Note on the Geometries in which Straight Lines are represented by Circles.

By DUNCAN M. Y. SOMMERVILLE, M.A., D.Sc.

§0. In a recent paper read before the Society, Professor Carslaw gave an account, from the point of view of elementary geometry, of the well-known and beautiful concrete representation of hyperbolic geometry in which the non-Euclidean straight lines are represented by Euclidean circles which cut a fixed circle orthogonally. He also considered the case in which the fixed circle vanishes to a point, and showed that this corresponds to Euclidean geometry. The remaining case, in which the fixed circle is imaginary and which corresponds to elliptic or spherical geometry, is not open to the same elementary geometrical treatment, and Professor Carslaw therefore omitted any reference to it. As this might be misleading, the present note has been written primarily to supply this gap. It has been thought best, however, to give a short connected account of the whole matter from the foundation, from the point of view of analysis, omitting the detailed consequences which properly find a place in Professor Carslaw's paper.

§1. Let us first find the conditions under which the straight lines of a geometry may be represented by Euclidean circles.

The defining characteristic of a straight line is that it is in general uniquely determined by two points. Hence if the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

represents a straight line, the constants, g, f, c, must be connected by a linear relation, of which the general form may be written

$$2gg' + 2ff' = c + c'.$$

But this relation expresses that the circle cuts orthogonally the fixed circle

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

Hence *the circles which represent the straight lines of a geometry form a linear system cutting a fixed circle orthogonally*.

§2. Thus we find at once that there are three forms of geometry, according as the fundamental circle is real, vanishing, or imaginary. In the first case two orthogonal circles cut in two points, which may be real, coincident, or imaginary; in the second case every orthogonal circle passes through one fixed point, and they cut in pairs in one other real point which may be coincident with the fixed point; in the third case they cut always in two real points which may coincide. Thus we have the three sorts of line-pairs, intersecting, non-intersecting, and parallel. If we fix the condition that two straight lines cut in only one point, it will be convenient to consider the point-pair (which are inverse points with respect to the fundamental circle) as being the representative of a point.* Then the geometry with a real circle is called Hyperbolic, and the geometry with an imaginary circle is called Elliptic. If we abandon the condition that two straight lines are only to cut in one point, we get in the latter case a geometry which we shall see is identical with Spherical geometry. In the former case we see from §5 that the distance between two points which are separated by the fixed circle is unreal, so that we cannot consider them as distinct real points. When the fundamental circle vanishes the geometry is called Parabolic, and we shall see that it is identical with Euclidean geometry.

§3. Next, to fix the representation, we have to consider the measurement of distances and angles.

Let us make the condition that angles are to be the same in the geometry and in its representation, *i.e.* that the representation is to be conform.† We shall find that this fixes also the distance function.

---

* When the fundamental circle is real we may employ another artifice and consider only the points in the interior (or exterior) of the fundamental circle. When the fundamental circle vanishes to a point, this point may be considered as the representative of all the points at infinity. (See §7.)

† C. E. Stromquist, in a paper " On the geometries in which circles are the shortest lines," New York, Trans. Amer. Math. Soc., 7 (1906), 175–183, has shown that "the necessary and sufficient condition that a geometry be such that extremals are perpendicular to their transversals is that the geometry be obtained by a conformal transformation of some surface upon the plane." The language and his methods are those of the calculus of

First let us find how a circle is represented. A circle is the locus of points equidistant from a fixed point, or it is the orthogonal trajectory of a system of concurrent straight lines. Now a system of concurrent straight lines will be represented by a linear one-parameter system of circles, *i.e.* a system of coaxal circles. The orthogonal system is also a system of coaxal circles, and the fixed circle belongs to this system. Hence *a circle is represented always by a circle, and its centre is the pair of limiting (or common) points of the coaxal system determined by the circle and the fixed circle.*

The distance function has thus to satisfy the condition that the points upon the circle which represents a circle are to be at a constant distance from the point which represents its centre. To determine this function let us consider motions. A motion is a point-transformation in which circles remain circles ; and further, the fundamental circle must be transformed into itself, and angles must be unchanged.

§4. The equation of any circle may be written*

$$z\bar{z} + \bar{p}z + p\bar{z} + c = 0$$

where $z = x + iy$, $p = g + if$ and $\bar{z}$, $\bar{p}$ are the conjugate complex numbers. Now the most general transformation which preserves angles and leaves the form of this equation unaltered is

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad z' = \frac{\overline{\alpha z + \beta}}{\gamma z + \delta} .$$

This is a conformal transformation since any transformation between two complex variables is so.

---

variations. The *extremals* are the curves along which the integral which represents the distance function is a minimum, *i.e.* the curves which represent shortest lines ; and the *transversals* are the curves which intercept between them arcs along which the integral under consideration has a constant value. Thus in ordinary geometry, where the extremals are straight lines, the transversals to a one-parameter system of extremals are the involutes of the curve which is the envelope of the system. In particular, when the straight lines pass through a fixed point the transversals are concentric circles.

* Cf. Liebmann, Nichteuklidische Geometrie (Leipzig, 1905), §§ 8, 11.

To find the relations between the coefficients in order that the fundamental circle may be unchanged, let its equation be

$$x^2 + y^2 + k = 0 \quad \text{or} \quad z\bar{z} + k = 0.$$

This becomes $(az + \beta)(\overline{az} + \overline{\beta}) + k(\gamma z + \delta)(\overline{\gamma z} + \overline{\delta}) = 0.$

Hence $$a\overline{\beta} + k\gamma\overline{\delta} = 0$$

and $$k(a\overline{a} + k\gamma\overline{\gamma}) = \beta\overline{\beta} + k\delta\overline{\delta}$$

$$= k\delta\overline{\delta} + k^2 \frac{\gamma\overline{\gamma} \cdot \delta\overline{\delta}}{a\overline{a}},$$

therefore $$a\overline{a} = \delta\overline{\delta},$$

so that $$\frac{a}{\overline{\delta}} = \frac{\delta}{\overline{a}} = -\frac{k\gamma}{\overline{\beta}} = -\frac{\beta}{k\overline{\gamma}} = \lambda.$$

We have $a = \lambda\overline{\delta}$ and $\overline{a} = \overline{\lambda}\delta$, and also $\overline{a} = \frac{1}{\lambda}\delta$

therefore $$|\lambda| = 1.$$

The general transformation is therefore *

$$z' = \lambda \frac{az - k\beta}{\overline{\beta}z + \overline{a}}, \text{ where } |\lambda| = 1.$$

By any such homographic transformation the cross-ratio of four numbers remains unchanged, *i.e.*,

$$(z_1 z_2, z_3 z_4) = (z_1' z_2', z_3' z_4').$$

To find the condition that this cross-ratio may be real, let θ_{13} be the amplitude, and r_{13} the modulus of $z_1 - z_3$ and so on, then

$$(z_1 z_2, z_3 z_4) = \frac{r_{13}}{r_{14}} \frac{r_{24}}{r_{23}} e^{i(\theta_{13} - \theta_{14} + \theta_{24} - \theta_{23})}.$$

Hence we must have

$$\theta_{13} - \theta_{14} + \theta_{24} - \theta_{23} = m\pi,$$

and the four points z_1, z_2, z_3, z_4 are concyclic.

§5. Now to find the function of two points which is invariant during a motion; the two points determine uniquely an orthogonal circle, and if the transformation leaves this circle unaltered it leaves

---

* When, as is often taken to be the case, the fundamental circle is the x-axis the conditions are simply that the coefficients a, β, γ, δ, be all real numbers.

unaltered the two points where it cuts the fixed circle. Hence if these points are x, y the cross-ratio (z_1z_2, xy) for all points on this circle depends only on z_1 and z_2. If the distance function is $(PQ) = f\{(z_1z_2, xy)\}$ or, as we may write it, $f(z_1, z_2)$, then for three points P, Q, R, $(PQ) + (QR) = (PR)$, or

$$f(z_1, z_2) + f(z_2, z_3) = f(z_1, z_3).$$

This is a functional equation by which the form of the function is determined. Differentiating with respect to z_1, which may for the moment be regarded simply as a parameter, we have

$$f'(z_1, z_2) \cdot \frac{QY}{QX} \cdot \frac{\partial}{\partial z_1}\left(\frac{PX}{PY}\right) = f'(z_1, z_3) \cdot \frac{RY}{RX} \cdot \frac{\partial}{\partial z_1}\left(\frac{PX}{PY}\right).$$

Hence $\quad \dfrac{f'(z_1, z_2)}{f'(z_1, z_3)} = \dfrac{QX}{QY}\dfrac{RY}{RX} = \left(\dfrac{PX}{PY}\dfrac{RY}{RX}\right) \div \left(\dfrac{PX}{PY}\dfrac{QY}{QX}\right) = \dfrac{(z_1z_3, xy)}{(z_1z_2, xy)},$

and $\qquad (z_1z_2, xy)f'\{(z_1z_2, xy)\} = \text{const.} = \mu.$

Integrating, we have

$$f(z_1, z_2) = \mu\log(z_1z_2, xy) + C.$$

The constant of integration, C, is determined $= 0$ by substituting in the original equation. Hence

$$(PQ) = \mu\log(z_1z_2, xy) = \mu\log\left(\frac{PX}{PY}\frac{QY}{QX}\right).$$

§6. The expression for the line-element can now be found by making PQ infinitesimal.

We have, by Ptolemy's Theorem,

$$PX \cdot QY = PQ \cdot XY + PY \cdot QX.$$

Hence $\qquad ds = \mu\log\left(1 + \dfrac{PQ \cdot XY}{PY \cdot QX}\right) = \mu\dfrac{XY}{PX \cdot PY} \cdot PQ.$

Let OP (Fig. 1) cut the circle PXY again in R and the fixed circle in A, B. Then R is a fixed point so that PR is constant.

Also $\qquad \dfrac{RX}{PX} = \dfrac{RY}{PY} = a \text{ fixed ratio} = e,$

and $\qquad PR \cdot XY = PX \cdot RY + PY \cdot RX = 2e \cdot PX \cdot PY.$

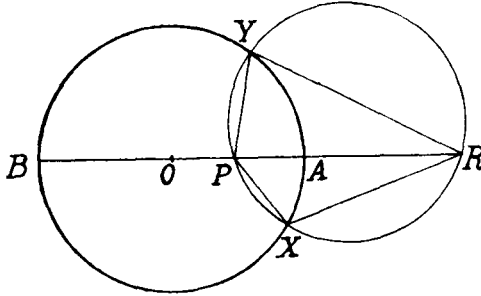Therefore $\dfrac{XY}{PX.PY} = \dfrac{2e}{PR}$ and is therefore a function of the position of P alone.

Fig. 1.

To find its value we may take any orthogonal circle through P, say the straight line PR.

Then

$$\frac{XY}{PX.PY} = \frac{AB}{PA.PB} = \frac{2\sqrt{-k}}{k+x^2+y^2}.$$

Hence

$$ds = \frac{2\mu\sqrt{-k}}{x^2+y^2+k}\sqrt{dx^2+dy^2}.$$

§7. The distance function is thus periodic with period $2i\mu\pi$. If P, P′ are inverse with respect to the fixed circle

$$(PP') = \mu\log\!\left(\frac{PA}{PB}\,\frac{P'B}{P'A}\right) = \mu\log(-1) = i\mu\pi,$$

and

$$(PQ') = \mu\log\!\left(\frac{PX}{PY}\,\frac{Q'Y}{Q'X}\right) = \mu\log\!\left(-\frac{PX}{PY}\,\frac{QY}{QX}\right)$$

$$= (PQ) + i\mu\pi.$$

When Q is on the fixed circle $(PQ) = \infty$. The fundamental circle is thus the assemblage of points at infinity.

If the fundamental circle is imaginary, k is positive and μ is purely imaginary and may be put $=i$. Then if inverse points are considered distinct their distance is π and the period is 2π, but if inverse points are identified the period must be taken as π.

If the fundamental circle is real, k is negative and μ is real and may be put $= 1$. Then the period must be taken as $i\pi$ and inverse points must be identified, otherwise we should have two real points with an imaginary distance. In this geometry there are three sorts of point-pairs, real, coincident, and imaginary, or actual, infinite, and ultra-infinite or ideal.

§8. Now if we change x, y into x', y' with the help of an additional variable z' by the equations

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z' - \mathrm{R}}{\sqrt{k}} = \frac{2\mathrm{R}\sqrt{k}}{x^2 + y^2 + k}$$

Then $$x'^2 + y'^2 + z'^2 = \mathrm{R}^2,$$

so that (x, y) is the stereographic projection of the point (x', y', z') on a sphere of radius R.

Obtaining the differentials dx', dy', dz', we find

$$ds^2 = dx'^2 + dy'^2 + dz'^2 = \frac{4\mathrm{R}^2 k}{(x^2 + y^2 + k)^2}(dx^2 + dy^2).$$

Hence $$\mathrm{R}^2 = -\mu^2.$$

Hence when k is positive and μ purely imaginary and $= i\mathrm{R}$, the geometry is the same as that upon a sphere of radius R, and the representation is by taking the stereographic projection.

When k is negative the sphere has an imaginary radius, but such an imaginary sphere can be conformly represented (by an imaginary transformation) upon a real surface of constant negative curvature, such as the surface of revolution of the tractrix about its asymptote (the pseudo-sphere).*

When k is zero μ must be infinite and the sphere becomes a plane.

Let $2\mu \sqrt{-k} = p$.

Then $$ds = \frac{p}{x^2 + y^2} \sqrt{dx^2 + dy^2} = \frac{p}{r^2} \sqrt{dr^2 + r^2 d\theta^2}.$$

By the transformation $$r' = \frac{p}{r}, \; \theta' = \theta$$

this becomes $$ds^2 = dr'^2 + r'^2 d\theta'^2 = dx'^2 + dy'^2.$$

---

* Cf. Darboux, Théorie des surfaces, vii., chap. XI.
  Also Klein, Nichteuklidische Geometrie, Vorlesungen.

Hence when k is zero the geometry is the same as that upon a plane, *i.e.* Euclidean geometry, and the representation is by inversion.

§9. Let us now return to the consideration of motions and investigate the nature of the general displacement of a rigid plane figure.* In ordinary space the general displacement of a rigid plane figure is equivalent to a rotation about a definite point, and this again is equivalent to two successive reflexions in two straight lines through the point. Now the operation which corresponds to reflexion in a straight line is inversion in an orthogonal circle. The formulae for inversion in the circle

$$z\bar{z} + \bar{p}z + p\bar{z} - k = 0,$$

which is any circle cutting $z\bar{z} + k = 0$ orthogonally, are

$$\frac{x'+g}{x+g} = \frac{y'+f}{y+f} = \frac{g^2+f^2+k}{(x+g)^2+(y+f)^2} = \frac{(x'+g)^2+(y'+f)^2}{g^2+f^2+k}.$$

or, using complex numbers,

$$z' + p = \frac{(p\bar{p}+k)(z+p)}{(z+p)(\bar{z}+\bar{p})}.$$

Whence

$$z' = \frac{k - p\bar{z}}{\bar{z}+\bar{p}}.$$

A second inversion in the circle

$$z\bar{z} + \bar{q}z + q\bar{z} - k = 0$$

gives

$$z'' = \frac{(k+p\bar{q})z + k(p-q)}{-(\bar{p}-\bar{q})z + (k+p\bar{q})}.$$

This will not hold when the circle of inversion is a straight line, $\theta = \phi$. Here inversion becomes reflexion and the formula is

$$z' = ze^{2i(\phi-\theta)} = \bar{z}e^{2i\phi}.$$

This combined with an inversion gives

$$z'' = \frac{-\bar{p}z + k}{z+p}\, e^{2i\phi}.$$

* Cf. Weber u. Wellstein, Encyklopädie der Elementar-Mathematik (2 Aufl. Leipzig, 1907), Bd. 2, Abschn. 2. Also, Klein u. Fricke, Vorlesungen über die Theorie der automorphen Functionen (Leipzig, 1897), Bd. 1.

Now these transformations are always of the general form

$$z' = \lambda \frac{az - k\beta}{\bar{\beta}z + \bar{a}}, \text{ where } |\lambda| = 1.$$

In fact this transformation is always of one or other of the two forms

$$z' = z\, e^{2i\phi}$$

(when $\beta = 0$) or

$$z' = \frac{-\bar{p}z + k}{z + p}e^{2i\phi}$$

(by dividing above and below by $\bar{\beta}$).

Hence *the general displacement of a plane figure is equivalent to a pair of inversions in two orthogonal circles.*

§10. In the general transformation there are always two points which are unaltered, for if $z' = z$ we have the quadratic equation

$$\bar{\beta}z^2 + (\bar{a} - \lambda a)z + k\lambda\beta = 0.$$

These form the centre of rotation, and the circles with these points as limiting points are the paths of the moving points.

There are three kinds of motions according as the roots of this quadratic are real, equal, or imaginary, or according as the centre of rotation is real, upon the fundamental circle, or imaginary. The first case is similar to ordinary rotation. In the second the paths are all circles touching the fundamental circle. In the third the paths all cut the fundamental circle; one of these paths is an orthogonal circle, the other paths are the equidistant curves.

§11. In conclusion, let us consider the connection between the representation by circles and the representation by straight lines. In the representation by straight lines, or circles with their centres at infinity, the fixed circle is the line infinity. If the representation is to be conformal, distances as well as angles must be represented unaltered, and the geometry can only be Euclidean.

Abandoning the conformal representation, the transformation which changes circles orthogonal to $x^2 + y^2 + k = 0$ into straight lines is

$$\theta' = \theta, \quad r' = \frac{2pr}{r^2 - k}.$$

The points (r, θ), $(-\dfrac{k}{r}, \theta)$ are both represented by the same point, so that this transformation gives a $(1, 1)$ correspondence between the pairs of real points which are inverse with respect to the circle $x^2 + y^2 + k = 0$ and the points which lie within the circle

$x^2 + y^2 + \dfrac{p^2}{k} = 0$, since for real values of r, $r'^2 < -\dfrac{p^2}{k}$. Every point

upon the circle $r^2 + k = 0$ is thus to be considered double. To a pair $\upmu$ imaginary points corresponds a point outside the new fixed circle   Any circle, not orthogonal, is transformed into a conic having contact with the circle $kr^2 + p^2 = 0$ at the two points which correspond to the intersections of the circle with the fixed circle $r^2 + k = 0$.

In fact, any curve in the r'-plane which cuts the fixed circle at a finite angle is represented in the r-plane by a curve cutting the fixed circle orthogonally, and any curve in the r-plane which cuts the fixed circle at a finite angle other than a right angle corresponds in the r'-plane to a curve touching the fixed circle.

Let the equation of a curve in the r'-plane be $f(r', \theta') = 0$.

Then $\dfrac{d\theta'}{dr'} = -\dfrac{\partial f}{\partial r'} \Big/ \dfrac{\partial f}{\partial \theta'}.$

But $\dfrac{\partial f}{\partial r'} = \dfrac{\partial f}{\partial r} \cdot \dfrac{dr}{dr'} = -\dfrac{(r^2 - k)^2}{2p(r^2 + k)} \cdot \dfrac{\partial f}{\partial r}$, and $\dfrac{\partial f}{\partial \theta'} = \dfrac{\partial f}{\partial \theta}.$

Therefore $\dfrac{d\theta'}{dr'} = -\dfrac{(r^2 - k)^2}{2p(r^2 + k)} \cdot \dfrac{d\theta}{dr}.$

Hence when $r^2 + k = 0$, $\dfrac{d\theta'}{dr'} = \infty$ unless $\dfrac{d\theta}{dr} = 0$, which proves the results.

§12. This transformation receives its simplest expression through the medium of the sphere.

Let a point Q be projected stereographically into P and centrally upon the same plane or a parallel plane into P' (Fig. 2).

Then $\theta' = \theta,$

and $\quad r = OP = OS\tan\phi = c\tan\phi, \; r' = OP' = OC\tan2\phi = c'\tan2\phi,$

therefore $\qquad\qquad\qquad r' = \dfrac{2cc'r}{c^2 - r^2},$

which agrees with the former equation if $c^2 = k$ and $cc' = -p$, so that $c'^2 = \dfrac{p^2}{k} = k'$, say.
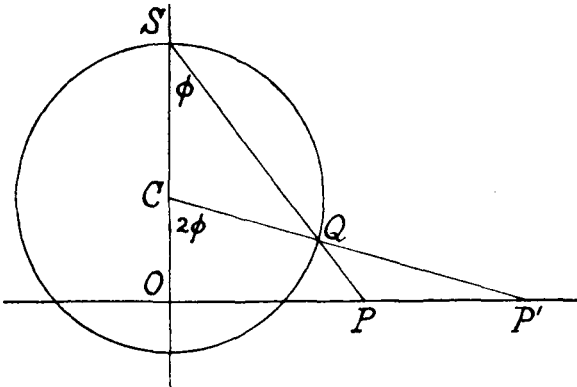

Fig. 2.

Hence as the representation by circles corresponds to stereographic projection, the representation by straight lines corresponds to central projection.

The transformation from the sphere to the plane is in this case given by the equations

$$\frac{x}{x'} = \frac{y}{y'} = \frac{\sqrt{k'}}{z'} = \frac{1}{R}\sqrt{x^2 + y^2 + k'},$$

where $\qquad\qquad\qquad x'^2 + y'^2 + z'^2 = R^2.$

Then* $\quad ds^2 = dx'^2 + dy'^2 + dz'^2 = R^2 \dfrac{k'(dx^2 + dy^2) + (y\,dx - x\,dy)^2}{(x^2 + y^2 + k')^2}$

---

* It may be noticed that the line-element can be expressed in terms of x', y' alone. Thus expressing z', dz' in terms of x', y' by means of the equation $x'^2 + y'^2 + z'^2 = R^2$, we have

$$ds^2 = \frac{R^2(dx'^2 + dy'^2) - (y'\,dx' - x'\,dy')^2}{R^2 - x'^2 - y'^2}.$$

§13. To determine the distance and angle functions in this representation we have first the relation between the angles from §11.

$$\tan\phi' = -\tan\phi \cdot \frac{(r^2-k)^2}{2p(r^2+k)} \cdot \frac{r'}{r} = -\tan\phi \cdot \frac{r^2-k}{r^2+k} = -\tan\phi \cdot \sqrt{\frac{k'}{r'^2+k'}}$$

where ϕ is the angle which the tangent at P to the curve $f(r,\theta)=0$ makes with the initial line.

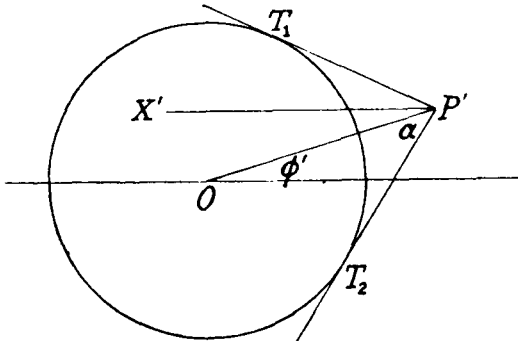Draw the tangents $P'T_1$, $P'T_2$ from P' to the circle (Fig. 3) and

Fig. 3.

let $\angle OP'T_1 = OP'T_2 = a$. Also draw $P'X'$ parallel to the x-axis. Then

$$\tan a = \sqrt{\frac{-k'}{r'^2+k'}}.$$

Therefore

$$e^{2i\phi} = \frac{\sin(a-\phi')}{\sin(a+\phi')} = \frac{\sin X'P'T_1}{\sin X'P'T_2} \cdot \frac{\sin OP'T_2}{\sin OP'T_1}$$
$$= P'(X'\Theta,\ T_1T_2).$$

Here x', y', $\dfrac{z'}{R}$ are the so-called Weierstrass' coordinates. Let the position of a point P on the sphere be fixed by its distances ξ, η from two fixed great circles intersecting at right angles at Ω, and let $\Omega P = \rho$, all the distances being measured on the sphere along arcs of great circles. Then

$$x' = R\sin\frac{\xi}{R},\ y' = R\sin\frac{\eta}{R},\ z' = R\cos\frac{\rho}{R}.$$

On the pseudosphere the circular functions become hyperbolic functions. (See Killing, Die nichteuklidischen Raumformen, Leipzig 1885, p. 17.)

Thus the true angle ϕ is given by

$$\phi = \frac{i}{2}\log(\mathrm{OX'}, \ \mathrm{T_1T_2}).$$

Hence the angle between two lines $\mathrm{P'X'}$, $\mathrm{P'Y'}$ through $\mathrm{P'}$ is given by

$$\frac{i}{2}\log(\mathrm{OY'}, \ \mathrm{T_1T_2}) - \frac{i}{2}\log(\mathrm{OX'}, \ \mathrm{T_1T_2}) = \frac{i}{2}\log(\mathrm{X'Y'}, \ \mathrm{T_1T_2}).$$

Next to determine the distance function; let P, Q become $\mathrm{P'}$, $\mathrm{Q'}$ (Fig. 4). The orthogonal circle PQXY becomes a straight
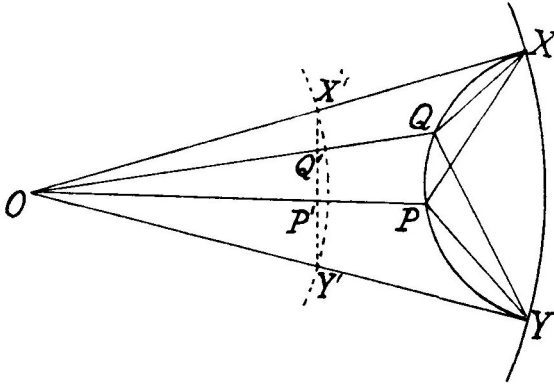

Fig. 4.

line $\mathrm{P'Q'X'Y'}$, and $\mathrm{OPP'}$, $\mathrm{OQQ'}$, etc., are collinear since angles at O are unaltered.

We have then

$$(PQ) = \mu\log\left(\frac{\mathrm{PX}}{\mathrm{PY}} \cdot \frac{\mathrm{QY}}{\mathrm{QX}}\right).$$

But
$$\frac{\mathrm{PX}}{\mathrm{OP}} = \frac{\sin\mathrm{XOP}}{\sin\mathrm{OXP}}, \ \frac{\mathrm{PY}}{\mathrm{OP}} = \frac{\sin\mathrm{YOP}}{\sin\mathrm{OYP}}$$

and
$$\frac{\mathrm{PX}}{\mathrm{PY}} \cdot \frac{\mathrm{QY}}{\mathrm{QX}} = \frac{\sin\mathrm{XOP}}{\sin\mathrm{YOP}} \cdot \frac{\sin\mathrm{YOQ}}{\sin\mathrm{XOQ}} \cdot \frac{\sin\mathrm{OYP}}{\sin\mathrm{OXP}} \cdot \frac{\sin\mathrm{OXQ}}{\sin\mathrm{OYQ}}$$

$$= \left(\frac{\mathrm{P'X'}}{\mathrm{P'Y'}} \cdot \frac{\mathrm{Q'Y'}}{\mathrm{Q'X'}}\right)\left(\frac{\mathrm{PY}}{\mathrm{PX}} \cdot \frac{\mathrm{QX}}{\mathrm{QY}}\right)$$

therefore
$$\left(\frac{\mathrm{PX}}{\mathrm{PY}} \cdot \frac{\mathrm{QY}}{\mathrm{QX}}\right)^2 = \frac{\mathrm{P'X'}}{\mathrm{P'Y'}} \cdot \frac{\mathrm{Q'Y'}}{\mathrm{Q'X'}}.$$

Hence we have the true distance (PQ) given by

$$(PQ) = \mu \log\left(\frac{PX}{PY} \cdot \frac{QY}{QX}\right) = \tfrac{1}{2}\mu\log\left(\frac{P'X'}{P'Y'} \cdot \frac{Q'Y'}{Q'X'}\right)$$
$$= \tfrac{1}{2}\mu\log(P'Q', \, X'Y') = (P'Q').$$

Then the line-element can be obtained in a manner similar to that of §6.

We find as before that $(PQ, XY) = 1 + \dfrac{XY}{PY \cdot QX} \cdot PQ$;

but in this case $\qquad PX \cdot PY = x^2 + y^2 + k'$

and $\qquad XY^2 = -4\{k'(dx^2 + dy^2) + (ydx - xdy)^2\}/(dx^2 + dy^2)$;

so that $\qquad ds^2 = -\mu^2 \cdot \dfrac{k'(dx^2 + dy^2) + (ydx - xdy)^2}{(x^2 + y^2 + k')^2}$.

Comparing this with the expression in §12 we find

$$R^2 = -\mu^2.$$

§14. Finally, this representation may be transformed projectively (distances and angles being unaltered as they are functions of cross-ratios), and we get the usual generalised representation in which the fixed circle or absolute becomes any conic ; straight lines are represented by straight lines, and distances, and angles in circular measure, are expressed by the formulae

$$(PQ) = \frac{\mu}{2}\log(PQ, \, XY)$$

$$(pq) = \frac{i}{2}\log(pq, \, xy)$$

where X, Y are the points in which the straight line PQ cuts the conic, and x, y are the tangents from the point of intersection of the lines p, q to the conic.