ON A DIFFERENTIAL EQUATION FOR ELECTROMAGNETIC WAVE TRANSMISSION IN FLARE STARS AND THE POSSIBLE EXISTANCE OF COHESIVE WAVE SOLUTIONS

by

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Abstract

Ambartsumian's celebrated hypothesis that stellar flares and other phenomena of stellar instability are due to a novel source of energy and a novel means of transporting this energy to the outer layers of the star has drawn the attention of electrodynamicists to a number of fundamental problems. One of these problems, namely the energy transport problem, is the subject of this communication. Herein, by assuming that the matter of the star is an isotropic collisionless plasma, from Maxwell's field equations and Newton's equation of motion with nonlinear Lorentz driving force, we have derived a vector differential equation for electromagnetic wave propagation. This equation contains the Debye radius and the plasma frequency as parameters, and reduces to the well-known wave equation when its nonlinear terms are neglected. We have indicated that the nonlinear equation has cohesive (solitary) wave solutions for both the longitudinal and transverse components of the electromagnetic field. Such cohesive waves are appropriate for transporting energy from the prestellar core of the star to its outer layers since they hold their shape, are free from dispersive distortion, and can carry energy in discrete amounts.

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Introduction

In a previous communication[1] based on Ambartsumian's famous hypothesis on stellar instabilities, we pictured a flare star as a kind of transformer that converts the low-entropy energy of the star's pre-stellar matter into the high-entropy energy of the star's flare radiation, and we reasoned that energy is drawn from the pre-stellar matter and deposited in discrete amounts on the outer layers of the star by means of cohesive waves having the form of solitary waves or solitons.

In the present communication we focus our attention on the cohesive waves and submit that such waves are mathematically possible if the nonlinearity of the plasma comprising the star is taken into account.

We proceed by deriving from Maxwell's equations and Newton's equation of motion a differential equation for the propagation of electromagnetic waves in an isotropic collisionless electronic plasma which we suppose resembles closely the plasma of the star. We find that the resulting equation is a nonlinear vector equation. To handle such an equation, we scalarize it and obtain a nonlinear system of two coupled scalar equations. And it is these coupled equations that we regard as the mathematical starting point of the problem.

Nonlinear Vector Differential Equation

We assume that the star's plasma is an isotropic collisionless electronic plasma; and we recall that for such a plasma the electric field $\mathbf{E}$, in the linear approximation, must satisfy the well-known equation[2]

$$c^2\nabla \times \nabla \times \mathbf{E} - 3\alpha^2\omega_p^2\ \nabla(\nabla \cdot \mathbf{E}) + \frac{\partial^2 \mathbf{E}}{\partial t^2} + \omega_p^2\ \mathbf{E} = 0\ , \tag{1}$$

where c denotes the velocity of light, ω_p denotes the plasma frequency, and α denotes the Debye radius. We also recall

$$m\alpha^2\omega_p^2 = \kappa T\ , \tag{2}$$

where κ is Boltzmann's constant, T is the temperature, and m is the electronic mass. The vector $\mathbf{E}$ can be expressed as the sum of a transverse field $\mathbf{E}^T$ and a longitudinal field $\mathbf{E}^L$. That is,

$$\mathbf{E} = \mathbf{E}^T + \mathbf{E}^L\ , \tag{3}$$

where, by definition,

$$\nabla \cdot \mathbf{E}^T = 0 \quad \text{and} \quad \nabla \times \mathbf{E}^L = 0 \,. \tag{4}$$

Accordingly, from equation (1) it follows that for transverse waves

$$\nabla^2 \mathbf{E}^T - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}^T}{\partial t^2} - \frac{\omega_p^2}{c^2} \mathbf{E}^T = 0 \,, \tag{5}$$

and for longitudinal waves

$$3\alpha^2 \omega_p^2 \, \nabla^2 \mathbf{E}^L - \frac{\partial^2 \mathbf{E}^L}{\partial t^2} - \omega_p^2 \mathbf{E}^L = 0 \,. \tag{6}$$

Since equation (5) does not involve $\mathbf{E}^L$ and since equation (6) does not involve $\mathbf{E}^T$ we see that in the linear approximation there is no interaction between the longitudinal and transverse waves. In other words, if a wave is initially transverse, it remains transverse, and if a wave is initially longitudinal it remains longitudinal. This is true in the linear approximation but not in the nonlinear case.

Taking into account the nonlinearity of the plasma that is quadratic with respect to the electric field $\mathbf{E}$ one can show that $\mathbf{E}$ must now satisfy the equation[3]

$$c^2 \nabla \times \nabla \times \mathbf{E} - 3\alpha^2 \omega_p^2 \nabla(\nabla \cdot \mathbf{E}) + \frac{\partial^2 \mathbf{E}}{\partial t^2} + \omega_p^2 \, \mathbf{E} = \frac{\omega_p^2 \, e}{2m} \frac{\partial}{\partial t} (\mathbf{Z} + 2 \mathbf{\Psi} \nabla \cdot \mathbf{\Phi}) \,, \tag{7}$$

where e denotes the electronic charge and where the vectors $\mathbf{Z}$, $\mathbf{\Psi}$, $\mathbf{\Phi}$ satisfy

$$\frac{\partial \mathbf{Z}}{\partial t} = \nabla(\mathbf{\Psi} \cdot \mathbf{\Psi}), \quad \frac{\partial \mathbf{\Psi}}{\partial t} = \mathbf{E}, \quad \frac{\partial^2 \mathbf{\Phi}}{\partial t^2} = \mathbf{E} \,. \tag{8}$$

The right side of equation (7) expresses the quadratic nonlinearity of the plasma.

The derivation of equation (7) is based on the Maxwell field equations and on the Newton equation of motion for the electrons of the plasma. From Maxwell's equations we have

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{4\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t} \,. \tag{9}$$

The current density $\mathbf{J}$ is given by

$$\mathbf{J} = ne\mathbf{v} \,, \tag{10}$$

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where n is the electron density and $\mathbf{v}$ is the electron velocity, and the conservation of charge is given by

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \,. \tag{11}$$

From Newton's equation of motion we have

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{e}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}) - \frac{1}{mn}\nabla p \,. \tag{12}$$

The left side of this equation is the convective derivative of the velocity, and the first term on the right side is the Lorentz force, and the second term on the right side is the force due to electron pressure. With the aid of equations (10), (11), and (12) the current $\mathbf{J}$ can be expressed in terms of $\mathbf{E}$, and by substituting the resulting equation into the right side of equation (9) we can obtain equation (7).

When we neglect the nonlinear terms $(\mathbf{v} \cdot \nabla)\mathbf{v}$ and $\mathbf{v} \times \mathbf{H}$ in the equation of motion we obtain equation (1), but if take these nonlinear terms into account we obtain equation (7). The second term on the left side of equation (7) comes from the electron pressure term ∇p of equation (12), and the right side of equation (7) comes from the nonlinear terms.

The nonlinear vector differential equation (7) is the equation we must solve to see whether or not cohesive (solitary) wave solutions are possible. To make this equation mathematically tractable it must be scalarized.

Coupled Scalar Equations

To reduce the vector wave equation to scalar form we orient the Cartesian coordinates x, y, z so that the x-axis becomes the longitudinal direction (the direction of propagation) and the z-axis becomes the transverse direction. From equation (8) we see that in component form $\mathbf{E}$ is given by

$$\mathbf{E} = \left[\frac{\partial^2 \Phi^L}{\partial t^2},\ 0,\ \frac{\partial^2 \Phi^T}{\partial t^2} \right] , \tag{13}$$

or by

$$\mathbf{E} = \left[\frac{\partial \Psi^L}{\partial t},\ 0,\ \frac{\partial \Psi^T}{\partial t} \right] . \tag{14}$$

Here Φ^L and Φ^T are the longitudinal and transverse components of the vector $\boldsymbol{\Phi}$, Ψ^L and Ψ^T are the longitudinal and transverse components of the vector $\boldsymbol{\Psi}$, and all four scalars Φ^L, Φ^T, Ψ^L, Ψ^T are functions of only x and t.

In view of representation (13) we can write equation (7) as two coupled scalar equations:

$$\mathcal{L}_1\,\Phi^L = \frac{e}{m}\omega_p^2\,(2\Phi_{xt}^T\Phi^L + \Phi_{tt}^L\Phi_x^L + \Phi_{tx}^T\Phi_t^T)\,, \tag{15}$$

$$\mathcal{L}_2\,\Phi^T = \frac{e}{m}\omega_p^2\,(\Phi_{tt}^T\Phi_x^L + \Phi_{xt}^L\Phi_t^T)\,, \tag{16}$$

where the subscripts x and t denote partial differentiation with respect to x and t, and where

$$\mathcal{L}_1 = \frac{\partial^4}{\partial t^4} - 3\alpha^2\omega_p^2\,\frac{\partial^4}{\partial x^2\partial t^2} + \omega_p^2\,\frac{\partial^2}{\partial t^2}\,, \tag{17}$$

$$\mathcal{L}_2 = \frac{\partial^4}{\partial t^4} - c^2\,\frac{\partial^4}{\partial x^2\partial t^2} + \omega_p^2\,\frac{\partial^2}{\partial t^2}\,, \tag{18}$$

are linear operators.

Since we are interested in a wave profile that is moving at a constant speed U, the x and t derivatives are related to each other linearly, i.e.

$$\frac{\partial}{\partial t} = -U\frac{\partial}{\partial x}\,. \tag{19}$$

For waves that satisfy relation (19) we can rewrite equation (15) and (16) in terms of Ψ^L and Ψ^T. That is we can write

$$\mathcal{M}_1\,\Psi^L = \frac{e}{m}\omega_p^2\,(3\Psi_x^L\Psi^L + \Psi_x^T\Psi^T)\,, \tag{20}$$

$$\mathcal{M}_2\,\Psi^T = \frac{e}{m}\omega_p^2\,(\Psi^T\Psi^L)_x\,, \tag{21}$$

where the linear operators $\mathcal{M}_1$ and $\mathcal{M}_2$ are given by

$$\mathcal{M}_1 = \frac{\partial^3}{\partial t^3} - 3\alpha^2\omega_p^2\,\frac{\partial^3}{\partial x^2\partial t} + \omega_p^2\,\frac{\partial}{\partial t}\,, \tag{22}$$

$$\mathcal{M}_2 = \frac{\partial^3}{\partial t^3} - c^2\,\frac{\partial^3}{\partial x^2\partial t} + \omega_p^2\,\frac{\partial}{\partial t}\,. \tag{23}$$

The coupled scalar equations (20) and (21) comprise the mathematical starting point of the problem. By inspection of these equations one can see that $\Psi^L = 0$ implies that $\Psi^T = 0$ but $\Psi^T = 0$ does not imply $\Psi^L = 0$; that is, purely longitudinal waves may exists whereas purely transverse waves may not. When transverse waves exist they are accompanied by longitudinal waves; and the interaction between transverse and longitudinal waves is due to the inclusion of the nonlinearity of the plasma.

Cohesive Wave Solutions

Using the nonlinear coupled equations (20) and (21) as a point of departure, we have shown elsewhere[3] that cohesive (solitary) vector wave solutions are possible. We do not reproduce the calculations here because they are tedious and irrelevant to the matter at hand. The point of importance is that cohesive wave solutions can exist. For such waves the distortions produced by plasma dispersion are cancelled by distortions produced by plasma nonlinearity and the waves can travel through the plasma at a constant velocity and without any change of profile shape.

References

(1) C. H. Papas," On the electrodynamical implications of flare stars", in: L. V. Mirzoyan, Ed., *Flare Stars*, Academy of Sciences, Yerevan (1977) 175-180.

(2) V. L. Ginzburg, *Propagation of Electromagnetic Waves in Plasmas*, Pergamon Press, Oxford (1970); V. N. Tsytovich, *Nonlinear Effects in Plasmas*, Plenum, New York (1970).

(3) J. Z. Tatoian and C. H. Papas, " On solitary waves in plasma", Wave Motion, **8** (1986) 415-438.

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