$ *with* $\widetilde{\mathrm{O}}(p, q)$ *acting diagonally on the left-hand side.*

(2) $\omega_{n,p,q} \otimes \omega_{n,p',q'} \simeq \omega_{n,p+p',q+q'}$ *as representations of* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \times \widetilde{\mathrm{O}}(p, q) \times \widetilde{\mathrm{O}}(p', q')$, *with* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ *acting diagonally on the left-hand side.*

Proof. There are obvious isomorphisms between the polynomial Fock spaces $\omega_{n,p,q}$ in the statements. We need to check the equivariance. Assertion (1) for $\widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ and assertion (2) for $\widetilde{\mathrm{O}}(p, q) \times \widetilde{\mathrm{O}}(q, p)$ follow from the explicit descriptions of the actions in [15] on the smooth models.

If (X, Y) is a complete polarization of W then $\widetilde{\mathrm{GL}}(X)$ acts on $S(X)$ in the oscillator representation by

$$\omega(\psi)(g, \varepsilon)(\phi)(x) = |\det(g)|^{-(1/2)} \chi(\psi)(g, \varepsilon)^{-1} \phi(g^{-1}x).$$

Since the action of $\widetilde{\mathrm{O}}(p, q)$ is via a homomorphism to $\tilde{\mathrm{GL}}(X \otimes V)$, it acts in $\omega_{n,p,q}$ by translation tensored with ξ. Assertion (1) for $\widetilde{\mathrm{O}}(p, q)$ follows immediately, the twist if n is odd coming from $\chi(\psi)^2 = \mathrm{sgn}$. The proof of assertion (2) is similar. $\square$

The first part of the next Lemma is due to Rallis [14] and Przebinda [13]. The second is the result obtained by applying the same technique in the other direction. It says that the duality correspondence is a bijection when all $\mathrm{O}(p, q)$ with fixed discriminant (and $p + q = 2n + 1$) are considered at once. Thus we are reduced to proving occurence, and computing the correspondence explicitly.

LEMMA 1.8.

(1) *Suppose* π *is a representation of* $\widetilde{\mathrm{O}}(p, q)$, *and* $\theta(\psi)(\pi) \neq 0$. *Then* $\theta(\psi)(\pi \otimes \mathrm{sgn}) = 0$.

(2) *Let* π *be a genuine representation of* $\widetilde{\mathrm{Sp}}(2m, \mathbb{R})$, *and suppose* $\theta(\psi, p, q)(\pi) \neq 0$. *Then* $\theta(\psi)_{p',q'}(\pi) = 0$ *for all* $(p', q') \neq (p, q)$ *with* $q' \equiv q \bmod(2)$.

Proof. Suppose both π and $\pi \otimes \mathrm{sgn}$ are quotients of $\omega_{n,p,q}$ restricted to $\widetilde{\mathrm{O}}(p, q)$. By Lemma 1.7(1) this implies $\pi \otimes \pi \otimes \mathrm{sgn}$ is a quotient of $\omega(\psi)_{2n,p,q} \otimes \mathrm{sgn}^{\mathrm{n}}$. Since $\pi^* \simeq \pi \otimes \mathrm{sgn}^n$, and the trivial representation is a quotient of $\pi \otimes \pi^*$, this implies that sgn is a quotient of $\omega_{2n,p,q}$. However this is impossible since (cf. Proposition 2.1) the sgn K-type of $\mathrm{O}(p, q)$ does not occur in the space of joint harmonics when paired with $\mathrm{Sp}(4n, \mathbb{R})$. This proves (1).

Now suppose π is a quotient of both $\omega_{n,p,q}$ and $\omega_{n,p',q'}$ restricted to $\widetilde{\mathrm{Sp}}(2m, \mathbb{R})$. By Lemmas 1.5(1) and 1.7(2) this implies that $\pi \otimes \pi^*$ is a quotient of $\omega_{n,p+q',q+p'}$. As in the proof of (1) this implies that the trivial representation is a quotient of $\omega_{n,p+q',q+p'}$. This can also be ruled out by K-types. By [6], cf. ([11], I.4), the trivial K-type for $\widetilde{\mathrm{Sp}}(2m, \mathbb{R})$ occurs in this space only if $\mathrm{O}(p+q', q+p')$ is quasi-split, i.e. $p + q' \Leftrightarrow q \Leftrightarrow p' = 0, \pm 1, \pm 2$. This together with $p + q = q + p$ gives $q = q' \pm 0, 1$, and since the discriminants are equal $q = q'$. Therefore $p = p', q = q'$, proving (2). $\square$

Groups will be denoted $G, K, T, \ldots$, their Lie algebras by $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{t}_0, \ldots$, and their complexified Lie algebras by $\mathfrak{g}, \mathfrak{k}, \mathfrak{t} \ldots$. For G reductive, a Cartan involution will be denoted θ with fixed points $K, \mathfrak{k}_0$ and $\mathfrak{k}$ respectively in $G, \mathfrak{g}_0$ and $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as usual. For $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ (always θ-stable) we will denote a system of roots by $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ with positive system Δ^+, and $\rho = \rho(\Delta^+) = \frac{1}{2}\sum_\alpha \alpha$. This notation will be extended in various standard ways, for example ρ_c denotes one-half the sum of the positive compact imaginary roots, and $\rho(\mathfrak{u})$ one-half the sum of the roots of a (nilpotent) subalgebra $\mathfrak{u}$. For $\widetilde{G}$ a covering group of a group G and H a subgroup of G, $\widetilde{H}$ will generally denote the inverse image of H in $\widetilde{G}$. This notation will occasionally conflict with the definition of $\widetilde{\mathrm{O}}(p,q)$ earlier; the meaning should be clear from the context. Unless otherwise stated G will denote $\mathrm{Sp}(2n, \mathbb{R})$, $\widetilde{G}$ will denote $\widetilde{\mathrm{Sp}}(2m, \mathbb{R})$. These groups have maximal compact subgroups K and $\widetilde{K}$ as chosen in Section 2. Similarly, G' will denote $\mathrm{O}(p,q)$ with corresponding $\widetilde{G}', K'$ and $\widetilde{K}'$.

We now describe the semisimple orbits and Cartan subgroups for $\mathrm{Sp}(2n, \mathbb{R})$. We begin by choosing representatives for the conjugacy classes of Cartan subgroups as in [2].

For nonnegative integers m, r, s with $2m + r + s = n$ we define a Cartan subgroup $H_{\mathrm{Sp}}^{m,r,s}$ of $\mathrm{Sp}(2n, \mathbb{R})$ with Lie algebra $\mathfrak{h}_{\mathrm{Sp}0}^{m,r,s}$. Write $W = \mathbb{R}^{2n} = W_1 \oplus W_2 \oplus W_3$, where W_1 is spanned by $\{e_i, f_i \mid 1 \leqslant i \leqslant 2m\}$, W_2 by $\{e_i, f_i \mid 2m+1 \leqslant i < 2m + r\}$ and W_3 by $\{e_i, f_j \mid 2m + r + 1 \leqslant i \leqslant n\}$. We identify $\mathrm{Sp}(W_i)$ and $\mathfrak{sp}(W_i)$ with their images in $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathfrak{sp}(2n, \mathbb{R})$. For $z_i = x_i + iy_i \in \mathbb{C}$, $1 \leqslant i \leqslant m$ let

$$\mathfrak{h}_{\mathrm{Sp}}^{m,0,0}(z_1, \ldots, z_m) = \begin{pmatrix} & X & Y & \\ X & & & -Y \\ -Y & & & -X \\ & Y & -X & \end{pmatrix} \in \mathfrak{sp}(W_1) \tag{1.9a}$$

where $X = \mathrm{diag}(x_1, \ldots, x_m)$ and $Y = \mathrm{diag}(y_1, \ldots, y_m)$. For $\theta_i \in \mathbb{R}$ $(1 \leqslant i \leqslant r)$ we let

$$\mathfrak{h}_{\mathrm{Sp}}^{0,r,0}(\theta_1, \ldots, \theta_r) = \begin{pmatrix} & X \\ -X & \end{pmatrix} \in \mathfrak{sp}(W_2) \tag{1.9b}$$

with $X = \mathrm{diag}(\theta_1, \ldots, \theta_r)$, and for $x_i \in \mathbb{R}$ $(1 \leqslant i \leqslant s)$ let

$$\mathfrak{h}_{\mathrm{Sp}}^{0,0,s}(x_1, \ldots, x_s) = \mathrm{diag}(x_1, \ldots, x_s, -x_1, \ldots, -x_s) \in \mathfrak{sp}(W_3). \tag{1.9c}$$

Taking the sum of these elements gives us an element

$$\mathfrak{h}_{\mathrm{Sp}}^{m,r,s}(z_1, \ldots, z_m, \theta_1, \ldots, \theta_r, x_1, \ldots, x_s) \in \mathfrak{sp}(2n, \mathbb{R}) \tag{1.9d}$$

and this defines the Cartan subalgebra $\mathfrak{h}_{\mathrm{Sp}0}^{m,r,s}$ of $\mathfrak{sp}(2n,\mathbb{R})$, with complexification $\mathfrak{h}_{\mathrm{Sp}}^{m,r,s}$. The compact Cartan subalgebra is $\mathfrak{t}_0=\mathfrak{h}_0^{0,n,0}$. Let

$$H_{\mathrm{Sp}}^{m,r,s}\simeq(\mathbb{C}^*)^m\times(S^1)^r\times(\mathbb{R}^*)^s \tag{1.9e}$$

be the Cartan subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ with Lie algebra $\mathfrak{h}_{\mathrm{Sp}0}^{m,r,s}$. These are representatives for the conjugacy classes of Cartan subgroups of $\mathrm{Sp}(2n,\mathbb{R})$. The compact Cartan subgroup is $T=H_{\mathrm{Sp}}^{0,n,0}$. Write the elements of $H_{\mathrm{Sp}}^{m,r,s}$ accordingly as

$$H_{\mathrm{Sp}}^{m,r,s}(z_1,\dots,z_m,u_1,\dots,u_r,x_1,\dots,x_2) \tag{1.9f}$$

$(z_i\in\mathbb{C}^*,u_i\in S^1,x_i\in\mathbb{R}^*)$.

The Weyl group of $\mathfrak{h}_{\mathrm{Sp}0}^{m,r,s}$ in $\mathfrak{sp}(2n,\mathbb{R})$ is generated by all permutations of $\{z_i\}$, $z_i\to\overline{z}_i,\Leftrightarrow z_i$, all permutations of $\{\theta_i\}$, and all permutations and sign changes of $\{x_i\}$. This describes the semisimple orbits. Note that two semisimple elements $\mathfrak{h}_{\mathrm{Sp}}^{m,r,s}(z_1,\dots,)$ and $\mathfrak{h}_{\mathrm{Sp}}^{m,r,s}(z_1',\dots,)$ are in the same orbit if and only if they have the same eigenvalues, and $\theta_1,\dots,\theta_r$ and $\theta_1',\dots,\theta_r'$ are the same up to permutation.

We write $H=TA$ with $\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{a}$, $T=H\cap K$ and $A=\exp(\mathfrak{a}_0)$ as usual. The centralizer of A is

$$M\cong\mathrm{GL}(1,\mathbb{R})^s\times\mathrm{GL}(2,\mathbb{R})^m\times\mathrm{Sp}(2r,\mathbb{R}). \tag{1.10a}$$

Let

$$\overline{M}=\widetilde{\mathrm{GL}}(1,\mathbb{R})^s\times\widetilde{\mathrm{GL}}(2,\mathbb{R})^m\times\widetilde{\mathrm{Sp}}(2r,\mathbb{R}) \tag{1.10b}$$

with double covers of GL and Sp as at the beginning of this section. There is a natural surjection $\overline{M}\twoheadrightarrow\widetilde{M}$.

Let $\widetilde{H}$ (respectively $\overline{H}$) be the inverse image of H in $\widetilde{M}$ (resp. $\overline{M}$). Then $\widetilde{H},\overline{H}$ are Cartan subgroups of $\widetilde{M},\overline{M}$. Furthermore

$$\overline{H}\simeq(\widetilde{\mathbb{R}^*})^s\times(\widetilde{S^1})^r\times(\mathbb{C}^*\times\mathbb{Z}/2\mathbb{Z})^m, \tag{1.10c}$$

where $\widetilde{\mathbb{R}^*}$ is the two-fold cover of $\mathbb{R}^*$ defined by the Hilbert symbol, and $\widetilde{S^1}$ is the connected two-fold cover of S^1 given by $z\to z^2$, $|z|=1$.

We now turn to a description of the Cartan subgroups and semisimple orbits for $\mathrm{O}(p,q)$. We follow [1].

Suppose $2m+s\leqslant\min(p,q)$. Write $V=V_1\oplus V_2\oplus V_3$ where $V_1=\mathrm{span}\{v_i,v_j'\mid 1\leqslant i,j\leqslant 2m\}$, $V_2=\mathrm{span}\{v_i,v_j'\mid 2m+1\leqslant i,j\leqslant 2m+s\}$ and $V_3=\mathrm{span}\{v_i,v_j'\mid 2m+s<i\leqslant p, 2m+s<j\leqslant q\}$. Then $\mathrm{SO}(V_i)$ is embedded naturally in $\mathrm{SO}(V)$

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and we identify $\mathrm{SO}(V_i)$ and $\mathfrak{so}(V_i)$ with their images in $\mathrm{SO}(V)$ and $\mathfrak{so}(V)$. For $w_j = x_j + iy_j \in \mathbb{C}$ let

$$\mathfrak{h}^{m,0,0}(w_1, \ldots, w_m) = \begin{pmatrix} & Y & & X \\ \Leftrightarrow Y & & X & \\ & X & & \Leftrightarrow Y \\ X & & Y & \end{pmatrix} \in \mathfrak{so}(V_1), \tag{1.11a}$$

where $X = \mathrm{diag}(x_1, \ldots, x_m), Y = \mathrm{diag}(y_1, \ldots, y_m)$. For $c_j \in \mathbb{R}$, let

$$\mathfrak{h}^{0,0,s}(c_1, \ldots, c_s) = \begin{pmatrix} & X \\ X & \end{pmatrix} \in \mathfrak{so}(V_2), \tag{1.11b}$$

where $X = \mathrm{diag}(c_1, \ldots, c_s)$. Finally let $r_1 = [(p \Leftrightarrow 2m \Leftrightarrow s)/2]$, $r_2 = [(q \Leftrightarrow 2m \Leftrightarrow s)/2]$, and for $\theta_i, \phi_j \in \mathbb{R}$ let

$$\begin{aligned} &\mathfrak{h}^{0,r_1+r_2,0}(\theta_1, \ldots, \theta_{r_1}, \phi_1, \ldots, \phi_{r_2}) \\ &\quad = \mathrm{diag}(\hat{\theta}_1, \ldots, \hat{\theta}_{r_1}, \hat{\phi}_1, \ldots, \hat{\phi}_{r_2}) \in \mathfrak{so}(V_3) \end{aligned} \tag{1.11c}$$

with $\hat{\theta} = \left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)$.

Taking the sum of these elements gives us an element

$$\begin{aligned} X &= \mathfrak{h}^{m,r,s}_{p,q}(w_1, \ldots, w_m, \theta_1, \ldots, \theta_{r_1}, \phi_1, \ldots, \phi_{r_2}, c_1, \ldots, c_s) \\ &\in \mathfrak{so}(p,q) \end{aligned} \tag{1.11d}$$

and this defines a Cartan subalgebra $\mathfrak{h}^{m,r,s}_{p,q0}$, with complexification $\mathfrak{h}^{m,r,s}_{p,q}$. Let

$$H^{m,r,s}_{p,q} \simeq (\mathbb{C}^*)^m \times (S^1)^r \times (\mathbb{R}^*)^s \tag{1.11e}$$

be the Cartan subgroup of $\mathrm{SO}(p,q)$ with Lie algebra $\mathfrak{h}^{m,r,s}_{p,q0}$. This gives a set of representatives of the conjugacy classes of Cartan subgroups of $\mathrm{SO}(p,q)$. The compact Cartan subgroup T is $H^{0,n,0}_{p,q}$. According to the decomposition (1.11e), we write elements of $H^{m,r,s}_{p,q}$ as

$$H^{m,r,s}_{p,q}(z_1, \ldots, z_m, u_1, \ldots, u_{r_1}, v_1, \ldots, v_{r_2}, x_1, \ldots, x_s) \tag{1.11f}$$

with $z_i \in \mathbb{C}^*, u_i, v_i \in S^1, x_i \in \mathbb{R}^*$.

The Weyl group of $\mathfrak{h}^{m,r,s}_{p,q0}$ in $\mathfrak{o}(p,q)$ is similar to the case of Sp. The only change is that on $u_1, \ldots, u_{r_1}, v_1, \ldots, v_{r_2}$ it is of type $B_{r_1} \times B_{r_2}$ acting by permutation and sign changes on $\{u_i\}$ and $\{v_i\}$ separately.

The corresponding Cartan subgroup of $\mathrm{O}(p,q)$ is isomorphic to $H^{m,r,s}_{p,q} \times \mathcal{Z}$ where $\mathcal{Z}$ is the center of $\mathrm{O}(p,q)$.

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The centralizer of A is

$$M' \cong \mathrm{GL}(1,\mathbb{R})^s \times \mathrm{GL}(2,\mathbb{R})^m \times \mathrm{O}(p',q') \tag{1.12a}$$

with $p' = p - s - 2m, q' = q - s - 2m$. The inverse image of M' in $\widetilde{\mathrm{O}}(p,q)$ is

$$\widetilde{M}' \cong \mathrm{GL}(1,\mathbb{R})^s \times \mathrm{GL}(2,\mathbb{R})^m \times \widetilde{\mathrm{O}}(p',q')[n]. \tag{1.12b}$$

It follows from the preceding discussion that there is a bijection (depending on the additive character ψ) between the regular semisimple adjoint orbits of $\mathrm{Sp}(2n,\mathbb{R})$ and the union of the regular semisimple adjoint orbits of $\mathrm{SO}(p,q)$ with $\delta = (-1)^q$ fixed. This is explained in more detail in [1], where it is described geometrically in terms of the orbit correspondence; here we resort to a simple explicit description.

Fix $\psi = \psi_a$ with $a > 0$. Let

$$X = \mathfrak{h}^{m,r,s}_{\mathrm{Sp}}(z_1,\dots,z_m,u_1,\dots,u_{r_1},v_1,\dots,v_{r_2},x_1,\dots,x_s) \tag{1.13a}$$

be a regular semisimple element, with $u_1 > \cdots > u_{r_1} > 0 > v_1 > \cdots > v_{r_2}$. Let $p = 2m + r_1 + s, q = 2m + r_2 + s + 1$ or $p = 2m + r_1 + s + 1, q = 2m + r_2 + s$, depending on δ. Then the orbit of $X \in \mathfrak{sp}(2n,\mathbb{R})$ corresponds to the orbit of $X' \in \mathfrak{so}(p,q)$, where

$$X' = \mathfrak{h}^{m,r,s}_{p,q}(z_1,\dots,z_m,u_1,\dots,u_{r_1},-v_{r_2},\dots,-v_1,x_1,\dots,x_s). \tag{1.13b}$$

If $\psi = \psi_a$ with $a < 0$, then the same result holds, with $u_1,\dots,u_{r_1},-v_{r_2},\dots,-v_1$ replaced by $v_1,\dots,v_{r_2},-u_{r_1},\dots,-u_1$.

By the preceding description of the semisimple orbits the following result is immediate. Let $\mathfrak{sp}(2n,\mathbb{R})_{ss}$ be the regular semisimple orbits of $\mathfrak{sp}(2n,\mathbb{R})$, and $\mathfrak{so}(p,q)_{ss}$ similarly.

LEMMA 1.14. *Fix ψ. There is a bijection between*

$$\mathfrak{sp}(2n,\mathbb{R})_{ss}$$

and

$$\bigcup_{\substack{p+q=2n+1\\(-1)^q=\delta}} \mathfrak{o}(p,q)_{ss}\,.$$

We refer to this as the *orbit correspondence*.

We write $X' = \mathcal{O}(\psi)(X)$ if the orbits of X and X' correspond as in Lemma 1.14. Dualizing, we obtain a correspondence $\lambda \leftrightarrow \lambda' = \mathcal{O}(\psi)(\lambda)$ of regular semisimple elements of the duals. Finally if $X' = \mathcal{O}(\psi)(X)$, let $\mathfrak{h},\mathfrak{h}'$ be the Cartan subalgebras centralizing X, X' respectively. The correspondence gives rise naturally to a correspondence of systems of positive roots, which we write $\Delta^+ \leftrightarrow \Delta'^+ = \mathcal{O}(\psi)(\Delta^+)$. As usual we drop ψ from the notation if it is fixed.

It is evident that every Cartan subgroup of $\mathrm{SO}(p,q)$ is isomorphic to a Cartan subgroup of $\mathrm{Sp}(2n,\mathbb{R})$. This correspondence preserves conjugacy classes, and is a bijection on conjugacy classes if $\mathrm{SO}(p,q)$ is quasisplit. We use the correspondence of semisimple orbits to choose these isomorphisms as follows.

Fix $\delta = \pm 1$ and ψ. Let $(\mathfrak{h}_0, \Delta^+)$ be a pair consisting of a Cartan subalgebra of $\mathfrak{sp}(2n,\mathbb{R})$ and a system of positive roots. Let $\phi\colon \mathfrak{h}_0 \to \mathfrak{h}'_0 \subset \mathfrak{so}(p,q)$ be an isomorphism. By abuse of notation we write $\phi(\Delta^+)$ for the natural system of positive roots of $\mathfrak{h}'_0$. More precisely, fix $X \in \mathfrak{h}$ so that $\Delta = \{\alpha \mid \alpha(X) > 0\}$. Then $\phi(\Delta^+) = \{\alpha' \mid \alpha'(\phi(X)) > 0\}$.

PROPOSITION 1.15. *Given* $(\mathfrak{h}_0, \Delta^+)$, *there exist* p, q, *and a pair* $(\mathfrak{h}'_0, \Delta'^+)$ *such that* $\Delta^{+'} = \mathcal{O}(\Delta^+)$ *and* $\mathfrak{h}'_0$ *is isomorphic to* $\mathfrak{h}_0$. *This determines* p, q *uniquely (subject to* $(-1)^q = \delta$*). Furthermore the isomorphism* $\phi\colon \mathfrak{h}_0 \to \mathfrak{h}'_0$ *may be chosen so that* $\phi(\Delta^+) = \Delta'^+$. *This determines* ϕ *up to conjugation by* $\mathrm{Sp}(2n,\mathbb{R})$ *and* $\mathrm{O}(p,q)$.

Furthermore ϕ *lifts to an isomorphism* $\phi\colon H \to H' \cap \mathrm{SO}(p,q)$. *Write* $H = TA$, $M = \mathrm{Cent}_G(A)$ *as usual, and similarly for* H'. *Then* ϕ *extends to an isomorphism of the* GL *factors of* M *and* M' (*cf.* (1.10a) *and* (1.12a)).

2. Maximal compact subgroups and joint harmonics

We first consider $G = \mathrm{Sp}(2n,\mathbb{R})$, $\widetilde{G} = \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Recall W and J as in Section 1. Then

$$G := \{g \in \mathrm{GL}(W) \mid {}^t g J g = J\}.$$

We choose the maximal compact subgroup K of G to be

$$K := \{g \in G \mid gJ = Jg\}.$$

Since $J^2 = -\mathrm{Id}$, it defines a complex structure on W. Let $W_{\mathbb{C}} \simeq \mathbb{C}^n$ denote the resulting complex space. Then $W_{\mathbb{C}}$ admits a positive definite symmetric Hermitian form $(v,w) = \langle Jv, w\rangle + i\langle v, w\rangle$. This gives an isomorphism of K with the isometry group $U(\mathbb{W}_{\mathbb{C}}, (\ ,\ ))$. We define the determinant character of K to be the pullback of the determinant character of this unitary group by the explicit isomorphism chosen.

The inverse image $\widetilde{K}$ of K in $\widetilde{G}$ is connected, and its representations may be studied by passing to the Lie algebra. To be explicit, $\widetilde{K}$ is isomorphic to the $\det^{1/2}$ cover of K, i.e. to $\overline{K} = \{(g,z) \mid g \in U(n), z \in \mathbb{C}^*, \det(g) = z^2\}$. The character $\tau\colon (g,z) \to z$ of $\overline{K}$ satisfies $\tau^2(g) = \det(g)$ and is denoted $\det^{1/2}$. We choose an isomorphism, unique up to conjugation, of $\widetilde{K}$ with $\overline{K}$ so the character of $\widetilde{K}$ acting on the unique $\widetilde{K}$-fixed line in $\omega(\psi)$ goes to $\det^{1/2}$. (This line is spanned by the Gaussian in the Schroedinger model, or the constants in the Fock

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model.) We fix the Cartan subgroup T of K as in (1.9), with the usual positive system $\Delta(\mathfrak{t},\mathfrak{k})$. Then $\widetilde{\widetilde{K}}_{\text{genuine}}$ is parametrized by certain dominant weights $\lambda \in i\mathfrak{t}_0^*$; in the usual coordinates $\lambda = (a_1,\ldots,a_n)$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and $a_i \in \mathbb{Z}+\frac{1}{2}$. The distinguished character $\det^{1/2}$ corresponds to the weight $(\frac{1}{2},\ldots,\frac{1}{2})$. This parametrization depends on ψ.

Now consider $G' = \mathrm{O}(p,q)$, $\widetilde{G}' = \widetilde{\mathrm{O}}(p,q)$, with maximal compact subgroups $K' = \mathrm{O}(p)\times\mathrm{O}(q)$, and $\widetilde{K}'$. We fix the Cartan subgroup T of K' as in (1.11), with the usual positive system. We identify an irreducible representations of $\mathrm{O}(p)$ with its 'highest weight' $\lambda = (\lambda_0;\varepsilon)$. Here $\lambda_0 = (a_1,\ldots,a_{p_0}) \in i\mathfrak{t}_0^*$ is the usual highest weight of a finite dimensional representation of $\mathrm{SO}(p)$. We are following [23], where $\varepsilon = 1$ (resp. $\varepsilon = \Leftrightarrow 1$) corresponds to the length of the first column less than (resp. greater than) or equal to p_0. If p is odd, then $\Leftrightarrow \mathrm{Id}$ acts by $(\Leftrightarrow 1)^{\Sigma a_i+\Sigma b_i}\varepsilon$ in this representation. If p is even and $a_{p_0} \neq 0$ then $\varepsilon = \pm 1$ give the same representation; in all other cases they are distinct. Furthermore $(0;\Leftrightarrow 1)$ is the one-dimensional representation $\mathrm{sgn}(g) = \mathrm{sgn}(\det(g)) = \det(g)$, and $(a_1,\ldots,a_{p_0};\varepsilon)\otimes\mathrm{sgn} = (a_1,\ldots,a_{p_0};\Leftrightarrow\varepsilon)$.

A similar discussion holds for $\mathrm{O}(q)$, and the irreducible finite dimensional representations of K' are parametrized by $(a_1,\ldots,a_{p_0};\varepsilon)\otimes(b_1,\ldots,b_{q_0};\eta)$. The irreducible genuine representations of $\widetilde{K}'$ are also parametrized in the same way, by tensoring with the genuine character ξ of $\widetilde{K}'$ as in (1.2g). (Here ξ is the character of $\widetilde{\mathrm{O}}(p,q)$ given by (1.2g), restricted to $\widetilde{K}'$.)

The action of $\widetilde{K}\times\widetilde{K}'$ on the space of joint harmonics gives a bijection between certain irreducible representations of $\widetilde{K}$ and $\widetilde{K}'$ [5]. If a $\widetilde{K}$-type μ corresponds to a $\widetilde{K}'$-type μ', we write $\mu' = \mathcal{H}(\psi,p,q)(\mu)$ and $\mu = \mathcal{H}(\psi)(\mu')$. As usual we drop ψ from the notation if it has been fixed.

The next result follows from [6], as in ([11], I.4) and ([3], Proposition 1.4).

PROPOSITION 2.1. (1) *The correspondence on the space of joint harmonics is as follows.*

$$\begin{aligned}\mu' &= (a_1,\ldots,a_{p_0};1)\otimes(b_1,\ldots,b_{q_0};1)\to\mathcal{H}(\psi)(\mu')\\ &= (a_1,\ldots,a_{p_0},\Leftrightarrow b_{q_0},\ldots,\Leftrightarrow b_1)+\left(\frac{p\Leftrightarrow q}{2},\ldots,\frac{p\Leftrightarrow q}{2}\right)\end{aligned}$$

$$\begin{aligned}\mu' &= (a_1,\ldots,a_k,0,\ldots,0;\Leftrightarrow 1)\otimes(b_1,\ldots,b_\ell,0,\ldots,0;1)\to\mathcal{H}(\psi)(\mu')\\ &= (a_1,\ldots,a_k,\overbrace{1,\ldots,1}^{p-2k},0,\ldots,0,\Leftrightarrow b_\ell,\ldots,\Leftrightarrow b_1)+\left(\frac{p\Leftrightarrow q}{2},\ldots,\frac{p\Leftrightarrow q}{2}\right)\end{aligned}$$

with $p\Leftrightarrow k+\ell\leqslant n$,

$$\mu' = (a_1,\ldots,a_k,0,\ldots,0;1)\otimes(b_1,\ldots,b_\ell,0,\ldots,0;\Leftrightarrow 1)\to\mathcal{H}(\psi)(\mu')$$

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$$= (a_1, \ldots, a_k, 0, \ldots, 0, \overbrace{\Leftrightarrow 1, \ldots, \Leftrightarrow 1}^{q-2\ell}, \Leftrightarrow b_\ell, \ldots, \Leftrightarrow b_1) + \left(\frac{p \Leftrightarrow q}{2}, \ldots, \frac{p \Leftrightarrow q}{2}\right)$$

with $q + k \Leftrightarrow \ell \leqslant n$.

(2) *The* p, q*-degree of* $\mu = (a_1, \ldots, a_n)$ *is* $\sum_{i=1}^n |a_i \Leftrightarrow ((p \Leftrightarrow q)/2)|$. *The degree of*

$$\mu' = (a_1, \ldots, a_k, 0, \ldots, 0; \varepsilon) \otimes (b_1, \ldots, b_\ell, 0, \ldots, 0; \eta)$$

is equal to $\sum_i a_i + ((1 \Leftrightarrow \varepsilon)/2)(p \Leftrightarrow 2k) + \sum b_i + ((1 \Leftrightarrow \eta)/2)(q \Leftrightarrow 2\ell)$.

Note that the dependence on ψ is via the dependence of the parametrization of $\widetilde{K}'$-types on ψ.

The images of $\mathcal{H}(\psi, p, q)$ and $\mathcal{H}(\psi)$ are described by the next Proposition.

PROPOSITION 2.2. (1) *Let* μ *be any (genuine)* $\widetilde{K}$*-type for* $\widetilde{\mathrm{Sp}}(2n)$. *Consider the groups* $\mathrm{O}(p, q)$ *with fixed discriminant. Then there is a unique choice of* p *and* q *such that* μ *is* p, q*-harmonic*.

(2) *Let* $\mu' = (a_1, \ldots, a_k, 0, \ldots, 0; \varepsilon) \otimes (b_1, \ldots, b_\ell, 0, \ldots, 0; \eta)$ *be a (genuine)* $\widetilde{K}'$*-type for* $\widetilde{\mathrm{O}}(p, q)$. *Then* μ' *is in the space of joint harmonics if and only if* $k + ((1 \Leftrightarrow \varepsilon)/2)(p \Leftrightarrow 2k) + \ell + ((1 \Leftrightarrow \eta)/2)(q \Leftrightarrow 2\ell) \leqslant n$. *This holds for precisely one of* μ' *and* $\mu' \otimes \mathrm{sgn}$.

Proof. Part (2) is an immediate consequence of Proposition 2.1, and we omit the details. For part (1), we claim that we may write μ uniquely in the form

$$(a_1, \ldots, a_r, \mu_0, b_1, \ldots, b_s) \tag{2.3a}$$

where

$$\mu_0 = (r \Leftrightarrow s, \ldots, r \Leftrightarrow s) + (\overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{x}, \overbrace{\Leftrightarrow\tfrac{1}{2}, \ldots, \Leftrightarrow\tfrac{1}{2}}^{y}) \tag{2.3b}$$

with $a_r \geqslant r \Leftrightarrow s + \frac{1}{2}, r \Leftrightarrow s \Leftrightarrow \frac{1}{2} \geqslant b_1$, and at least one of these inequalities is strict. The algorithm in Chapter 6 of [18] (see Section 6 for more detailed calculations) attaches to μ an element $\lambda^G(\mu) \in \mathfrak{t}_c^*$. It is of the form

$$\lambda^G(\mu) = (\alpha_1, \ldots, \alpha_r, \overbrace{0, \ldots, 0}^{t}, \beta_1, \ldots, \beta_s) \tag{2.4}$$

with $\alpha_1 \geqslant \cdots \geqslant \alpha_r > 0 > \beta_1 \geqslant \cdots \geqslant \beta_s$. Thus r, s if they exist, are uniquely determined. The same holds for x, y from the particular form of μ. Running the algorithm in reverse on the λ's as in (2.4), we see that every μ must be of the form (2.3a) with some choice of (x, y) (essentially Chapter 6 in [18]).

Given this form, μ corresponds to μ' in the space of joint harmonics for $\mathrm{O}(2r+t+1, 2s+t)$, with

$$\begin{aligned}\mu' &= (a_1 \Leftrightarrow r + s \Leftrightarrow \tfrac{1}{2}, \ldots, a_r \Leftrightarrow r + s \Leftrightarrow \tfrac{1}{2}, 0, \ldots, 0; 1)\\ &\quad \otimes(\Leftrightarrow b_s + r \Leftrightarrow s + \tfrac{1}{2}, \ldots, \Leftrightarrow b_1 + r \Leftrightarrow s + \tfrac{1}{2}, \overbrace{1, \ldots, 1}^{j}, 0, \ldots, 0; \varepsilon),\end{aligned}$$

where

$$(j, \varepsilon) = \begin{cases} (y, +1) & 0 \leqslant y \leqslant \left[\dfrac{t}{2}\right] \\ (x, \Leftrightarrow 1) & \left[\dfrac{t+1}{2}\right] \leqslant y \leqslant s \quad \left(\Leftrightarrow 0 \leqslant x \leqslant \left[\dfrac{t}{2}\right]\right). \end{cases}$$

Similarly μ corresponds to μ' in the space of joint harmonics for $\mathrm{O}(2r+t, 2s+t+1)$, with

$$\begin{aligned}\mu' &= (a_1 \Leftrightarrow r + s + \tfrac{1}{2}, \ldots, a_r \Leftrightarrow r + s + \tfrac{1}{2}, \overbrace{1, \ldots, 1}^{j}, 0, \ldots, 0; \varepsilon)\\ &\quad \otimes(\Leftrightarrow b_s + r \Leftrightarrow s \Leftrightarrow \tfrac{1}{2}, \ldots, \Leftrightarrow b_1 + r \Leftrightarrow s \Leftrightarrow \tfrac{1}{2}, 0, \ldots, 0; +1),\end{aligned}$$

where

$$(j, \varepsilon) = \begin{cases} (x, +1) & 0 \leqslant x \leqslant \left[\dfrac{t}{2}\right] \\ (y, \Leftrightarrow 1) & \left[\dfrac{t+1}{2}\right] \leqslant x \leqslant s \quad \left(\Leftrightarrow 0 \leqslant y \leqslant \left[\dfrac{t}{2}\right]\right). \end{cases}$$

Note that for t even and $x = y = [t/2]$ the two cases agree, since for even orthogonal groups $(a_1, \ldots, 1; +1) \cong (a_1, \ldots, 1; \Leftrightarrow 1)$.

It remains to show μ is p, q-harmonic for at most one choice of q with $(\Leftrightarrow 1)^q = \delta$. Given δ, to determine (p, q) it is enough to find p_0. Assume $\delta = 1$, the other case is similar. Suppose μ is p, q-harmonic corresponding to a μ' as in Proposition 2.1. Then in the expression $\mu = (\mu_1, \ldots, \mu_n)$, we must have $\mu_{p_0} = (p \Leftrightarrow q)/2$ if $\varepsilon = 1$, or $\mu_{p_0} = ((p \Leftrightarrow q)/2) + 1$ if $\varepsilon = \Leftrightarrow 1$. This is the same as $\mu_{p_0} + (n + 1 \Leftrightarrow 2p_0) = \frac{3}{2}$ or $\frac{5}{2}$. But the sequence $r_i = \mu_i + (n + 1 \Leftrightarrow 2i)$ decreases monotonically by at least 2 each consecutive term, so there is at most one i such that $r_i = \frac{5}{2}$ or $\frac{3}{2}$, never both. If all $r_i > \frac{5}{2}$ then $p_0 = n$, if all $r_i < \frac{1}{2}$, then $p_0 = 0$. This proves (1). $\square$

Remark. This Proposition also follows naturally from the calculations in the proof of Proposition 6.1. Namely if μ and μ' correspond, then $\lambda^G(\mu)$ and $\lambda^G(\mu')$ correspond in a simple fashion, implying in addition that r, s, t given by formula (2.4) for μ, and the (r', s', t') coming from formula (6.6) coincide. Thus $p = 2r + t + 1$ or $p = 2r + t$ according to the parity of δ.

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EXAMPLE 2.4. The example of small weights ([18], Definition 5.3.24) is important. These are weights of the form

$$\mu = (\overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{x}, \overbrace{\Leftrightarrow\tfrac{1}{2}, \ldots, \Leftrightarrow\tfrac{1}{2}}^{y}).$$

Then μ is small, *i.e.* $\mu_0 = \mu$ and it corresponds to a μ' in the split group $\mathrm{O}(n+1, n)$ as well as $\mathrm{O}(n, n+1)$. Specifically μ corresponds to μ' for $\mathrm{O}(n+1, n)$ with

$$\mu' = (0, \ldots, 0; 1) \otimes (\overbrace{1, \ldots, 1}^{r}, 0, \ldots, 0; \varepsilon)$$

with

$$(r, \varepsilon) = \begin{cases} (y, 1) & 0 \leqslant y \leqslant \left[\dfrac{n}{2}\right] \\ (x, \Leftrightarrow 1) & \left[\dfrac{n+1}{2}\right] \leqslant y \leqslant n. \end{cases}$$

On the other hand it goes to μ' for $\mathrm{O}(n, n+1)$ with

$$\mu' = (\overbrace{1, \ldots, 1}^{r}, 0, \ldots, 0; \varepsilon) \otimes (0, \ldots, 0; 1)$$

with

$$(r, \varepsilon) = \begin{cases} (x, 1) & 0 \leqslant x \leqslant \left[\dfrac{n}{2}\right] \\ (y, \Leftrightarrow 1) & \left[\dfrac{n+1}{2}\right] \leqslant x \leqslant n. \end{cases}$$

3. Discrete series

A genuine discrete series representation π of $\widetilde{\mathrm{Sp}}(2n)$ is determined by its Harish-Chandra parameter λ. In coordinates we write

$$\lambda = (a_1, \ldots, a_k, b_1, \ldots, b_\ell) \in i\mathfrak{t}_0^*, \quad (\mathfrak{t} \text{ as in } 1.9)$$

with $a_1 > \cdots > a_k > 0 > b_1 > \cdots > b_\ell,\ \ a_i, b_j \in \mathbb{Z} + \frac{1}{2}$ and $a_i + b_j \neq 0$ for all i, j. Then π has lowest K-type $\lambda + \rho(\Delta^+) \Leftrightarrow 2\rho_c(\Delta^+)$ where Δ^+ is the system for which λ is dominant.

The genuine limits of discrete series for $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ are obtained by allowing λ to be singular with respect to a set of simple noncompact roots. Explicitly these

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representations are parametrized by pairs (Δ^+, λ) where λ is dominant with respect to the roots in Δ^+. In coordinates it is of the form

$$(\overbrace{a_1 \dots, a_1}^{m_1}, \overbrace{a_2 \dots, a_2}^{m_2}, \dots, \overbrace{a_r \dots, a_r}^{m_r}, \overbrace{-a_r \dots, -a_r}^{n_r}, \dots,$$
$$\overbrace{-a_2 \dots, -a_2}^{n_2}, \overbrace{-a_1 \dots, -a_1}^{n_1})$$

with $a_i \in \mathbb{Z} + \frac{1}{2}$, $a_1 > \cdots > a_r > 0$, and $|m_i - n_i| \leqslant 1$ for all i. The lowest K-type of π has the same form as for discrete series, $\lambda + \rho(\Delta^+) - 2\rho_c(\Delta^+)$.

Similarly a discrete series representation of $\mathrm{SO}(p, q)$ is given by its Harish-Chandra parameter $\lambda = (a_1, \dots, a_{p_0}, b_1, \dots, b_{q_0})$ with $a_i, b_j \in \mathbb{Z} + \frac{1}{2}$ satisfying $a_1 > \cdots > a_{p_0} > 0, b_1 > \cdots > b_{q_0} > 0,\ a_i \neq b_j\ \forall i, j$. Assume for the moment that p is odd and q is even. The lowest K'-type $\mu = \lambda + \rho(\Delta^+) - 2\rho_c(\Delta^+)$ is of the form

$$\mu = (x_1, \dots, x_k, 0, \dots, 0; \pm 1) \otimes (y_1, \dots, y_{q_0}; \pm 1) \quad x_k, y_j \in \mathbb{Z} > 0.$$

The second ± 1 has no effect since $y_{q_0} > 0$. The two representations given by the first ± 1 have the same restriction to $S(\mathrm{O}(p) \times \mathrm{O}(q))$ since $\mathrm{sgn} \otimes \mathrm{sgn}$ of $\mathrm{O}(p) \times \mathrm{O}(q)$ is trivial on this subgroup. Passing to $\mathrm{O}(p, q)$ we obtain the following Lemma.

LEMMA 3.1. *The discrete series representations* π *of* $\mathrm{O}(p, q)$ *are parametrized by*

$$\lambda = (\lambda_0; \varepsilon) = (a_1, \dots, a_{p_1}, b_1, \dots, b_{q_0}; \varepsilon)$$

with $a_1 > \cdots > a_{p_0} > 0; b_1 > \cdots > b_{q_0} > 0; a_i,\ b_j \in \mathbb{Z} + \frac{1}{2}$, *and* $a_i - b_j \neq 0$ *for all* i, j. *Here* π *is determined by its Harish-Chandra parameter* λ_0 *and its lowest* K'*-type* μ *which is of the form*

$$\mu = \begin{cases} (x_1, \dots, x_{p_0}, 1) \otimes (y_1, \dots, y_\ell, 0, \dots, 0; \varepsilon) \quad (x_{p_0} > 0) \\ \quad p \textit{ even, } q \textit{ odd} \\ (x_1, \dots, x_k, 0, \dots, 0; \varepsilon) \otimes (y_1, \dots, y_{q_0}; +1) \quad (y_{q_0} > 0) \\ \quad p \textit{ odd, } q \textit{ even.} \end{cases} \tag{3.2}$$

The genuine discrete series of $\widetilde{\mathrm{O}}(p, q)$ are obtained by tensoring the discrete series of $\mathrm{O}(p, q)$ with ξ, so we use the same parameters.

We will refer to $\lambda = (\lambda_0; \varepsilon)$ as a Harish-Chandra parameter for $\mathrm{O}(p, q)$ or $\widetilde{\mathrm{O}}(p, q)$. The limits of discrete series are parametrized as for $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, by pairs (Δ^+, λ) where λ_0 is Δ^+-dominant and λ_0 of the form

$$(\overbrace{a_1 \dots, a_1}^{m_1}, \overbrace{a_2 \dots, a_2}^{m_2}, \dots,$$
$$\overbrace{a_r \dots, a_r}^{m_r}, \overbrace{a_1 \dots, a_1}^{n_1}, \overbrace{a_2 \dots, a_2}^{n_2}, \dots, \overbrace{a_r \dots, a_r}^{n_r})$$

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with $a_i \in \mathbb{Z} + \frac{1}{2}$, $a_1 > \cdots > a_r > 0$, and $|m_i - n_i| \leqslant 1$ for all i. Again μ is of the form (3.2).

THEOREM 3.3. *Fix ψ and $\delta = \pm 1$.* (1) *Let π be a genuine discrete series representation of $\widetilde{\mathrm{Sp}}(2n)$ with Harish-Chandra parameter λ. Choose p, q so that λ occurs in the orbit correspondence for the dual pair $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$. Recall* (*Lemma* 1.14) *p, q is uniquely determined, subject to $(-1)^q = \delta$.*

Let $\lambda' = \mathcal{O}(\lambda)$ be a corresponding element of $\mathfrak{so}(p, q)^*$. Then π occurs in the representation correspondence with $\widetilde{\mathrm{O}}(p, q)$, and $\theta_{p,q}(\pi)$ is the discrete series representation with Harish-Chandra parameter $(\lambda'; +1)$. Furthermore π does not occur in the correspondence for any other $\widetilde{\mathrm{O}}(r, s)$ (with $(-1)^s = \delta$).

If μ is the lowest $\widetilde{K}$-type of π, then μ is of lowest p, q-degree, and $\mathcal{H}(\mu)$ is the lowest $\widetilde{K}'$-type of π'.

Conversely every discrete series of $\widetilde{\mathrm{O}}(p, q)$ with Harish-Chandra parameter $(*; +1)$ corresponds to a discrete series of $\widetilde{\mathrm{Sp}}(2n)$, and those of the form $(*; -1)$ do not occur in the correspondence.

(2) *The same results as in* (a) *holds for limits of discrete series, where if π is given by data (Δ^+, λ) then $\theta_{p,q}(\pi)$ is given by $(\mathrm{O}(\Delta^+), (\mathcal{O}(\lambda); 1))$.*

Explicitly (cf. 1.13) let $\psi = \psi_a$ with $a > 0$, and suppose $\lambda = (a_1, \ldots, a_{p_0}, b_1, \ldots, b_{q_0})$ with $a_1 > \cdots > a_{p_0} > 0 > b_1 > \cdots > b_{q_0}$. Then $p = 2p_0 + 1, q = 2q_0$ or $p = 2p_0, q = 2q_0 + 1$ and

$$\lambda' = ((a_1, \ldots, a_{p_0}, -b_{q_0}, \ldots, -b_1); +1).$$

Theorem 3.3 will be proved in Section 9.

Note. The minimal K-type of a discrete series representation or a limit of discrete series is unique, and such a representation is determined by its minimal K-type. This follows from [6] or in our case from the results of Section 6.

4. Standard modules

We use the version of the Langlands classification of [21], which is valid for disconnected and non-linear groups of Harish-Chandra's class. Throughout this section G will denote $\mathrm{O}(p, q)$ $(p = 2n + 1)$ or $\mathrm{Sp}(2n, \mathbb{R})$, with maximal compact subgroup K and covering groups $\widetilde{G}$ and $\widetilde{K}$.

We first consider $G = \mathrm{O}(p, q)$. Let $H = TA$ be a θ-stable Cartan subgroup of G. Recall from Section 1 that H is isomorphic to

$$(\mathbb{C}^*)^m \times (S^1)^r \times (\mathbb{R}^*)^s \times \mathcal{Z} \tag{4.1a}$$

with $2m+s \leqslant \min(p,q)$, $\mathcal{Z}$ the center of $\mathrm{O}(p,q)$. The centralizer M of A is isomorphic to

$$\begin{aligned}&\mathrm{GL}(1,\mathbb{R})^s \times \mathrm{GL}(2,\mathbb{R})^m \times \mathrm{O}(p',q'),\\ &(p' = p \Leftrightarrow s \Leftrightarrow 2m,\ \ q' = q \Leftrightarrow s \Leftrightarrow 2m)\end{aligned} \tag{4.1b}$$

and the inverse image $\widetilde{M}$ of M in $\widetilde{\mathrm{O}}(p,q)$ is isomorphic to

$$\mathrm{GL}(1,\mathbb{R})^s \times \mathrm{GL}(2,\mathbb{R})^m \times \widetilde{\mathrm{O}}(p',q')[n]. \tag{4.1c}$$

For inducing data on $\widetilde{M}$ we take an irreducible representation $\sigma = \beta\otimes\gamma\otimes\tau$ of $\widetilde{M}$. Here $\beta = \beta_1\otimes\cdots\otimes\beta_s$ is a product of characters, $\gamma = \gamma_1\otimes\cdots\otimes\gamma_m$ is a product of relative limits of discrete series representations and τ is a limit of discrete series representation of $\widetilde{\mathrm{O}}(p',q')$. The restriction of σ to A is a multiple of a character ν; choose a parabolic subgroup $\widetilde{P} = \widetilde{M}N$ so that $\mathrm{Re}\langle\alpha,\nu\rangle \geqslant 0$ for all roots α of $\mathfrak{a}$ in $\mathfrak{n}$.

The *standard module* for $\widetilde{\mathrm{O}}(p,q)$ associated to this data is

$$X(\widetilde{P},\sigma) = \mathrm{Ind}_{\widetilde{M}N}^{\widetilde{\mathrm{O}}(p,q)}(\sigma) \tag{4.1d}$$

(here and elsewhere we extend σ to $\widetilde{M}N$ trivially on N). This has the same type as does τ. If $(\widetilde{M},\sigma)$ also satisfy condition (F-2) of [21], which we make explicit in Lemma 4.3, then this module has a unique irreducible quotient, and every irreducible representation is obtained this way. The data $(\widetilde{M},\sigma)$ are unique up to conjugation by $\widetilde{K}$ and will be called *inducing data* for π.

We next describe standard modules for genuine representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Recall from Section 1 that a Cartan subgroup H of $\mathrm{Sp}(2n,\mathbb{R})$ is isomorphic to

$$(\mathbb{C}^*)^m \times (S^1)^r \times (\mathbb{R}^*)^s \quad (r+s+2m = n) \tag{4.2a}$$

in which case M is isomorphic to

$$\mathrm{GL}(1,\mathbb{R})^s \times \mathrm{GL}(2,\mathbb{R})^m \times \mathrm{Sp}(2s,\mathbb{R}). \tag{4.2b}$$

Let

$$\overline{M} = \widetilde{\mathrm{GL}}(1,\mathbb{R})^s \times \widetilde{\mathrm{GL}}(2,\mathbb{R})^m \times \widetilde{\mathrm{Sp}}(2s,\mathbb{R}). \tag{4.2c}$$

For inducing data we take $\overline{\sigma} = \beta\otimes\gamma\otimes\tau$ with $\beta = \beta_1\otimes\cdots\otimes\beta_s$ a product of genuine characters, $\gamma = \gamma_1\otimes\cdots\otimes\gamma_m$ a product of genuine relative limits of discrete series representations, and τ a genuine limit discrete series representation. Then $\overline{\sigma}$ factors to a genuine representation σ of $\widetilde{M}$, and every genuine representation σ of

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$\widetilde{M}$ comes from a unique such $\widetilde{\sigma}$. Choosing N satisfying the positivity condition as above, the standard module associated to the data $(\overline{M}, \overline{\sigma})$ or $(\widetilde{M}, \sigma)$ is

$$X(\widetilde{P}, \sigma) = \mathrm{Ind}_{\widetilde{M}N}^{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(\sigma). \tag{4.2d}$$

It has the same properties as in the case of $\widetilde{\mathrm{O}}(p, q)$. We freely pass back and forth between $(\overline{M}, \overline{\sigma})$ and $(\widetilde{M}, \sigma)$ without further comment.

Condition (F-2) of [21] for $\widetilde{\mathrm{O}}(p, q)$ and $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ is made explicit as follows. Fix a genuine character χ of $\widetilde{\mathrm{GL}}(1)$, and write

$$\beta_i(x, \varepsilon) = |x|^{\nu_i}\,\mathrm{sgn}(x)^{\delta_i} \cdot \begin{cases} 1 & \widetilde{\mathrm{O}}(p, q) \\ \chi(x, \varepsilon) & \widetilde{\mathrm{Sp}}(2n, \mathbb{R}). \end{cases} \tag{4.2e}$$

A limit of discrete series representation of $\mathrm{GL}(2, \mathbb{R})$ is parametrized by (k, μ) with $k \in \mathbb{N}$ and $\mu \in \mathbb{C}$; the lowest K-type of this representation has highest weight $k + 1$ for $\mathrm{O}(2)$. The genuine limit of discrete series representations of $\widetilde{\mathrm{GL}}(2, \mathbb{R})$ are parametrized the same way by tensoring with $\chi(\det)$; this is independent of χ since for such a representation $\pi \otimes \mathrm{sgn} \simeq \pi$.

LEMMA 4.3. *Let* $G = \widetilde{\mathrm{O}}(p, q)$ *or* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ *as before. The data* $(\widetilde{M}, \sigma)$ *satisfy condition* (F-2) *of* [21] *if and only if*

(1) *For each* $\mathrm{GL}(2, \mathbb{R})$*-factor,* $\mu = 0$ *implies* $k \in \mathbb{Z}$,
(2) $\nu_i = \pm\nu_j$ *implies* $\delta_i = \delta_j$.

In this case, $X(\widetilde{P}, \sigma)$ *has a unique irreducible quotient.*

We will prove this in Section 7.

We also use character data for these groups as described in [21], which refers to [18, 20]. Unexplained notation is as in [21].

A *limit character* for $\widetilde{G}$ is a pair $(\widetilde{H}, \gamma)$. Here $\widetilde{H}$ is a Cartan subgroup of $\widetilde{G}$, and γ is a triple

$$(\Psi, \Gamma, \overline{\gamma}) \tag{4.4a}$$

consisting of a positive system Ψ for the imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$, a character Γ of $\widetilde{H}$, and an element $\overline{\gamma}$ of $\mathfrak{h}^*$. These must satisfy two conditions. First of all $\langle \alpha, \overline{\gamma} \rangle \geqslant 0$ for all $\alpha \in \Psi$, and $d\Gamma = \overline{\gamma} + \rho(\Psi) \Leftrightarrow 2\rho_c(\Psi)$.

A limit character is called *final* if in addition it satisfies the following two conditions. First of all if α is a simple root of Ψ then

$$\langle \alpha, \overline{\gamma} \rangle = 0 \text{ implies } \alpha \text{ is non-compact.} \tag{4.4b}$$

Secondly if α is a real root of $\mathfrak{h}$ in $\mathfrak{g}$ then

$$\langle \alpha, \overline{\gamma} \rangle = 0 \text{ implies } \Gamma(m_\alpha) \neq \varepsilon_\alpha \tag{4.4c}$$

for $m_\alpha \in H$ and $\varepsilon_\alpha = \pm 1$ as in [18, 8.3.11], *i.e.* α does not satisfy the parity condition. We will make condition (4.4c) explicit and relate this data to inducing data in Section 6.

Attached to a final limit character γ is a standard module $X(\gamma)$ which has an irreducible quotient $\overline{X}(\gamma)$, and the $\widetilde{K}$-conjugacy classes of final limit characters thereby parametrize the admissible dual of $\widetilde{G}$. The central character of $X(\gamma)$ is the restriction of Γ to the center of G; in particular $X(\gamma)$ is genuine if and only if Γ is genuine.

5. Main results

We consider the dual pairs $(\widetilde{\mathrm{O}}(p,q), \widetilde{\mathrm{Sp}}(2n,\mathbb{R}))$ with $p+q = 2n+1$. Throughout this section we fix $\delta = \pm 1$, and a nontrivial additive character ψ of $\mathbb{R}$. Recall (1.2) $\chi = \chi(\psi)$ is a genuine character of $\widetilde{\mathrm{GL}}(m,\mathbb{R})$ for any m. Also recall for V an orthogonal space of signature (p,q), the genuine character $\chi_V = \chi(\psi, V)$ of $\widetilde{\mathrm{GL}}(m,\mathbb{R})$ satisfies $\chi_V = \chi(\psi)^{-p+q}$. We write $\theta_{p,q} = \theta(\psi,p,q)$ for the θ-correspondence as in (1.3).

THEOREM 5.1. *Let π be a genuine irreducible representation of $\widetilde{\mathrm{Sp}}(2n)$, with inducing data* (cf. Section 4)

$$\overline{M} = \widetilde{\mathrm{GL}}(1,\mathbb{R})^s \times \widetilde{\mathrm{GL}}(2,\mathbb{R})^m \times \widetilde{\mathrm{Sp}}(2r,\mathbb{R}) \quad \sigma = \alpha \otimes \beta \otimes \tau.$$

By Theorem 3.3, *there exist p', q' satisfying $p' + q' = 2c + 1$ and $(\Leftrightarrow 1)^s = \delta(\Leftrightarrow 1)^m$, such that τ is in the domain of $\theta_{p',q'}$. Let $\eta = \theta_{p',q'}(\tau)$.*

Let $p = a + 2b + p', q = a + 2b + q'$. Then $p + q = 2n + 1, (\Leftrightarrow 1)^q = \delta$, $\theta_{p,q}(\pi) \neq 0$ and π is in the domain of $\theta_{p,q}$. The inducing data for $\theta_{p,q}(\pi)$ are given by

$$\begin{aligned} \widetilde{M}' &= \mathrm{GL}(1,\mathbb{R})^s \times \mathrm{GL}(2,\mathbb{R})^m \times \widetilde{\mathrm{O}}(p',q')[n] \\ \sigma' &= \alpha^* \chi_V \otimes \beta^* \chi_V \otimes \eta' \end{aligned}$$

In these formulas, $\alpha^*\chi_V$ means $\alpha_1^*\chi_V \otimes \cdots \otimes \alpha_k^*\chi_V$, similarly for β, and η' is given by

$$\eta' = \begin{cases} \eta & a \text{ even}, \\ \eta\chi & a \text{ odd}, \ n \text{ even}, \\ \eta\chi^{-1} & a \text{ odd}, \ n \text{ odd}. \end{cases}$$

Note. To define σ' in Theorem 5.1 we have identified the $\mathrm{GL}(1)$ and $\mathrm{GL}(2)$ factors of M and M' as in Section 1.

We summarize some useful properties of this correspondence which follow immediately from Theorem 5.1 and its proof.

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COROLLARY 5.2.

(1) *Let μ be a lowest $\widetilde{K}$-type of π. Then μ is of lowest p, q-degree in π.*

(2) *p, q are the unique choice with $(\Leftrightarrow 1)^q = \delta$ and $\mathcal{H}_{p,q}(\mu) \neq 0$.*

(3) *Let μ be a minimal $\widetilde{K}$-type of π, and write the element $\lambda^G(\mu)$ of $\mathfrak{t}$ associated to μ by the Vogan algorithm (cf. Section 6) as*

$$\lambda^G(\mu) = (\lambda_1, \dots, \alpha_r, \overbrace{0, \dots, 0}^{t}, \beta_1, \dots, \beta_s)$$

with $\alpha_1 \geqslant \cdots \geqslant \alpha_r > 0 > \beta_1 \geqslant \cdots \geqslant \beta_s$. Then $p, q = 2r + t + 1, 2s + t$ or $2r + t, 2s + t + 1$.

(4) *$\mu' = \mathcal{H}_{p,q}(\mu)$ is a lowest $\widetilde{K}'$-type of π'. In particular μ' has multiplicity one in π' and the standard module of π'.*

(5) *$\mathcal{H}_{p,q}$ defines a bijection between the lowest $\widetilde{K}$-types of π and the lowest $\widetilde{K}'$-types of π'.*

(6) *A representation π of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ occurs in the correspondence for the dual pair $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$ if and only if some (equivalently every) minimal $\widetilde{K}$-type μ is p, q-harmonic.*

(7) *A representation π' of $\widetilde{\mathrm{O}}(p, q)$ occurs in the correspondence if and only if some (equivalently every) minimal $\widetilde{K}'$-type is harmonic.*

A comment is also in order due to our choice of coverings of orthogonal groups (Section 1). The group $\widetilde{\mathrm{O}}(p', q')$ in $\widetilde{M}'$ is $\widetilde{\mathrm{O}}(p', q')[n]$, while η is defined on $\widetilde{\mathrm{O}}(p', q')[c]$. Since $n \Leftrightarrow c \equiv a \bmod (2)$, there is an identification in the definition of η' if a is odd. Strictly speaking it should read $\eta' = (\eta\chi)\xi$ (n even) or $\eta' = (\eta\xi)\chi^{-1}$ (n odd) where ξ is the genuine character $1 \otimes \mathrm{sgn}$ of the trivial cover $\mathrm{O}(r, s) \times \mathbb{Z}/2\mathbb{Z}$ of $\mathrm{O}(r, s)$ (cf. 1.2).

Let $\mathrm{SO}(p, q)\widehat{\ }$ be the admissible dual of $\mathrm{SO}(p, q)$, *i.e.* the set of equivalence classes of irreducible admissible Harish-Chandra modules for $\mathrm{SO}(p, q)$, and let $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})\widehat{\ }_{\text{genuine}}$ be the genuine admissible dual of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$.

COROLLARY 5.3. *Fix δ and ψ. Then the representation correspondence gives a bijection*

$$\widetilde{\mathrm{Sp}}(2n, \mathbb{R})\widehat{\ }_{\text{genuine}} \xleftrightarrow{1\Leftrightarrow 1} \bigcup_{\substack{p+q=2n+1 \\ (-1)^q=\delta}} \mathrm{SO}(p, q)\widehat{\ }.$$

More precisely, if π is a genuine irreducible representation of $\widetilde{\mathrm{Sp}}(2n)$, let $\pi_0' = \theta(\psi, p, q)(\pi)$ be the θ-lift of π to $\widetilde{\mathrm{O}}(p, q)$ for the unique choice of p, q for which this is non-zero. Then $\pi_0' \otimes \xi^{-1}$ factors to $\mathrm{O}(p, q)$, and let π' be the restriction to $\mathrm{SO}(p, q)$. Then $\pi \to \pi'$ gives one direction of the bijection.

Conversely if π' is an irreducible representation of $\mathrm{SO}(p, q)$, extend π' to an irreducible representation of $\mathrm{O}(p, q)$ (there are two such choices), and tensor with

ξ. Precisely one such choice of representation π_0' is in the domain of $\theta(\psi)$; let $\pi = \theta(\psi)(\pi_0')$.

6. Some calculations

In this section we do some calculations involving K-types. The main results are Propositions 6.1, 6.18, 6.21 and 6.29. Throughout this section we fix $\psi = \psi_a$ with $a > 0$ (cf. Section 1).

Let μ be a K-type for a group G. Proposition 5.3.3 of [18] produces an element $\lambda \in \mathfrak{t}^*$ where $\mathfrak{t}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$. We refer to this map as the Vogan algorithm, and denote it $\mu \to \mathcal{V}(\mu) = \lambda$.

PROPOSITION 6.1. *Let μ be a $\widetilde{K}$-type for $\widetilde{\mathrm{Sp}}(2n)$, and suppose μ is p, q-harmonic. Then the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{Sp}(2n) & \mathrm{O}(p,q) \\
\mu \arrow[r, "\mathcal{H}"] \arrow[d, "\mathcal{V}"'] & \mu' \arrow[d, "\mathcal{V}"] \\
\lambda \arrow[r, "\mathcal{O}"] & \lambda'
\end{tikzcd}$$

A small but useful observation is that for the purposes of computation it is better to compute the inverse of $\mathcal{V}$, *i.e.* the multi-valued map $\lambda \to \mu$. With this in mind we summarize some standard theory [6, 18].

Let G be a reductive group with a compact Cartan subgroup T_c. We use $\mathfrak{t}_c$ as a fixed complex Cartan subalgebra of $\mathfrak{g}$ (an 'abstract' Cartan subalgebra in the sense of [19]). Let $\alpha_1, \ldots, \alpha_k$ be a set of strongly orthogonal non-compact roots of $\mathfrak{t}_c$ in $\mathfrak{g}$. Associated to this set is a G-conjugacy class of Cartan subgroups of G. We choose H in this conjugacy class, and writing $H = TA$ as usual we may and do assume $\mathfrak{t} \subset \mathfrak{t}_c$, and $\mathfrak{t}^* \subset \mathfrak{t}_c^*$.

The Cartan involution of $\mathfrak{g}$ carried back to $\mathfrak{t}_c$ via a Cayley transform gives an involution σ of $\mathfrak{t}_c$. Let $\overline{\gamma} = (\lambda, \nu) \in i\mathfrak{t}_{c,0}^* \times \mathfrak{t}_c^*$ satisfying $\sigma(\lambda) = \lambda$ and $\sigma(\nu) = \Leftrightarrow\nu$. Then the Cayley transform identifies λ (resp. ν) with an element of $i\mathfrak{t}_0^*$ (resp. $\mathfrak{a}^*$), and $\overline{\gamma}$ with an element of $\mathfrak{h}^*$.

Let π be an irreducible representation of G with character data $(H, \gamma) = (\Psi, \Gamma, \overline{\gamma})$ (cf. Section 4). Write $\overline{\gamma} = (\lambda, \nu)$ with $\lambda \in \mathfrak{t}_c^*$, and let $\mathfrak{q} = \mathfrak{q}(\lambda) = \mathfrak{l} \oplus \mathfrak{u}$ be the θ-stable parabolic subalgebra of $\mathfrak{g}$ defined by λ ([18], Definition 5.2.1). The normalizer L of $\mathfrak{q}$ in G is quasi-split. The minimal K-types of π are of the form

$$\mu = \lambda + \rho(\mathfrak{u} \cap \mathfrak{p}) \Leftrightarrow \rho(\mathfrak{u} \cap \mathfrak{k}) + \mu_L \tag{6.2}$$

for some fine $L \cap K$-type μ_L.

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Proof of Proposition 6.1. Let $\widetilde{G} = \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, $\widetilde{G}' = \widetilde{\mathrm{O}}(p, q)$, *etc.*, with maximal compact subgroups and compact Cartan subgroups chosen as in Section 2. Given μ, let $\pi = \overline{X}(\gamma)$ be an irreducible representation with lowest $\widetilde{K}$-type μ. Then $\overline{\gamma} = (\lambda, \nu)$ with $\lambda = \mathcal{V}(\mu) \in \mathfrak{t}_c^*$. By the above discussion $\mu = \lambda + \rho(\mathfrak{u} \cap \mathfrak{p}) \Leftrightarrow \rho(\mathfrak{u} \cap \mathfrak{k}) + \mu_L$. To avoid covering groups we work on the Lie algebras whenever possible.

After conjugating by $W(\mathfrak{t}_c, \mathfrak{g})$ we may write

$$\begin{aligned} \lambda = {} & (\overbrace{\lambda_1, \ldots, \lambda_1}^{x_1}, \ldots, \overbrace{\lambda_r, \ldots, \lambda_r}^{x_r}, \overbrace{0, \ldots, 0}^{m_0}, \\ & \overbrace{\Leftrightarrow\lambda_r, \ldots, \Leftrightarrow\lambda_r}^{y_r}, \ldots, \overbrace{\Leftrightarrow\lambda_1, \ldots, \Leftrightarrow\lambda_1}^{y_1}) \end{aligned} \tag{6.3}$$

with $\lambda_1 > \cdots > \lambda_r > 0$; $x_i, y_i \geqslant 0$.

The fine $\mathfrak{k} \cap \mathfrak{l}_0$-types μ_L for $\mathfrak{l}_0 \simeq \prod_{i=1}^r \mathfrak{u}(x_i, y_i) \times \mathfrak{sp}(2m_0, \mathbb{R})$ are described as follows. If $x_i \neq y_i$ then μ_L is trivial on this factor. If $x_i = y_i$, then μ_L is trivial, or has highest weight

$$\pm(\overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{x_i}, \overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{y_i}), \tag{6.4a}$$

on this factor. Finally on $\mathfrak{sp}(2m_0, \mathbb{R})$, μ_L has highest weight of the form

$$(1, \ldots, 1, 0, \ldots, 0) \ \text{ or } \ (0, \ldots, 0, \Leftrightarrow 1, \ldots, \Leftrightarrow 1) \tag{6.4b}$$

or

$$(\overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{u}, \overbrace{\Leftrightarrow\tfrac{1}{2}, \ldots, \Leftrightarrow\tfrac{1}{2}}^{v}). \tag{6.4c}$$

In the case of a genuine representation π of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, μ_L will have form (6.4c) on this factor.

A straightforward computation now gives

$$\begin{aligned} \mu = {} & (\overbrace{\alpha_1, \ldots, \alpha_1}^{x_1}, \ldots, \overbrace{\alpha_r, \ldots, \alpha_r}^{x_r}, \overbrace{\widetilde{x} \Leftrightarrow \widetilde{y} + \tfrac{1}{2}, \ldots, \widetilde{x} \Leftrightarrow \widetilde{y} + \tfrac{1}{2}}^{m_1}, \\ & \overbrace{\widetilde{x} \Leftrightarrow \widetilde{y} \Leftrightarrow \tfrac{1}{2}, \ldots, \widetilde{x} \Leftrightarrow \widetilde{y} \Leftrightarrow \tfrac{1}{2}}^{m_2}, \overbrace{\beta_r, \ldots, \beta_r}^{y_r}, \ldots, \overbrace{\beta_1, \ldots, \beta_1}^{y_1}). \end{aligned} \tag{6.5a}$$

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Here

$$\begin{aligned}
&\widetilde{x}_k = \sum_{i=1}^{k} x_i, \widetilde{y}_k = \sum_{i=1}^{k} y_i,\\
&\alpha_i = \lambda_i + (\widetilde{x}_{i-1} \Leftrightarrow \widetilde{y}_{i-1}) + \tfrac{1}{2}(x_i \Leftrightarrow y_i) + \tfrac{1}{2} + \frac{\varepsilon_i}{2} \quad (\varepsilon_i = 0, \pm 1),\\
&\beta_i = \Leftrightarrow \lambda_i + (\widetilde{x}_{i-1} \Leftrightarrow \widetilde{y}_{i-1}) + \tfrac{1}{2}(x_i \Leftrightarrow y_i) \Leftrightarrow \tfrac{1}{2} + \frac{\varepsilon_i}{2},\\
&\varepsilon_i = \begin{cases} \pm 1 & x_i = y_i \text{ and } \lambda_i \in \mathbb{Z} + \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{6.5b}$$

and m_1 and m_2 are any non-negative integers with $m_1 + m_2 = m_0$. We set $\widetilde{x} = \widetilde{x}_r, \widetilde{y} = \widetilde{y}_r$.

We now let $\lambda' = \mathcal{O}(\lambda)$, and do the corresponding calculation on the orthogonal group. It follows from Proposition 2.2 that μ is p, q-harmonic with $p = 2x + m_0 + 1$, $q = 2y + m_0$ or $p = 2x + m_0$, $q = 2y + m_0 + 1$. We consider only the first case, the second is similar.

From 1.13 we have

$$\begin{aligned}
\lambda' \;=\; &(\overbrace{\lambda_1, \ldots, \lambda_1}^{x_1}, \ldots, \overbrace{\lambda_r, \ldots, \lambda_r}^{x_r}, \overbrace{0, \ldots, 0}^{m_0^+};\\
&\overbrace{\lambda_1, \ldots, \lambda_1}^{y_1}, \ldots, \overbrace{\lambda_r, \ldots, \lambda_r}^{y_r}, \overbrace{0, \ldots, 0}^{m_0^-})
\end{aligned} \tag{6.6}$$

with $m_0^+ = [(m_0 + 1)/2]$ and $m_0^- = [m_0/2]$.

As before with $\mathfrak{q}' = \mathfrak{q}'(\lambda') = \mathfrak{l}' \oplus \mathfrak{u}'$

$$\mu' = \lambda' + \rho(\mathfrak{u}' \cap \mathfrak{p}') \Leftrightarrow \rho(\mathfrak{u}' \cap \mathfrak{k}') + \mu_{L'}. \tag{6.7}$$

We assume first that m_0 is even, and compute

$$\begin{aligned}
\mu' \;=\; &(\overbrace{\alpha'_1, \ldots, \alpha'_1}^{x_1}, \ldots, \overbrace{\alpha'_r, \ldots, \alpha'_r}^{x_r}, \overbrace{0, \ldots, 0}^{m_1})\\
&\otimes(\overbrace{\beta'_1, \ldots, \beta'_1}^{x_1}, \ldots, \overbrace{\beta'_r, \ldots, \beta'_r}^{x_r}, \overbrace{0, \ldots, 0}^{m_2}) + \mu_{L'}.
\end{aligned} \tag{6.8a}$$

Here

$$\begin{aligned}
&\alpha'_i = \lambda_i \Leftrightarrow \frac{p \Leftrightarrow q}{2} + (\widetilde{x}_{i-1} \Leftrightarrow \widetilde{y}_{i-1}) + \tfrac{1}{2}(x_i \Leftrightarrow y_i) + \tfrac{1}{2},\\
&\beta'_i = \lambda_i + \frac{p \Leftrightarrow q}{2} + (\widetilde{y}_{i-1} \Leftrightarrow \widetilde{x}_{i-1}) + \tfrac{1}{2}(y_i \Leftrightarrow x_i) + \tfrac{1}{2}.
\end{aligned} \tag{6.8b}$$

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Now $\mathfrak{l}_0' \simeq \prod_{i=1}^r \mathfrak{u}(x_i, y_i) \oplus \mathfrak{o}(m_0+1, m_0)$, and on the unitary group factors fine $\mathfrak{k}' \cap \mathfrak{l}_0'$-types are as in (6.4a). On $\mathrm{O}(m_0+1, m_0)$, the fine $L' \cap K'$ type $\mu_{L'}$ is of the form

$$(0, \ldots, 0; \delta) \otimes (\overbrace{1, \ldots, 1}^{s}, 0, \ldots, 0; \varepsilon). \tag{6.9}$$

A similar statement holds for $\widetilde{\mathrm{O}}(m_0+1, m_0)$, upon tensoring with ξ.

It follows from Proposition 2.1 that if μ is any $\widetilde{K}$-type for $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ of the form (6.2), then $\mu' = \mathcal{H}(\mu)$ is of the form (6.7). In (6.7) we take $\mu_{L'}$ to be the same as μ_L on the unitary group factors, and on $\mathrm{O}(m_0+1, m_0)$ it is given by (6.9) with

$$(\delta, s, \varepsilon) = \begin{cases} (+1, m_2, +1) & 0 \leqslant 2m_2 \leqslant m_0 \\ (+1, m_1, \Leftrightarrow 1) & m_0 < 2m_2 \leqslant 2m_0. \end{cases}$$

The other case $(p = 2x + m_0)$ is similar. This completes the proof of Proposition 6.1. □

PROPOSITION 6.10. (a) *Let* $\pi = \overline{X}(\gamma)$ *be an irreducible genuine representation of* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$. *Write* $\overline{\gamma} = (\lambda, \nu)$ *and* λ *as in* (6.3), *and let* $\mathfrak{q} = \mathfrak{q}(\lambda) = \mathfrak{l} \oplus \mathfrak{u}$. *Then the lowest* K*-types of* $X(\gamma)$ *are of the form*

$$\mu = \lambda + \rho(\mathfrak{u} \cap \mathfrak{p}) \Leftrightarrow \rho(\mathfrak{u} \cap \mathfrak{k}) + \mu_L$$

such that all the possible μ_L *have the same restriction to* $\mathfrak{sp}(2m_0, \mathbb{R})$. *Thus* μ_L *is trivial except on* $\mathfrak{sp}$ *and on factors* $\mathfrak{u}(x_i, y_i)$ *of* $\mathfrak{l}_0$ *with* $x_i = y_i$ *and* $\lambda_i \in \mathbb{Z} + \frac{1}{2}$.

(b) *The analogous statement holds for* $\widetilde{\mathrm{O}}(p, q)$.

Proof. This follows from the preceding discussion, and the following Lemma.

LEMMA 6.11. *Let* π *be a genuine principal series representation of* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ *or* $\widetilde{\mathrm{O}}(n+1, n)$. *Then* π *contains a unique fine* K*-type.*

Proof. For $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, let $\overline{A} = (\widetilde{\mathbb{R}^*})^n$ as in Section 1. We consider Γ as a character of $\overline{A}$. Write

$$\Gamma = |x|^{\nu_i}\, \mathrm{sgn}(x)^{\delta_i} \chi^{-1}$$

on the ith factor $(\nu_i \in \mathbb{C}, \delta_i = 0, 1)$. Let $n_1 = \sum_i \delta_i$, and $n_0 = n \Leftrightarrow n_1$. Then by Frobenius reciprocity

$$(\overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{n_0}, \overbrace{\Leftrightarrow\tfrac{1}{2}, \ldots, \Leftrightarrow\tfrac{1}{2}}^{n_1})$$

is the unique fine K-type in the corresponding induced representation.

The proof for $\widetilde{\mathrm{O}}(p, q)$ is similar. We omit the details. This completes the proof of Proposition 6.10. □

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We now describe character data in more detail, and relate this to inducing data (cf. Section 4). Let $(\widetilde{H},\gamma)$ be a limit character for $G=\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. As in (6.3) write

$$\begin{aligned}\lambda \;=\; &(\overbrace{\lambda_1,\dots,\lambda_1}^{x_1},\dots,\overbrace{\lambda_r,\dots,\lambda_r}^{x_r},\overbrace{0,\dots,0}^{m_0},\\ &\overbrace{-\lambda_r,\dots,-\lambda_r}^{y_r},\dots,\overbrace{-\lambda_1,\dots,-\lambda_1}^{y_1}),\quad 2\lambda_i\in\mathbb{Z}.\end{aligned}\tag{6.12a}$$

and corresponding to this write

$$\begin{aligned}\nu \;=\; &(\alpha_1^1,\dots,\alpha_1^{x_1},\dots,\alpha_r^1,\dots,\alpha_r^{x_r},\nu_1,\dots,\nu_{m_0},\\ &\beta_r^1,\dots,\beta_r^{y_r},\dots,\beta_1^1,\dots,\beta_1^{y_1}).\end{aligned}\tag{6.12b}$$

For any i, (because we may conjugate by the stabilizer of λ in K) we may assume $\beta_i^j=\alpha_i^j$ for all $j\leqslant\min(x_i,y_i)$. For the parameter to be genuine, we also need $x_i\neq y_i\Rightarrow\lambda_i\in\mathbb{Z}+\frac{1}{2}$, $x_i>y_i\Rightarrow\alpha_i^{x_i}=0$, and $y_i>x_i\Rightarrow\beta_i^{y_i}=0$.

For each i let

$$\begin{aligned}&\ell_i=\begin{cases}x_i=y_i & \lambda_i\in\mathbb{Z},\\ |\{j\leqslant\min(x_i,y_i)\mid\alpha_i^j\neq 0\}| & \lambda_i\in\mathbb{Z}+\frac{1}{2},\end{cases}\\ &x_i'=x_i-\ell_i,\qquad y_i'=y_i-\ell_i,\qquad t_i=x_i'+y_i'.\end{aligned}\tag{6.13a}$$

Then set

$$\ell=\sum_i\ell_i,\qquad t=\sum_i t_i.\tag{6.13b}$$

Let $\overline{H}$ be the covering group of H defined in Section 1, and let $M,\widetilde{M}$ and $\overline{M}$ be as in Section 4. In fact $\overline{H}$ and $\overline{M}$ are determined by $\overline{\gamma}$: $\overline{H}$ is isomorphic to

$$\left(\widetilde{\mathbb{R}^*}\right)^{m_0}\times(\mathbb{C}^*)^{\ell}\times\widetilde{U(1)}^{\,t}$$

and $\overline{M}$ is isomorphic to

$$\widetilde{\mathrm{GL}}(1)^{m_0}\times\widetilde{\mathrm{GL}}(2)^{\ell}\times\widetilde{\mathrm{Sp}}(2t,\mathbb{R}).$$

(That $\overline{H}$ is so determined is due to the condition that γ satisfies condition (4.4b); see the proof of Proposition 6.15.)

Now γ determines a (relative) discrete series representation $\overline{\sigma}$ of $\overline{M}$, explicitly described as follows. For each i there are ℓ_i limits of (relative) discrete series representations of $\widetilde{\mathrm{GL}}(2)$, all with lowest K-type $(2\lambda_i+1)\chi$, and the center acting by $|\det|^{\alpha_i^j}$.

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On $\widetilde{\mathrm{Sp}}(2t,\mathbb{R})$ there is a limit of discrete series representation with Harish-Chandra parameter

$$(\overbrace{\lambda_1,\ldots,\lambda_1}^{x'_1},\ldots,\overbrace{\lambda_r,\ldots,\lambda_r}^{x'_r},\overbrace{\Leftrightarrow\lambda_r,\ldots,\Leftrightarrow\lambda_r}^{y'_r},\ldots,\overbrace{\Leftrightarrow\lambda_1,\ldots,\Leftrightarrow\lambda_1}^{y'_1}). \tag{6.14a}$$

The positive imaginary roots on this factor are the corresponding restriction of Ψ.

The character Γ of $\widetilde{H}$ satisfies $d\Gamma = \overline{\gamma} + \rho(\Psi) \Leftrightarrow 2\rho_c(\Psi)$. Thus Γ is determined by $\overline{\gamma}$ and Ψ except on the cover of the $\mathbb{R}^*$ factors. We consider Γ as a character of $\overline{H}$, genuine on each factor for which the cover is non-trivial (so Γ factors to a genuine character of $\widetilde{H}$). For $i = 1,\ldots,m_0$ write Γ on the corresponding factor of $\widetilde{\mathbb{R}^*}$ as

$$\Gamma(x,\varepsilon) = |x|^{\nu_i}\operatorname{sgn}(x)^{\delta_i}\chi(x,\varepsilon)^{-1}. \tag{6.14b}$$

PROPOSITION 6.15. $(\widetilde{H},\gamma)$ *satisfies condition* (4.4b) *if and only if* $\nu_i = \pm\nu_j \Rightarrow \delta_i = \delta_j$.

Proof. Suppose $\langle\alpha,\overline{\gamma}\rangle = 0$. If α is a long root and γ is data for a genuine representation, then $\Gamma(m_\alpha) = \pm i$, and (4.4b) is immediate for these roots. If α is a short root on a factor of $\mathbb{C}^*$ then a straightforward calculation shows that (4.4b) holds if and only if $\lambda_i \in \mathbb{Z}$. This is taken care of by our choice of H: the short real roots for which $\alpha(\overline{\gamma}) = 0$ and $\lambda_i \in \mathbb{Z} + \frac{1}{2}$ are imaginary. If α is a short real root on the factors of $\mathbb{R}^*$ then a similar calculation shows (4.4b) is equivalent to the condition stated in the Proposition.

The preceding steps may be reversed to express character data in terms of inducing data. □

We turn next to an orthogonal group $\mathrm{O}(p,q) = \mathrm{O}(2p_0+1,2q_0)$. Let γ be a limit character, and write

$$\begin{aligned}\lambda \;&=\; (\overbrace{\lambda_1,\ldots,\lambda_1}^{x_1},\ldots,\overbrace{\lambda_r,\ldots,\lambda_r}^{x_r},\overbrace{0,\ldots,0}^{m_0^+})\\ &\quad\otimes(\overbrace{\lambda_1,\ldots,\lambda_1}^{y_1},\ldots,\overbrace{\lambda_r,\ldots,\lambda_r}^{y_r},\overbrace{0,\ldots,0}^{m_0^-})\end{aligned} \tag{6.16a}$$

as in (6.6). Then write the real part of the parameter as

$$\begin{aligned}\nu \;&=\; (\alpha_1^1,\ldots,\alpha_1^{x_1},\ldots,\alpha_r^1,\ldots,\alpha_r^{x_r},\nu_1,\ldots,\nu_{m_0^+})\\ &\quad\otimes(\beta_1^1,\ldots,\beta_1^{y_1},\ldots,\beta_r^1,\ldots,\beta_r^{y_r},\nu_{m_0^++1},\ldots,\nu_{m_0^++m_0^-}).\end{aligned} \tag{6.16b}$$

The corresponding Cartan subgroup H of $\mathrm{SO}(p,q)$ is isomorphic to

$$(\mathbb{R}^*)^{m_0^++m_0^-}\times(\mathbb{C}^*)^{\ell}\times U(1)^t$$

where ℓ and t are defined as in the previous case (cf. (6.12c)). Again Γ is determined by $\overline{\gamma}$ except on the copies of $\mathbb{R}^*$. Write $\Gamma(x) = |x|^{\nu_i}\,\mathrm{sgn}(x)^{\delta_i}$ on these terms.

LEMMA 6.17. *γ satisfies condition* (4.4b) *if and only if* $\nu_i = \pm\nu_j \Rightarrow \delta_i = \delta_j$.

Proof. If α is a short root then $m_\alpha = 1$ and $\varepsilon_\alpha = \Leftrightarrow 1$ [18, 8.3.8 and 8.3.11]. The first fact comes down to the isomorphism $\mathrm{SO}(2,1) \simeq \mathrm{PGL}(2,\mathbb{R})$, and the second from a straightforward calculation that the integers d_i of [18, 8.3.9] are even. Thus (4.4b) is automatic for these roots. The proof for the other roots is the same as for the symplectic group. We omit the details. □

This result extends in the obvious way to $\mathrm{O}(p,q)$ and $\widetilde{\mathrm{O}}(p,q)$.

PROPOSITION 6.18. *Let* $\widetilde{G} = \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ *or* $\widetilde{\mathrm{O}}(p,q)$ *and let*

$$\mathrm{Ind}_{\widetilde{M}N}^{\widetilde{G}}(\sigma) \tag{6.18a}$$

be a standard module for $\widetilde{G}$ (*cf. Section* 4). *Let* μ *be a minimal* $\widetilde{K}$*-type of* (6.18), *and suppose* μ *is* p,q*-harmonic. Let* μ_M *be the* (*unique*) *minimal* $\widetilde{K\cap M}$*-type of* σ. *Then* $\deg_{p,q}(\mu) = \deg_{p,q}(\mu_M)$, *and* μ_M *is contained in the restriction of* μ *to* $\widetilde{K\cap M}$.

Proof. We first consider $\widetilde{G} = \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Write $\mathcal{V}(\mu) = \lambda$ as in (6.3), μ as in (6.5a), and other notation as in (6.5b). By the proof of Lemma 6.11 we have $m_1 = \sum_i \delta_i$ with Γ written as in (6.13).

Then $p = 2\widetilde{x} + m_0 + 1$, $q = 2\widetilde{y} + m_0$ or $p = 2\widetilde{x} + m_0$, $q = 2\widetilde{y} + m_0 + 1$. We consider only the first case, the second is similar. Let $z = (p \Leftrightarrow q)/2 = \widetilde{x} \Leftrightarrow \widetilde{y} + \frac{1}{2}$.

By Proposition 2.1, $\deg_{p,q}(\mu) = \sum_i x_i|\alpha_i \Leftrightarrow z| + \sum_i y_i|\beta_i \Leftrightarrow z| + m_2$.

It is not hard to see that $\alpha_i \Leftrightarrow z \geqslant 0 \geqslant \beta_i \Leftrightarrow z$. This implies the degree of μ is the sum of the following terms

$$= \sum_i \ell_i(2\lambda_i + 1) \tag{6.19a}$$

$$+ \sum_i x_i'(\alpha_i \Leftrightarrow z) \tag{6.19b}$$

$$\Leftrightarrow \sum_i y_i'(\beta_i \Leftrightarrow z) \tag{6.19c}$$

$$+ m_2 \tag{6.19d}$$

On the other hand with σ described preceding Proposition 6.15 we compute the lowest $\widetilde{K\cap M}$-type μ_M of σ. We pull this back to the group $\overline{M}$. The degree of $\overline{\mu}_M$ is the sum of the degrees of the factors. With notation as in (6.13), on each of the ℓ_i factors of type $\widetilde{\mathrm{GL}}(2)$, the degree of μ_M is $2\lambda_i + 1$. This contributes

$$\sum_i \ell_i(2\lambda_i + 1) \quad \text{(independent of } \varepsilon_i) \tag{6.20a}$$

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to the degree of $\overline{\mu}_M$.

On the $\widetilde{\mathrm{Sp}}(2t,\mathbb{R})$ factor, σ is a limit of discrete series representation with Harish-Chandra parameter λ given by (6.14a). Define $\widetilde{x}', \widetilde{y}', \alpha_i'$ *etc.* by applying (6.5b). Then μ_M on this factor is $(2\widetilde{x}'+1, 2\widetilde{y}')$-harmonic. With $z' = (2\widetilde{x}'+1 \Leftrightarrow 2\widetilde{y}')/2$, we see that the degree of μ_M is the sum of

$$\sum_i x_i'|\alpha_i' \Leftrightarrow z'|, \tag{6.20b}$$

$$\sum_i y_i'|\beta_i' \Leftrightarrow z'|. \tag{6.20c}$$

By (6.13a) $x_i' \Leftrightarrow y_i' = x_i \Leftrightarrow y_i$ for all i. Therefore by (6.5), $\widetilde{x}_i' \Leftrightarrow \widetilde{y}_i' = \widetilde{x}_i \Leftrightarrow \widetilde{y}_i$, $\alpha_i' = \alpha_i$, and $\beta_i' = \beta_i$ for all i. Also $\widetilde{x}' \Leftrightarrow \widetilde{y}' = \widetilde{x} \Leftrightarrow \widetilde{y}$ and so $z' = z$. It follows that (6.20b) (resp. (c)) equals (6.16b) (resp. (c)).

Finally on each $\widetilde{\mathrm{GL}}(1)$ factor, the degree is δ_i, which gives a contribution of

$$m_2 \tag{6.20d}$$

to the degree of $\overline{\mu}_M$. Comparing (6.16) to (6.20) we conclude that $\deg(\mu) = \deg(\mu_M)$. This calculation also shows that the highest weight of μ_M is the same as the highest weight of μ (independent of the choice of μ), proving the last claim of the Proposition in this case.

The proof for $\widetilde{\mathrm{O}}(p,q)$ is similar. We won't use this fact, and so we leave the details to the reader. □

PROPOSITION 6.21. *Let μ be a lowest $\widetilde{K}$-type of an irreducible genuine representation π of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$, and choose p, q so that μ is p, q-harmonic. Then μ is of lowest p, q-degree in π, and in the standard module of π.*

Proof. Obviously it is enough to prove the second claim. We use notation as in the proof of Proposition 6.18. In particular write $\pi = X(\gamma)$, λ as in (6.3) and μ as in (6.5a). Let p, q and z be as in the proof of Proposition 6.18, and write $\deg = \deg_{p,q}$. For any k-tuple write $\deg_z(x_1,\dots,x_k) = \sum_i |x_i \Leftrightarrow z|$. With $\widetilde{x}, \widetilde{y}, m_0$ as in (6.5) (computed for μ), write any $\widetilde{K}$-type γ as

$$\gamma = (\gamma_+, \gamma_0, \gamma_-) = (a_1,\dots,a_{\widetilde{x}}, c_1,\dots,c_{m_0}, b_{\widetilde{y}},\dots,b_1). \tag{6.22}$$

Then $\deg(\gamma) = \deg_z(\gamma_+) + \deg_z(\gamma_0) + \deg_z(\gamma_-)$. By (6.5a) μ satisfies

$$\alpha_1,\dots,\alpha_{\widetilde{x}} \geqslant z \geqslant \beta_{\widetilde{y}},\dots,\beta_1. \tag{6.23}$$

The standard module $X(\gamma)$ may be realized as a derived functor module from the parabolic subalgebra $\mathfrak{q} = \mathfrak{q}(\lambda)$ of $\mathfrak{g}$ [18]. By the generalized Blattner formula ([18], Theorem 6.3.12) the highest weight of any $\widetilde{K}$-type μ' of π may be written

$$\mu' = \mu + \sum_\alpha m_\alpha \alpha \quad (m_\alpha \geqslant 0), \tag{6.24}$$

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where the sum runs over the roots of $\mathfrak{t}_c$ in $\mathfrak{l}$ and $\mathfrak{u}\cap\mathfrak{p}$, and μ is a lowest $\widetilde{K}$-type of π. Furthermore if the sum is restricted to roots in $\mathfrak{l}$, then the resulting weight is the highest weight of a $\widetilde{K\cap L}$-type in the corresponding principal series representation $X_L(\gamma)$ of $\widetilde{L}$.

In our coordinates these roots are those in the following table

$$\varepsilon_i \pm \varepsilon_j \quad 1 \leqslant i,j \leqslant \tilde{x}, \tag{6.25a}$$

$$\Leftrightarrow\varepsilon_i \pm \varepsilon_j \quad \tilde{x}+m_0 < i,j \leqslant n, \tag{6.25b}$$

$$\pm(\varepsilon_i + \varepsilon_j) \quad 1 \leqslant i \leqslant \tilde{x}, \quad \tilde{x}+m_0 < j \leqslant n, \tag{6.25c}$$

$$\varepsilon_i + \varepsilon_j, 2\varepsilon_i \quad 1 \leqslant i \leqslant \tilde{x} < j \leqslant \tilde{x}+m_0, \tag{6.25d}$$

$$\Leftrightarrow\varepsilon_i \Leftrightarrow\varepsilon_j, \Leftrightarrow 2\varepsilon_j \quad \tilde{x} < i \leqslant \tilde{x}+m_0 < j \leqslant n, \tag{6.25e}$$

$$\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_j \quad \tilde{x} < i,j \leqslant \tilde{x}+m_0. \tag{6.25f}$$

These roots also satisfy

$$\langle\alpha,\lambda\rangle \geqslant 0,\ \text{and if } \langle\alpha,\lambda\rangle > 0 \text{ then } \alpha \text{ is not of the form } \varepsilon_i \Leftrightarrow\varepsilon_j.$$

LEMMA 6.26. *If γ is any k-tuple satisfying* (6.23), *then*

$$\deg\left(\gamma + \sum_{a,b,c} m_\alpha\alpha\right) \geqslant \deg(\gamma)$$

for any $m_\alpha \geqslant 0$. Here $\sum_{a,b,c}$ denotes a sum over roots of the form (6.25a,b,c).

Proof. Adding roots of the form $\varepsilon_i+\varepsilon_j$ and $\Leftrightarrow\varepsilon_i \Leftrightarrow\varepsilon_j$ of type (6.25a–b) changes γ to a γ' satisfying (6.23) and such that $\deg(\gamma') \geqslant \deg(\gamma)$, with equality if and only if all $m_\alpha = 0$. So we may as well assume that no such roots occur. Then

$$\begin{aligned}
&\gamma + \sum_{a,b,c} m_\alpha\alpha \\
&\quad = (\ldots, a_i + k_i + \alpha_i, \ldots, \ldots, c_1, \ldots, c_{m_0}, \ldots b_j + \ell_j + \beta_j, \ldots)
\end{aligned}$$

where $\sum k_i = \sum l_j$ and $\sum \alpha_i = \sum \beta_j = 0$. Then

$$\begin{aligned}
&\sum |a_i + \alpha_i + k_i \Leftrightarrow z| + \sum |b_j + \beta_j + \ell_j \Leftrightarrow z| \\
&\quad \geqslant \sum (a_i \Leftrightarrow z + \alpha_i + k_i) + \sum (z \Leftrightarrow b_j \Leftrightarrow \beta_j \Leftrightarrow \ell_j) \\
&\quad = \sum (a_i \Leftrightarrow z) + \sum (z \Leftrightarrow b_j).
\end{aligned}$$

The claim follows. $\square$

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Lemma 6.26 applies to $\mu + \sum_{d,e,f} m_\alpha \alpha$, and gives

$$\deg(\mu') \geqslant \deg\left(\mu + \sum_{d,e,f} m_\alpha \alpha\right). \tag{6.27a}$$

The right-hand side of (6.27a) equals

$$\begin{aligned} &\deg_z\left(\left[\mu + \sum_d m_\alpha \alpha\right]_+\right) + \deg_z\left(\left[\mu + \sum_{d,e,f} m_\alpha \alpha\right]_0\right) \\ &\quad + \deg_z\left(\left[\mu + \sum_e m_\alpha \alpha\right]_-\right) && (6.27\text{b}) \\ &\geqslant \deg_z(\mu_+) + \deg_z\left(\left[\mu + \sum_{d,e,f} m_\alpha \alpha\right]_0\right) + \sum_{d,e} m_\alpha + \deg_z(\mu_-) && (6.27\text{c}) \\ &\geqslant \deg_z(\mu_+) + \deg_z\left(\left[\mu + \sum_f m_\alpha \alpha\right]_0\right) + \deg_z(\mu_-). && (6.27\text{d}) \end{aligned}$$

Here (d) follows from repeated applications of the inequality $\deg_z([\gamma + \alpha]_0) + 1 \geqslant \deg_z(\gamma_0)$ for any weight γ and α in (6.25d,e).

Separating the sum (6.24) into roots of $\mathfrak{l}$ and $\mathfrak{u} \cap \mathfrak{p}$ gives $[\mu + \sum_f m_\alpha \alpha]_0 = [\mu + \sum_{\alpha \in \Delta(\mathfrak{t}_c, \mathfrak{l})} m_\alpha \alpha]_0$, so by the discussion following (6.24) this is a $\widetilde{K \cap L}$-type of a principal series representation of $\widetilde{\mathrm{Sp}}(2m_0, \mathbb{R})$. This has (unique) lowest $\widetilde{K \cap L}$-type

$$(\overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{m_1}, \overbrace{\Leftrightarrow\tfrac{1}{2}, \ldots, \Leftrightarrow\tfrac{1}{2}}^{m_2})$$

(cf. (6.5a)).

LEMMA 6.28. *Let π be a minimal principal series representation of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, containing the* (*unique*) *fine K-type*

$$\mu = (\overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{m_1}, \overbrace{\Leftrightarrow\tfrac{1}{2}, \ldots, \Leftrightarrow\tfrac{1}{2}}^{m_2})$$

(*cf. Lemma* 6.11). *Then μ is of lowest $n+1, n$-degree in π.*

Proof. This follows easily from Frobenius reciprocity. We omit the details. □

Proposition 6.21 follows from (6.27d) and Lemma 6.28. □

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We only need part (1) of the next Proposition for the proof of the main results in Section 5.

PROPOSITION 6.29. *Let μ be a lowest $\widetilde{K}'$-type of an irreducible genuine representation π of $\widetilde{\mathrm{O}}(p,q)$.*

(1) *If π is a discrete series representation then μ is of lowest degree in π.*
(2) *For any π, assume μ occurs in the space of joint harmonics. Then μ is of lowest degree in π.*

Proof. We may safely ignore the covering groups, and for the remainder of this section we let $G = \mathrm{O}(p,q) = \mathrm{O}(2p_0+1, 2q_0)$, $K = \mathrm{O}(p) \times \mathrm{O}(q)$, *etc.*

Let π be a discrete series representation of G with Harish-Chandra parameter $\lambda = (\lambda_0; \varepsilon)$ (cf. Section 3) and lowest K-type μ. Let μ' be any K-type of π.

Suppose μ is of the form

$$\mu = (a_1, \ldots, a_r, 0, \ldots, 0; \varepsilon) \otimes (b_1, \ldots, b_{q_0}; \eta) \tag{6.30a}$$

for some $0 \leqslant r \leqslant p_0$ (cf. Section 3). Write any K-type γ as $\gamma = (\gamma_+, \gamma_0, \gamma_-)$ with $\gamma_+ = (a_1, \ldots, a_r)$, $\gamma_- = (b_1, \ldots, b_{q_0}; \eta)$ and $\gamma_0 = (a_{r+1}, \ldots, a_{p_0}; \varepsilon)$. Then $\deg(\gamma) = \deg_0(\gamma_+) + \deg(\gamma_0) + \deg(\gamma_-)$ where the second and third terms are for the groups $\mathrm{O}(p \Leftrightarrow 2r)$ and $\mathrm{O}(q)$ respectively.

It follows from the formula $\mu = \lambda_0 + \rho(\mathfrak{u} \cap \mathfrak{p}) \Leftrightarrow \rho(\mathfrak{u} \cap \mathfrak{p})$ that λ_0 may be written

$$\lambda_0 = (\alpha_1, \ldots, \alpha_r, p_0 \Leftrightarrow r \Leftrightarrow \tfrac{1}{2}, \ldots, \tfrac{3}{2}, \tfrac{1}{2}; \beta_1, \ldots, \beta_{q_0}). \tag{6.30b}$$

By induction by stages ([18], Corollary 6.3.10) it follows that π may be realized as a derived functor module for $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ with $L \simeq U(1)^{q_0+r} \times \mathrm{O}(p \Leftrightarrow 2r)$, from a one-dimensional representation π_L of L. The $\mathrm{O}(p \Leftrightarrow 2r)$ component of π_L is the one-dimensional representation $(0, \ldots, 0; \varepsilon)$ realized on the space $\mathbb{C}_\varepsilon$. By the Blattner formula (6.24) it follows that

$$\mu' = \mu + \sum_\alpha m_\alpha \alpha \tag{6.31}$$

with $\alpha \in \mathfrak{u} \cap \mathfrak{p}$.

The roots of $\mathfrak{u} \cap \mathfrak{p}$ are (among those) of the form

$$\pm(e_i \Leftrightarrow e_j), e_i + e_j, e_j \quad 1 \leqslant i \leqslant r, \quad p_0 + 1 \leqslant j \leqslant n \tag{6.32a}$$

$$\pm\varepsilon_i + \varepsilon_j, \quad r+1 \leqslant i \leqslant p_0, \quad p_0 + 1 \leqslant j \leqslant n \tag{6.32b}$$

As in (6.27a–c) it is immediate that

$$\deg(\mu') \geqslant \deg\left(\mu + \sum_b m_\alpha \alpha\right) \tag{6.33a}$$

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$$\geqslant \deg(\mu_+) + \deg\left(\left[\mu + \sum_b m_\alpha \alpha\right]_0\right) + \deg(\mu_-)$$
$$+ \sum_b m_\alpha \deg(\mu_-) \tag{6.33b}$$

where the subscripts denote the roots of (6.32a,b). Noting that $[\mu + \sum_b m_\alpha \alpha]_0 = \mu'_0$, it is enough to show

$$\deg(\mu'_0) + \sum_b m_\alpha \geqslant \deg(\mu_0). \tag{6.34}$$

This requires a refinement of the Blattner formula. We only consider the factor $\mathrm{O}(p \Leftrightarrow 2r)$ of L. Let $\tau,\ \tau'$ be the $\mathrm{O}(p \Leftrightarrow 2r)$ factor of $H^0(\mathfrak{u} \cap \mathfrak{k}, V_\mu)$ and $H^0(\mathfrak{u} \cap \mathfrak{k}, V_{\mu'})$ respectively. These are the finite dimensional $L \cap K$-modules with the same highest weight as μ and μ' respectively. It follows from [18, 6.3.12] (a sharpening of (6.31)) that (with $m = \sum_b m_\alpha$),

$$\mathrm{mult}[\tau' : \tau \otimes S^m(\mathfrak{u} \cap \mathfrak{p})|_{\mathrm{O}(p-2r)}] > 0. \tag{6.35}$$

Recall from Section 1 that we realized each representation of $\mathrm{O}(p \Leftrightarrow 2r)$ as the highest weight factor in a representation of $U(p \Leftrightarrow 2r)$. As a module for $\mathrm{O}(p \Leftrightarrow 2r)$, $\mathfrak{u} \cap \mathfrak{p}$ is isomorphic to a direct sum of copies of the standard module; thus it is a module for $U(p \Leftrightarrow 2r)$ as well. So we can decompose $S^m(\mathfrak{u} \cap \mathfrak{p})$ with respect to $U(p \Leftrightarrow 2r)$ and then restrict to $\mathrm{O}(p \Leftrightarrow 2r)$.

It follows that the highest weight of any irreducible summand of $S^m(\mathfrak{u} \cap \mathfrak{p})$, is of the form

$$(c_1, \dots, c_{p-2r}) \quad \text{with} \quad \sum c_i = m, \quad c_i \geqslant 0 \text{ for all } i,$$

when written as a weight for $U(p \Leftrightarrow 2r)$.

LEMMA 6.36. *Let τ, τ' be irreducible representations of* $\mathrm{O}(n)$, *with τ one-dimensional. Let γ be an irreducible representation of $U(n)$ with highest weight* $(c_1, \dots, c_n)$, $c_i \geqslant 0$. *Suppose*

$$\mathrm{mult}[\tau' : \tau \otimes (\gamma|_{\mathrm{O}(n)})] > 0.$$

Then $\deg(\tau') + \sum_{i=1}^n c_i \geqslant \deg(\tau)$.

Proof. This is obvious if τ is trivial, so assume $\tau = \mathrm{sgn}$. Replacing τ' with $\tau' \otimes \mathrm{sgn}$ it is enough to show

$$\mathrm{mult}[\tau' : \gamma|_{\mathrm{O}(n)}] > 0 \Rightarrow \deg(\tau' \otimes \mathrm{sgn}) + \sum c_i \geqslant n. \tag{6.37}$$

Write $\tau' = (a_1, \ldots, a_k, 0, \ldots, 0; \varepsilon)$, so $\deg(\tau') = \sum a_i + ((1 \Leftrightarrow \varepsilon)/2)(n \Leftrightarrow 2k)$. If the multiplicity is greater than zero, then γ must contain a vector of weight $(d_1, \ldots, d_n)$ which is a highest weight vector for τ'. It follows that

$$d_i \Leftrightarrow d_{n+1-i} = a_i, \quad i \leqslant k, \tag{6.38a}$$

$$d_i \Leftrightarrow d_{n+1-i} = 0, \quad k+1 \leqslant i \leqslant [\tfrac{1}{2}n], \tag{6.38b}$$

$$d_i \equiv d_{n+1-i} \equiv \frac{1 \Leftrightarrow \varepsilon}{2} \bmod (2), \quad k+1 \leqslant i \leqslant [\tfrac{1}{2}n], \tag{6.38c}$$

$$d_{(n+1)/2} \equiv \frac{1 \Leftrightarrow \varepsilon}{2} \bmod (2) \quad \text{if } n \text{ is odd.} \tag{6.38d}$$

In addition, the relations $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} c_i$ and $d_i \geqslant 0$ for all i, hold.

By (6.38a), $d_i \geqslant a_i \geqslant 1$ for $1 \leqslant i \leqslant k$. Thus

$$\sum_1^k (a_i + d_i) \geqslant 2k. \tag{6.39}$$

If $\varepsilon = 1$, then (6.37) becomes $\sum_1^k a_i + \sum_1^n d_i \geqslant 2k$, which is immediate from (6.38). Assume $\varepsilon = \Leftrightarrow 1$. We need to show $\sum_1^k (a_i + d_i) + \sum_{k+1}^n d_i \geqslant n$. By (6.38c) and (6.38d), we get $d_i \geqslant 1$ for $k+1 \leqslant i \leqslant n \Leftrightarrow k$. The assertion follows from this together with (6.39). □

This also completes the proof of Proposition 6.29(1). □

Part (2) of the Proposition may be proved similarly, using an extensionion of Lemma 6.36 to general τ, and a version of Lemma 6.28 for $\mathrm{O}(p,q)$. Since we won't need it we omit the details, but we note that it is also an immediate consequence of Theorem 5.1.

7. Occurence of the discrete series

In this section we prove that the entire genuine discrete series of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, and half of the genuine discrete series of $\widetilde{\mathrm{O}}(p,q)$, occur in the correspondence. We assume $p+q = 2n+1$ throughout, and fix ψ. The arguments hold for $p+q = 2n$ as well, recovering some of the results of [11]. We depart from our convention of Section 1 and let $\widetilde{\mathrm{O}}(p,q) = \mathrm{O}(p,q)$ if n is even, and we let $\widetilde{\mathrm{Sp}}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R})$ when considering a dual pair $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(p,q))$ with $p+q$ even.

PROPOSITION 7.1. (1) *Let* π *be a genuine discrete series representation of* $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. *Then, for any* $\delta = \pm 1$, π *occurs in the correspondence with some* $\widetilde{\mathrm{O}}(p,q)$, $(\Leftrightarrow 1)^q = \delta$.

(2) *Let* π *be a genuine discrete series representation of* $\widetilde{\mathrm{O}}(p,q)$. *Then precisely one of* π *and* $\pi \otimes \mathrm{sgn}$ *occurs in the correspondence with* $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$.

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This follows from a doubling of variables argument due to Kudla and Rallis. The proof is divided into a series of Lemmas.

For an irreducible representation π of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ (resp. $\widetilde{\mathrm{O}}(p,q)$), let $R_{n,p,q}$ denote the maximal quotient of $\omega_{n,p,q}$ on which $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ (resp. $\widetilde{\mathrm{O}}(p,q)$) acts by a multiple of π [5].

We embed $\mathrm{Sp}(2n,\mathbb{R})\times\mathrm{Sp}(2n,\mathbb{R})$ diagonally in $\mathrm{Sp}(4n,\mathbb{R})$. This induces a natural map $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})\times\widetilde{\mathrm{Sp}}(2n,\mathbb{R})\to\widetilde{\mathrm{Sp}}(4n,\mathbb{R})$. Similarly $\widetilde{\mathrm{O}}(p,q)\times\widetilde{\mathrm{O}}(q,p)$ maps to $\widetilde{\mathrm{O}}(p+q,p+q)$.

LEMMA 7.2. (1) *For any irreducible representation* π *of* $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$,

$$\theta_{p,q}(\pi)\neq 0\Leftrightarrow \mathrm{Hom}_{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})\times\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(R_{2n,p,q}(1\!\!1),\pi\otimes\pi)\neq 0. \tag{7.3a}$$

(2) *For any irreducible representation* π *of* $\widetilde{\mathrm{O}}(p,q)$,

$$\theta(\pi)\neq 0\Leftrightarrow \mathrm{Hom}_{\widetilde{\mathrm{O}}(p,q)\times\widetilde{\mathrm{O}}(q,p)}(R_{n,p+q,q+p}(1\!\!1),\pi\otimes\pi)\neq 0. \tag{7.3b}$$

Proof. We prove (1), the proof of (2) is similar. To conserve notation let $G=\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$, $G'=\widetilde{\mathrm{O}}(p,q)$, and let ω_n be the oscillator representation for the dual pair (G,G'). Then according to ([11], I.8),

$$\mathrm{Hom}_G(\omega_n,\pi)\neq 0\Leftrightarrow\mathrm{Hom}_{G\times G'}(\omega_n,\pi\otimes\pi')\neq 0\quad\text{for some }\pi'. \tag{7.4a}$$

Thus if $\mathrm{Hom}_G(\omega_n,\pi)\neq 0$, then $\mathrm{Hom}_{G\times G\times G'\times G'}(\omega_n\otimes\omega_n,\pi\otimes\pi\otimes\pi'\otimes\pi')\neq 0$. Since π' is a genuine representation of $\widetilde{\mathrm{O}}(p,q)$, $(\pi')^*\simeq\pi'\otimes\mathrm{sgn}^n$. Therefore sgn^n is a quotient of $\pi'\otimes\pi'$, which gives

$$\mathrm{Hom}_G(\omega_n,\pi)\neq 0\Rightarrow\mathrm{Hom}_{G\times G\times\Delta(G')}(\omega_n\otimes\omega_n,\pi\otimes\pi\otimes\mathrm{sgn}^n)\neq 0, \tag{7.4b}$$

where $\Delta(G')$ is the diagonal subgroup of $G'\times G'$. By Lemma 1.7 we may replace $\omega_n\otimes\omega_n$ with $\omega_{2n}\otimes\mathrm{sgn}^n$. Thus the right-hand side of (7.4b) is equivalent to

$$\begin{aligned}&\mathrm{Hom}_{G\times G\times\Delta(G')}(\omega_{2n},\pi\otimes\pi\otimes 1\!\!1)\neq 0\\ &\quad\Leftrightarrow\mathrm{Hom}_{G\times G}(R_{2n,p,q}(1\!\!1),\pi\otimes\pi)\neq 0.\end{aligned} \tag{7.4c}$$

Thius proves one direction of the statement. On the other hand, if (7.4c) holds (*i.e.* the right-hand side is nonzero), the same is true for (7.4b), and ignoring the $\Delta(G')$ action, we see that $\mathrm{Hom}_{G\times G}(\omega_n\otimes\omega_n,\pi\otimes\pi)\neq 0$. This is easily seen to imply $\mathrm{Hom}_G(\omega_n,\pi)\neq 0$, proving the Lemma. □

Let $P=MN$ be the stabilizer of the Lagrangian subspace $L_0=\langle e_1,\ldots,e_{2n}\rangle$, and $\widetilde{P}=\widetilde{M}N$ its inverse image in $\widetilde{\mathrm{Sp}}(4n,\mathbb{R})$. For $\alpha\in\mathbb{Z}/4\mathbb{Z}$ we consider the Harish-Chandra module of the induced representation

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(4n,\mathbb{R})}(\chi^\alpha). \tag{7.5a}$$

We are using normalized induction, so this representation is unitarily induced and completely reducible. Similarly, for $G' = \widetilde{\mathrm{O}}(p,q)$, we consider the induced representation

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{O}}(2n+1,2n+1)}(\chi), \tag{7.5b}$$

where the Levi component of $P = MN$ is isomorphic to $\mathrm{GL}(m)$, $\widetilde{M} \simeq M \times \mathbb{Z}/2\mathbb{Z}$, and $\chi = 1\!\!1 \otimes \mathrm{sgn}$.

LEMMA 7.6. (1) *For* $\alpha = \pm 1$

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(\chi^\alpha) \simeq \bigoplus_{\substack{p+q=2n+1\\ p-q=\alpha}} R_{2n,p,q}(1\!\!1).$$

(2)

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{O}}(2n+1,2n+1)}(\chi \otimes 1) \simeq R_{n,2n+1,2n+1}(1\!\!1) \oplus (R_{n,2n+1,2n+1}(1\!\!1) \otimes \mathrm{sgn}).$$

Proof. Part (1) is proved in [9], and both (1) and (2) are in [24]. □

LEMMA 7.7. (1) *Let* π *be a genuine irreducible representation of* $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. *Then* $\theta_{p,q}(\pi) \neq 0$ *for some* p, q *with* $(-1)^q = \delta$ *if and only if*

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})\times\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}[\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(\chi^\alpha), \pi \otimes \pi] \neq 0,$$
$$\textit{for}\ \ \alpha = \delta(-1)^n. \tag{7.7a}$$

(2) *For* π *an irreducible representation of* $\widetilde{\mathrm{O}}(p,q)$, $\theta(\pi) \neq 0$ *or* $\theta(\pi \otimes \mathrm{sgn}) \neq 0$ *if and only if*

$$\mathrm{Hom}_{\widetilde{\mathrm{O}}(p,q)\times\widetilde{\mathrm{O}}(q,p)}[\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{O}}(2n+1,2n+1)}(\chi), \pi \otimes \pi] \neq 0. \tag{7.7b}$$

Proof. This follows immediately from Lemmas 7.2 and 7.6.

LEMMA 7.8. *Let* π *be the Harish-Chandra module of a genuine discrete series representation of* $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ (resp. $\widetilde{\mathrm{O}}(p,q)$). *Then the space* (7.7a) (resp. (b)) *is non-zero*.

Proof. The two cases are similar, so we treat only $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Let $\mathcal{X}$ be the variety of Lagrangian subspaces of $\mathbb{R}^{4n}$. Then $\mathcal{X} \simeq \widetilde{\mathrm{Sp}}(2n,\mathbb{R})/\widetilde{P}$. Let $L^2-I_{2n}(\alpha)$ be the L^2-induced version of (7.5a). This is realized on L^2 sections of the induced bundle $\mathcal{B} = \widetilde{\mathrm{Sp}}(4n,\mathbb{R}) \times_{\widetilde{P}} (\chi^\alpha)$ over $\mathcal{X}$.

Let $L = \langle e_1 + e_{n+1}, \ldots, e_n + e_{2n}, f_1 + f_{n+1}, \ldots, f_n + f_{2n}\rangle$. Then the orbit $\mathcal{O}$ of L by $G = \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Sp}(2n,\mathbb{R})$ is open in $\mathcal{X}$. Let H be the stabilizer of L in G

so $\mathcal{O} \simeq G/H$. Then $H \simeq \mathrm{Sp}(2n,\mathbb{R})$ is embedded in $G \simeq \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Sp}(2n,\mathbb{R})$ via $g \to (g, \tau(g))$ where τ is the outer automorphism of $\mathrm{Sp}(2n,\mathbb{R})$ of 1.4. Let $Z \simeq \mathbb{Z}/2\mathbb{Z}$ be the kernel of the covering map $\widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \to \mathrm{Sp}(2n, R)$. Passing to the coverings we see that the map

$$\mathcal{O} \simeq \widetilde{G}/\widetilde{H} \to \widetilde{\mathrm{Sp}}(2n,\mathbb{R})/Z \tag{7.9a}$$

given by $(g,h,\varepsilon)\widetilde{H} \to (g\tau(h^{-1}),\varepsilon)Z$ is an isomorphism, and induces an isomorphism between the restriction of $\mathcal{B}$ to $\mathcal{O}$ and

$$\widetilde{G} \times_Z \chi|_Z. \tag{7.9b}$$

Under this isomorphism the action of $\widetilde{G}$ on $\mathcal{O}$ becomes

$$(g,h,\varepsilon)\cdot(x,\delta)Z = (gx\tau(h^{-1}),\varepsilon\delta)Z$$

$(g,h,x \in \mathrm{Sp}(2n,\mathbb{R}))$. Thus sections of the bundle (7.9b) are identified with $L^2_\chi(\widetilde{\mathrm{Sp}}(2n,\mathbb{R}))$, *i.e.* L^2 functions on $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ transforming by χ under Z, with $\widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \times \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ acting by conjugation twisted by τ. Since τ takes π to π^* it follows that the discrete spectrum of this space is precisely the sum of $\tau \otimes \tau$ where τ runs over the genuine discrete series representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$.

Therefore there is a nonzero map

$$\phi\colon L^2 \Leftrightarrow I_{2n}(\alpha) \xrightarrow{\text{restriction}} L^2_\chi(\widetilde{\mathrm{Sp}}(2n,\mathbb{R})) \to \pi \otimes \pi$$

intertwining the action of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \times \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$, where the first map is restriction of sections to $\mathcal{O}$.

To complete the proof we need to replace $L^2 \Leftrightarrow I_{2n}(\alpha)$ and π by their Harish-Chandra modules. Let $\tilde{\mathbb{K}}$ (resp. $\widetilde{K}$) be the maximal compact subgroup of $\widetilde{\mathrm{Sp}}(4n,\mathbb{R})$ (resp. $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$). The restriction of the $\tilde{\mathbb{K}}$-finite functions $I_{2n}(\chi)$ of $L^2 \Leftrightarrow I_{2n}(\alpha)$ is a dense subspace of the $\widetilde{K} \times \widetilde{K}$-finite functions on $\mathcal{O}$. Therefore ϕ restricted to $I_{2n}(\alpha)$ is nonzero. □

Proof of Proposition 7.1. Part (1), and the occurence of either π or $\pi \otimes \mathrm{sgn}$ in the correspondence in (2), is an immediate consequence of Lemma 7.8. The fact that both π and $\pi \otimes \mathrm{sgn}$ cannot occur in (2) was proved in Lemma 1.5. □

8. Induction principle

In this section we turn to a more general setting and let V be an orthogonal space of signature (p,q) and W be a symplectic space of dimension $2n$, with no further restrictions. Throughout this section we fix ψ and let ω be the corresponding oscillator representation for the dual pair $(\mathrm{O}(V), \mathrm{Sp}(W))$.

Suppose we are given a decomposition

$$W = W_1^+ \oplus \cdots \oplus W_r^+ \oplus W^0 \oplus W_1^- \oplus \cdots \oplus W_r^-, \tag{8.1a}$$

where all $W_j^{\pm}$ are isotropic, W_j^+ and W_j^- are in duality, and W^0 is a non-degenerate symplectic space or 0. Let $P = MN$ be the stabilizer in $\mathrm{Sp}(W)$ of the flags $0 \subset W_1^- \subset W_1^- \oplus W_2^- \subset \cdots \subset W^- = \sum_i W_i^-$. Let $\widetilde{P} = \widetilde{M}N$ be the inverse image of P in $\widetilde{\mathrm{Sp}}(W)$. There is a surjective map

$$\overline{M} = \widetilde{\mathrm{GL}}(W_1) \times \cdots \times \widetilde{\mathrm{GL}}(W_r) \times \widetilde{\mathrm{Sp}}(W^0) \twoheadrightarrow \widetilde{M}. \tag{8.1b}$$

Let

$$V = V_1^+ \oplus \cdots \oplus V_r^+ \oplus V^0 \oplus V_1^- \oplus \cdots \oplus V_r^- \tag{8.1c}$$

be a decomposition of V, and define $P' = M'N'$ and $\widetilde{P}' = \widetilde{M}'N'$ in an analogous manner as for the symplectic group. In this case $\widetilde{M}' \simeq \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_r) \times \widetilde{\mathrm{O}}(V^0)$.

Let ω_M denote the oscillator representation for (M, M'). This is the product of the oscillator representations for the dual pairs $(\mathrm{GL}(W_j), \mathrm{GL}(V_j))$ and $(\mathrm{Sp}(W^0), \mathrm{O}(V^0))$. It gives a correspondence between representations of $\overline{M}$ and

$$\overline{M}' = \widetilde{\mathrm{GL}}(V_1) \times \cdots \times \widetilde{\mathrm{GL}}(V_r) \times \widetilde{\mathrm{O}}(V^0). \tag{8.1d}$$

If one member of a dual pair is the trivial group, then we take the trivial representation for the oscillator representation for this pair.

Set $\chi = \chi(\psi)$ and $\chi_V = \chi(\psi, V)$ as in (1.2).

DEFINITION 8.2. ([7], 1.1.1). For $a, b \in \mathbb{Z}$ define the character $\zeta(a,b)$ of $\widetilde{\mathrm{GL}}(1)$ by

$$\begin{aligned}
\zeta(a,b)(x,\varepsilon) &= \chi(x^a, \varepsilon^a)\chi(x^b, \varepsilon^b)^{-1} \\
&= \gamma(x^a, \tfrac{1}{2}\psi)\gamma(x^b, \tfrac{1}{2}\psi)^{-1}\varepsilon^{a-b} \\
&= \begin{cases} 1 & a \equiv b(2), \\ \chi^{-1}(x,\varepsilon) & a \text{ even}, b \text{ odd}, \\ \chi(x,\varepsilon) & a \text{ odd}, b \text{ even}. \end{cases}
\end{aligned}$$

For $(g, \varepsilon) \in \widetilde{\mathrm{GL}}(m)$ let $\zeta(a,b)(g,\varepsilon) = \zeta(a,b)(\det(g), \varepsilon)$.

Note that $\zeta(a,b)$ is of type $a+b$.

Let $k_i = \dim(V_i)$, $\widetilde{k}_i = \sum_{j=1}^i k_j$, $k = \widetilde{k}_r$, $\ell_i = \dim(W_i)$, and $\tilde{\ell}_i = \sum_{j=1}^i \ell_j$, $\ell = \tilde{\ell}_r$.

DEFINITION 8.3. Let $m = p + q$, and define the character ξ of $\overline{M}$ by

$$\xi = \begin{cases} |\det|^{n-(m/2)-\tilde{\ell}_j+\tilde{k}_j+(1/2)\ell_j-(1/2)k_j+(1/2)}\zeta(k_j,k)\chi(\det^k,\varepsilon^k)\chi_V & \widetilde{\mathrm{GL}}(W_j), \\ 1 \quad \widetilde{\mathrm{Sp}}(W^0) \end{cases}$$

and ξ' of $\overline{M}\,'$ by

$$\xi' = \begin{cases} |\det|^{-n+(m/2)+\tilde{\ell}_j-\tilde{k}_j-(1/2)\ell_j+(1/2)k_j-(1/2)}\zeta(\ell_j,\ell)\chi(\det^\ell,\varepsilon^\ell) & \widetilde{\mathrm{GL}}(V_j), \\ \chi(\det^{-\ell},\varepsilon)(\Leftrightarrow 1,\det)_{\mathbb{R}}^{\ell(n-\ell)}\varepsilon & \widetilde{\mathrm{O}}(V^0). \end{cases}$$

THEOREM 8.4.: INDUCTION PRINCIPLE I. *Let σ be a representation of $\overline{M}$ and σ' a representation of $\overline{M}\,'$. Suppose there is a non-zero $\overline{M} \times \overline{M}\,'$ equivariant map*

$$\omega_M \to \sigma \otimes \sigma'. \tag{8.5a}$$

Then there is a non-zero $\widetilde{\mathrm{O}}(V) \times \widetilde{\mathrm{Sp}}(W)$ equivariant map

$$\phi\colon \omega \to \mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(W)}(\sigma\xi) \otimes \mathrm{Ind}_{\widetilde{P'}}^{\widetilde{\mathrm{O}}(V)}(\sigma'\xi'). \tag{8.5b}$$

Here $\sigma\xi$ factors to $\widetilde{M}$, and extends to $\widetilde{P}$ trivially on N, and $\sigma'\xi'$ factors to $\widetilde{M}'$ and extends to $\widetilde{P}'$ trivially on N'.

Note. The cover $\widetilde{\mathrm{O}}(V^0)$ of $\mathrm{O}(V^0)$ in (8.1d) (resp. (8.3)) is $\widetilde{\mathrm{O}}(V^0)[n \Leftrightarrow \ell]$ (resp. $\widetilde{\mathrm{O}}(V^0)[\ell]$). Then $\sigma'\xi'$ and the representation in (8.5b) are naturally representations of the covers of $\mathrm{O}(V^0)[n]$ and $\mathrm{O}(V)[n]$.

Proof. The proof is essentially the same as the proofs of ([7], Theorem 2.5) and ([3], Corollary 3.21). It follows from Frobenius reciprocity and the following two Lemmas.

LEMMA 8.5. *In the setting of Theorem* 8.4, *suppose $r = 1$ so that $V = V^+ \oplus V^0 \oplus V^-$, $P_V = M_V N_V$ is the stabilizer of V^-, $M_V \simeq \mathrm{GL}(V^+) \times \mathrm{O}(V^0)$, and similarly for $\mathrm{Sp}(W)$. Then there is a surjective $\widetilde{M}_V \times \widetilde{M}_W$ equivariant map*

$$\omega \twoheadrightarrow \omega_M \nu$$

where ν is the following character of $\overline{M}_V \times \overline{M}_W$.

$$\nu = \begin{cases} |\det|^{-n+(\ell/2)}\gamma(\det^\ell,\tfrac{1}{2}\psi)\varepsilon^\ell & (g,\varepsilon) \in \widetilde{\mathrm{GL}}(V^+), \\ |\det|^{-(m/2)+(k/2)}\gamma(\det^k,\tfrac{1}{2}\psi)\chi_V & (g,\varepsilon) \in \widetilde{\mathrm{GL}}(W^+), \\ \gamma(\det^\ell,\tfrac{1}{2}\psi)(\Leftrightarrow 1,\det)_{\mathbb{R}}^{n\ell}\varepsilon^\ell & (g,\varepsilon) \in \widetilde{\mathrm{O}}(V^0), \\ 1 & (g,\varepsilon) \in \widetilde{\mathrm{Sp}}(W^0). \end{cases}$$

Proof. The proof is the same as the proof of ([3], Proposition 3.13). □

LEMMA 8.6. *Let* $(\mathrm{GL}(V), \mathrm{GL}(W))$ *be a dual pair. Suppose* $V = V_1 \oplus V_2$, *and let* $P_V = M_V N_V$ *be the stabilizer of* V_1, *so* $M_V \simeq \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. *Let* $\overline{M} = \widetilde{\mathrm{GL}}(V_1) \times \widetilde{\mathrm{GL}}(V_2)$, *and let* $\widetilde{P} = \widetilde{M}N$ *be the inverse image of* P *in* $\widetilde{\mathrm{GL}}(V)$. *Similarly let* $W = W_1 \oplus W_2$, *etc. Let* ω_M *be the oscillator representation for the dual pair* (M_V, M_W). *Set* $k_i = \dim(V_i)$ *and* $\ell_i = \dim(W_i)$. *Then there is a surjective* $\widetilde{P} \times \widetilde{P}$ *equivariant map* $\omega \twoheadrightarrow \omega_M \nu$, *where* ν *is the following character*

$$\nu = \begin{cases} |\mathrm{det}|^{-(1/2)\ell_2}\zeta(\ell_1, \ell) & \widetilde{\mathrm{GL}}(V_1), \\ |\mathrm{det}|^{(1/2)\ell_1}\zeta(\ell_2, \ell) & \widetilde{\mathrm{GL}}(V_2), \\ |\mathrm{det}|^{-(1/2)k_2}\zeta(k_1, k) & \widetilde{\mathrm{GL}}(W_1), \\ |\mathrm{det}|^{(1/2)k_1}\zeta(k_2, k) & \widetilde{\mathrm{GL}}(W_2). \end{cases}$$

Proof. See the proof of ([3], Proposition 3.13). We omit the details. □

In the setting of Theorem 8.4, let $\overline{K}_{\bar{M}}$ be a maximal compact subgroup of $\overline{M}$. There is a surjective map from $\overline{K}_{\bar{M}}$ to the maximal compact subgroup $\widetilde{K \cap M}$ of $\widetilde{M}$. The $\overline{K}_{\bar{M}}$-type in the next result factors to, and is identified with, a representation of $\widetilde{K \cap M}$.

THEOREM 8.7.: INDUCTION PRINCIPLE II. *In the setting of Theorem* 8.4, *suppose* μ *is a* $\widetilde{K}$*-type for* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, *and* $\mu_{\bar{M}}$ *is a* $\overline{K}_{\bar{M}}$*-type for* $\overline{M}$ *satisfying the following properties.*

(1) μ_M *is of minimal degree in* σ,
(2) $\deg(\mu) = \deg(\mu_M)$, μ *contains* μ_M *in its restriction to* $\widetilde{K \cap M}$ *and is of minimal degree and multiplicity one in*

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{O}}(V)}(\sigma\alpha),$$

(3) *There exist characters* α *and* α' *of* $\overline{M}$ *and* $\overline{M}\,'$, *trivial on* $\widetilde{K \cap M}$ *and* $\widetilde{K' \cap M'}$ *such that* $\sigma\alpha \otimes \sigma'\alpha'$ *is also a quotient of* ω_M, *and* $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(\mathbb{W})(W)}(\sigma\xi\alpha)$ *is irreducible.*

Then $\mu \otimes \mathcal{H}(\mu)$ *is in the image of* ϕ.

Proof. The proof is the same as the proof of [3], Proposition 3.25. □

THEOREM 8.8. *In the setting of Theorem* 8.4, *assume* $\dim(V_i^+) = \dim(W_i^+) = k_i$ *for all* $1 \leqslant i \leqslant r$; *so with* $k = \sum_i k_i$,

$$\overline{M} \simeq \widetilde{\mathrm{GL}}(k_1) \times \cdots \times \widetilde{\mathrm{GL}}(k_r) \times \widetilde{\mathrm{Sp}}(2n \Leftrightarrow 2k, \mathbb{R})$$
$$\overline{M}' \simeq \mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_r) \times \widetilde{\mathrm{O}}(p \Leftrightarrow k, q \Leftrightarrow k).$$

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Let σ_i *be any irreducible representation of* $\mathrm{GL}(k_i)$*, and suppose* σ_0 *corresponds to* τ_0 *for the dual pair* $(\mathrm{O}(p \Leftrightarrow k, q \Leftrightarrow k), \mathrm{Sp}(2n \Leftrightarrow 2k, \mathbb{R}))$*. Then there is a non-zero map* Φ *from* ω *to the tensor product of*

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes \sigma_0)$$

and

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{O}}(p,q)}(\sigma_1^* \chi_V \otimes \cdots \times \sigma_r^* \chi_V \otimes \tau_0 \zeta(n, k+n)).$$

Proof. The existence of Φ follows from Theorem 8.4 and the fact that for the dual pair $(\mathrm{GL}(m), \mathrm{GL}(m))$ the correspondence is $\pi \to \pi \otimes (\Leftrightarrow 1, \det)_{\mathbb{R}}^m$ for all π. $\square$

Note. The oscillator representation for the dual pair $\mathrm{GL}(m), \mathrm{GL}(m)$ may be normalized so that the action of the dual pair factors to the linear groups. We are using the unnormalized oscillator representation, which accounts for the term $(\Leftrightarrow 1, \det)_{\mathbb{R}}^m$.

9. Proof of the main results

We prove the four results in Section 5, and Theorem 3.3. Throughout this section we fix ψ and $\delta = \pm 1$.

The most natural way to prove Theorem 5.1 would be to prove it first for the discrete series, and then in general using the induction principle (Section 8). Unfortunately, the results in Section 7 are not sharp enough to compute the correspondence of the discrete series.

Instead we proceed by induction on n. Given the result for $\widetilde{\mathrm{Sp}}(2n \Leftrightarrow 2, \mathbb{R})$, the induction principle computes the result for all representations but the discrete series of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ (and $\widetilde{\mathrm{O}}(p, q)$). This implies that the representation correspondence can only map discrete series to discrete series; since these are in the domain of the correspondence, it remains to match up parameters. This is a relatively simple matter using the results on harmonic K-types in Section 6.

Proof of Theorem 5.1. $n = 0$. This is not quite empty, but an exercise in the definitions and covering groups. Consider the dual pair $(\mathrm{O}(1,0), \mathrm{Sp}(0, \mathbb{R}))$. The group $\widetilde{\mathrm{O}}(1,0) \simeq \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathrm{O}(1,0) \times \mathbb{Z}/2\mathbb{Z}$, and $\widetilde{\mathrm{Sp}}(0, \mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$. This dual pair is mapped to $\widetilde{\mathrm{Sp}}(0, \mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$, and the correspondence is obtained by restricting the nontrivial character of this group to the dual pair. This takes the nontrivial character of $\widetilde{\mathrm{Sp}}(0, \mathbb{R})$ to $\xi = \mathbb{1} \otimes \mathrm{sgn}$ of $\widetilde{\mathrm{O}}(1,0)$. This is as predicted by Theorem 3.3, and Theorem 5.1 is immediate. The case of $\mathrm{O}(0,1)$ is the same. This completes the proof in this case.

Inductive step: Induced representations

Assume Theorems 3.3 and 5.1 for $\widetilde{\mathrm{Sp}}(2n \Leftrightarrow 2, \mathbb{R})$, and let π be a genuine irreducible representation of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, which is not a discrete series or a limit of discrete series.

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Let $\widetilde{M}$ and σ be inducing data for π as in Theorem 5.1. We choose $\widetilde{P} = \widetilde{M}N$ to be of the form in Theorem 8.8 (cf. 8.1), and so π is the (unique) irreducible quotient of the standard module

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(\sigma). \tag{9.1a}$$

By Theorem 8.8 there is a non-zero map Φ from ω to the tensor product of (9.1a) and

$$\mathrm{Ind}_{\widetilde{P}'}^{\widetilde{\mathrm{O}}(p,q)}(\sigma'). \tag{9.1b}$$

Here $p', q', \widetilde{M}'$ and σ' are as in Theorem 5.1, the twist by η coming from $\zeta(n, k+n)$ of Theorem 8.8.

Recall from (Section 4) that π is the unique irreducible constituent of (9.1a) containing a minimal $\widetilde{K}$-type μ, and similarly π'. It is enough to show

$$\mu \text{ is in the image of } \Phi. \tag{9.2}$$

Let $\pi \otimes \pi''$ be any irreducible quotient of the image of Φ. By [5] π'' contains $\mathcal{H}(\mu)$. Write the standard module (9.1a) as $X(\gamma)$, with $\overline{\gamma} = (\lambda, \nu)$, and similarly (9.1b). It is immediate from the calculations in Section 6 that $\mathcal{O}(\lambda) = \lambda'$ (cf. the proof of Theorem 5.5 below). By Proposition 6.1, $\mathcal{H}(\mu)$ is a minimal $\widetilde{K}'$-type of (9.1b), and it follows that $\pi'' = \pi'$, as we needed to show.

To see (9.2) we apply Theorem 8.7. Let μ_M be the minimal $\widetilde{K \cap M}$-type of σ. By Corollary 5.2 applied to $\widetilde{\mathrm{Sp}}(2r, \mathbb{R})$ the $\widetilde{\mathrm{Sp}}(2r, \mathbb{R})$ component of μ_M is p', q'-harmonic. By Proposition 6.21, and ([11], III.9) for the GL terms, μ_M is of lowest p', q'-degree in σ, so condition (1) of Theorem 8.7 holds. By Proposition 6.18, $\deg_{p,q}(\mu) = \deg_{p,q}(\mu_M)$, and since $p \Leftrightarrow q = p' \Leftrightarrow q'$ this equals $\deg_{p',q'}(\mu_M)$. Also by Proposition 6.18, the restriction of μ to $\widetilde{K \cap M}$ contains μ_M. By Proposition 6.21, μ is of lowest degree in (9.1a), and also of multiplicity one (this is a general fact about standard modules). This verifies (8.7)(2).

Take α to be a generic character of $\overline{M}$ given by a power of $|\det|$ on each of the GL terms, and let $\alpha' = \alpha^*$. Then $\sigma\alpha \otimes \sigma'\alpha'$ is a quotient of ω_M (cf. the proof of Theorem 8.8) and (9.1a) is irreducible by the usual argument. Thus (3) for Theorem 8.7 holds, and applying the theorem we conclude (9.2).

Inductive step: Limits of discrete series

Let π be a genuine limit of discrete series representation of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ *not* in the discrete series. Then π may be realized as the unique irreducible quotient of

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}(\sigma) \tag{9.3a}$$

containing the (unique) minimal $\widetilde{K}$-type μ. Here $\widetilde{M} \simeq \widetilde{\mathrm{GL}}(2)^\ell \times \widetilde{\mathrm{Sp}}(2t, \mathbb{R})$ and σ is a discrete series representation of $\widetilde{M}$. In the notation of (6.13), $\ell = \sum_i \min(x_i, y_i)$ and $t = \sum_i |x_i - y_i|$. This follows from the discussion in Section 6. As for (9.1a), by Theorem 8.8 there is a non-zero map from the tensor product of (9.3a) and

$$\mathrm{Ind}_{\widetilde{P}'}^{\widetilde{\mathrm{O}}(p,q)}(\sigma'), \tag{9.3b}$$

where σ' is now a discrete series representation of $\widetilde{M}'$. The same argument applied with (9.3a,b) in place of (9.1a,b) proves that the lowest $\widetilde{K}$ and $\widetilde{K}'$-types of (9.3a) and (9.3b) correspond. Theorem 5.1 reduces to Theorem 3.3(2) in this case, and holds from the calculations of Section 6.

Inductive step: Discrete series

Theorem 5.1 reduces to Theorem 3.3(1) in this case, so it is enough to prove Theorem 3.3(1).

Proof of Theorem 3.3(1). It is convenient to start on the orthogonal group. So let π' be in the discrete series representation of $\widetilde{\mathrm{O}}(p, q)$. By Proposition 6.29 and Proposition 2.1, π' occurs in the representation correspondence, while $\pi' \otimes$ sgn does not. The corresponding representation π of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ is also in the discrete series (the representation correspondence is the graph of a bijection, π' is in the domain and all but the discrete series in the range are accounted for).

Let μ_0' be the lowest $\widetilde{K}'$-type of π', and μ_0 the lowest $\widetilde{K}$-type of π. By Proposition 6.28(2), μ_0' is of lowest degree and occurs in the space of joint harmonics; let $\mu = \mathcal{H}(\mu')$. It is enough to show $\mu = \mu_0$.

We calculate the length of $\mathcal{V}(\mu)$, the element defined by the Vogan algorithm applied to μ (cf. Section 6). If λ is the Harish-Chandra parameter of π, then $\mathcal{V}(\mu_0) = \lambda$, and (the Weyl group orbit of) λ is the infinitesimal character. The relation between infinitesimal characters is given by the orbit map which preserves lengths (cf. Section 1), so $|\lambda| = |\lambda'|$, where λ' is the Harish-Chandra parameter for π'. Therefore

$$\begin{aligned} |\mathcal{V}(\mu_0)| &= |\lambda| \\ &= |\lambda'| \\ &= |\mathcal{V}(\mu_0')| \\ &= |\mathcal{O}(\mathcal{V}(\mu_0'))| \\ &= |\mathcal{V}(\mathcal{H}(\mu_0'))| \quad \text{by Proposition 6.1} \\ &= |\mathcal{V}(\mu)|. \end{aligned} \tag{9.4}$$

Thus μ_0 and μ are both lambda-lowest $\widetilde{K}$-types of π [18, Definition 5.4.1]. Therefore [17, Lemma 8.8] μ and μ_0 are both lowest $\widetilde{K}$-types, and therefore equal.

This accounts for all genuine discrete series representations of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$. This completes the proof of Theorem 3.3(1) as well as Theorem 5.1. □

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