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LAGRANGIAN SYSTEMS WITH NON-SMOOTH CONSTRAINTS

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Abstract. The Lagrange-d'Alembert equations with constraints belonging to $H^{1,\infty}$ have been considered. A concept of weak solutions to these equations has been built. A global existence theorem for Cauchy problem has been obtained.

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1. Introduction. The statement of the problem. In this paper, we consider a dynamical system with configuration space $M \subset \mathbb{R}^m = \{x = (x^1, \dots, x^m)^T\}$ being a bounded domain.

The system is described by the Lagrangian

$$L(x, \dot{x}) = T(x, \dot{x}) - V(x), \quad T = \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j = \frac{1}{2}\dot{x}^T G(x)\dot{x}, \quad x \in M$$

and (possibly non-holonomic) constraints

$$a(x)\dot{x} = a_l^k(x)\dot{x}^l = 0, \quad k = 1, \dots, n < m, \quad \operatorname{rang} a(x) = n. \tag{1}$$

1.1. Here and below use the Einstein summation convention. We also use the notation $c, c_1, c_2 \dots$ for inessential positive constants.

The function T is the kinetic energy of the system; it is a positive definite quadric in the variables $\dot{x}$. The matrix G is positive definite and determines a Riemann metric in M. The function V is a potential.

We watch the motion of this system on the time interval $I_\tau = [0, \tau]$.

In the classical situation, all these functions are smooth in M and the dynamics of the system is described by the Lagrange-d'Alembert principle:

$$\Big(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^j} - \frac{\partial L}{\partial x^j}\Big)\psi^j = 0. \tag{2}$$

By definition, the function $x(t)$ is a motion of the system iff it satisfies (1) and for any functions $\{\psi^j(t)\}$ such that $a^k_l(x(t))\psi^l(t)=0$ it satisfies (2).

To present systems (1) and (2) in the resolved with respect to the highest derivatives form, one must rewrite this system using Lagrange multipliers:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot x}-\frac{\partial L}{\partial x}=\Lambda(x,\dot x)a(x),\quad \Lambda(x,\dot x)=(\lambda_1,\dots,\lambda_n)(x,\dot x). \tag{3}$$

To express the Lagrange multipliers λ, one should take time derivative from the both sides of (1) and substitute there $\ddot x$ from (3).

After these transformations, one obtains

$$\Lambda^T(x,\dot x)=\Big(a(x)G^{-1}(x)a^T(x)\Big)^{-1}w\Big(\frac{\partial a}{\partial x},\frac{\partial G}{\partial x},a,G,x,\dot x\Big). \tag{4}$$

The function w is smooth in $x\in M$, and in the other arguments w is smooth in the whole space. Correspondingly, (1) is the equation of an invariant manifold to system (3).

Assume that

$$G,a\in H^{1,\infty}(M). \tag{5}$$

This particularly implies $G(x),a(x)\in C(M)$.

This situation, for example, takes place when the Chaplygin sleigh [7] moves on a very irregular surface, say $z=f(x,y)$ and the function f is constructed as follows. Let $\varphi(x)$ be a smooth function with compact support and such that $\varphi(x)=1$ for $x\in(-\delta,\delta)$. Then, we let

$$f(x,y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\varphi(x-x_k)(x-x_k)^4\cos\Big(\frac{1}{x-x_k}\Big) \tag{6}$$

and the sequence $\{x_k\}_{k\in\mathbb{N}}$ runs over all rationals $\mathbb{Q}$.

Actually, the function f depends only upon x and as a function of single variable, it belongs to $H^{2,\infty}(\mathbb{R})$ and in the same sense $\frac{\partial f}{\partial x}\in H^{1,\infty}(\mathbb{R})\subset C(\mathbb{R})$.

Indeed, in this problem,

$$T=\frac{m}{2}\Big(\dot x^2+\dot y^2+\Big(\frac{\partial f}{\partial x}(x,y)\dot x\Big)^2\Big)+\frac{1}{2}\langle J\vec\omega,\vec\omega\rangle$$

and in the differential form the constraint (6) looks as follows:

$$\dot z-\frac{\partial f}{\partial x}(x,y)\dot x=0.$$

In such a case, the Lagrange multipliers (4) cannot be defined correctly because the function

$$\frac{\partial^2 f}{\partial x^2}$$

belongs just to $L^\infty(M)$ and even it is not clear what the expression

$$\frac{\partial^2 f}{\partial x^2}(x(t),y(t))$$

means. Therefore, equation (3) also becomes impossible.

These difficulties are overcame by introducing a concept of weak solutions to the Lagrange-d'Alambert equations, see Definition 1. The point is that this definition does not require to have second-order derivatives of f but simultaneously in smooth (classical) case, it is reduced to the classical solutions. This effect is very specific to the Lagrange-d'Alembert equations because their deep connection to variational problems. (Although the non-holonomic mechanics does not enjoy the variational principle of Hamilton.)

One should not expect the same effects in general systems of ODE.

In this paper, we propose a definition of weak solutions to the Lagrange-d'Alembert equations. This definition allows us to overcome the described problem and prove an existence theorem for the weak solutions. Nevertheless, the question on the uniqueness remains open.

For general dynamical systems $\dot{x} = f(t, x)$ with non-Lipschitz f, there are a lot of works devoted to investigating different types of uniqueness conditions in case. As far as the author knows this activity has been started from Kamke [**5**] and Levy [**6**]. Their results have been generalized in different directions. See, for example, [**8**, **2**] and references therein.

The case when f belongs to Sobolev spaces (at least $H^{1,1}$) has been studied in [**3**] in connection with the Navier–Stokes equation. In that paper, the equations which have good invariant measure are mainly considered. The existence and uniqueness theorems for the flow are given in terms of the corresponding transport equation.

In this paper, we consider individual solutions to the Cauchy problem for general non-holonomic systems; such systems need not necessarily possess a good invariant measure.

Collisions in holonomic Lagrangian systems have been considered in [**4**]. Collisions provide a source of another type of singularities and generalized solutions in dynamics.

2. Main theorem. Let $\|\cdot\|$ stand for the l_2-norm in $\mathbb{R}^m$. We introduce the following subspace of the Sobolev space $H^1(I_\tau)$:

$$H^1_0(I_\tau) = \{u \in H^1(I_\tau) \mid u(0) = 0\}.$$

In the sequel by $c, c_1, c_2 \ldots$, we denote positive constants.

Give a precise description of our functions: $V \in C^2(M)$ and the functions g_{ij} are such that for almost all $x \in M$, the conditions

$$g_{ij}(x) = g_{ji}(x), \quad c_1\|\xi\|^2 \le g_{ij}(x)\xi^i\xi^j \le c_2\|\xi\|^2, \quad \xi \in \mathbb{R}^m$$

hold.

We also suppose that for some constant h, a domain

$$D_h = \{x \in M \mid V(x) < h\}$$

is non-void and $\overline{D}_h \subset M$.

A non-degeneracy condition is also applied:

$$A(x) = \left(a^k_l(x)\right)_{k,l=1,\ldots,n}, \quad \det A(x) \ne 0, \quad x \in M. \tag{7}$$

We introduce the energy of the system $H(x, \dot{x}) = T(x, \dot{x}) + V(x)$.

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Integrating (2) by parts, one obtains the Lagrange-d'Alembert principle in the integral form [**1**]. This justifies the following definition.

DEFINITION 1. We shall say that a function $x(t) \in H^1(I_\tau)$ is a weak solution to the system of Lagrange-d'Alembert equations and the equations of constraint iff the equation

$$\int_{I_\tau} \Big(\frac{\partial L}{\partial x}(x(t), \dot{x}(t))\psi(t) + \frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t))\dot{\psi}(t)\Big)\, dt \\ - \frac{\partial L}{\partial \dot{x}}(x(\tau), \dot{x}(\tau))\psi(\tau) = 0 \tag{8}$$

holds for any $\psi(t) = (\psi^1, \dots, \psi^m)^T(t) \in H^1_0(I_\tau)$ that satisfies

$$a(x(t))\psi(t) = 0, \tag{9}$$

and equation (1) holds for almost all $t \in I_\tau$ that is, $a(x(t))\dot{x}(t) = 0$.

Observe that due to compact embedding $H^1(I_\tau) \subset C(I_\tau)$, this definition implies $x(t) \in C(I_\tau)$.

THEOREM 2.1. *For any positive constant τ and for any initial conditions x_0, v such that*

$$a(x_0)v = 0$$

and

$$H(x_0, v) = h' < h, \tag{10}$$

there exists a weak solution $x(t), \quad x(0) = x_0, \quad \dot{x}(0) = v$ to the Lagrange-d'Alembert equations and the equations of constraint (1).

Moreover, $H(x(t), \dot{x}(t)) = h' \quad \forall \quad t \in I_\tau$ and $x \in C^{1,\alpha}(I_\tau)$ for any $\alpha \in (0, 1)$.

THEOREM 2.2. *Let $x(t)$ be a solution to the system of Lagrange-d'Alembert equations in the sense of Definition 1.*

Then, there exists a function $\gamma(t) = (\gamma_1, \dots, \gamma_n)(t) \in L^2(I_\tau)$ such that the equation

$$\int_{I_\tau} \Big(\frac{\partial L}{\partial x}(x(t), \dot{x}(t))\psi(t) + \frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t))\dot{\psi}(t)\Big)\, dt \\ - \frac{\partial L}{\partial \dot{x}}(x(\tau), \dot{x}(\tau))\psi(\tau) = \int_{I_\tau} \gamma(t)a(x(t))\psi(t)\, dt \tag{11}$$

holds for any $\psi \in H^1_0(I_\tau)$.

3. Proof of Theorem 2.1. Introduce matrices

$$Q(x) = \big(a_l^k(x)\big), \quad l = n+1, \dots, m, \quad k = 1, \dots, n,$$
$$B(x) = -A^{-1}(x)Q(x).$$

PROPOSITION 1. *Suppose that for some* $x_0 \in M$ *and* $v \in \mathbb{R}^m$, *one has* $a(x_0)v = 0$. *Then, there is a sequence* $G_i(x), a_i(x) \in C^\infty(M)$ *(this notation implies that all the components of matrices* G_i, a_i *belong to* $C^\infty(M)$*) such that*

$$\|G_i - G\|_{L^\infty(M)}, \quad \|a_i - a\|_{L^\infty(M)} \to 0 \quad as \quad i \to \infty, \tag{12}$$

and $a_i(x_0)v = 0$,

$$\left\|\frac{\partial G_i}{\partial x}\right\|_{L^\infty(M)}, \quad \left\|\frac{\partial a_i}{\partial x}\right\|_{L^\infty(M)} \leq c.$$

Proof. First, let us recall a standard fact.
There is a sequence $a_i^*(x) \in C^\infty(M)$ such that

$$\|a_i^* - a\|_{L^\infty(M)} \to 0 \quad \text{as} \quad i \to \infty,$$

and

$$\left\|\frac{\partial a_i^*}{\partial x}\right\|_{L^\infty(M)} \leq c.$$

The constant c does not depend on i. This follows from real analysis and formula (5).
Thus, if we find a sequence $\{b_i\}$ such that

$$\|b_i\| \to 0, \quad b_i v = -a_i^*(x_0)v$$

and put $a_i(x) = a_i^*(x) + b_i$, then the Proposition is proved.
Let $v = (v^1, \dots, v^m)^T$ and $v^1 \neq 0$ furthermore,

$$b_i = (b_{ij}^r), \quad a_i^*(x_0) = (w_{il}^k)$$

and observe that $w_{il}^k v^l \to 0, \quad i \to \infty$.
It remains to take $b_{ij}^r = 0, \quad j > 1$ and

$$b_{i1}^k = -\frac{w_{il}^k v^l}{v^1}.$$

The Proposition is proved. □

Let us approximate our initial problem by the smooth problems:

$$\frac{d}{dt}\frac{\partial L_i}{\partial \dot{x}} - \frac{\partial L_i}{\partial x} = \Lambda_i(x, \dot{x})a_i(x), \quad \Lambda_i(x, \dot{x}) = (\lambda_{i1}, \dots, \lambda_{in})(x, \dot{x}), \tag{13}$$

$$a_i(x)\dot{x} = 0, \quad L_i = \frac{1}{2}\dot{x}^T G_i(x)\dot{x} - V(x). \tag{14}$$

To express the Lagrange multipliers λ, one should take time derivative from the both sides of (14) and substitute there $\ddot{x}$ from (13).
After these transformations, one obtains

$$\Lambda_i^T(x, \dot{x}) = \left(a_i(x)G_i^{-1}(x)a_i^T(x)\right)^{-1} w\left(\frac{\partial a_i}{\partial x}, \frac{\partial G_i}{\partial x}, a_i, G_i, x, \dot{x}\right). \tag{15}$$

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The function w is smooth in $x \in M$, and in the other arguments w is smooth in the whole space.

Meanwhile system (13) with formula (15) takes the form

$$\ddot{x} = \phi_i(x, \dot{x}), \tag{16}$$

where $\phi_i \in C^\infty(M \times \mathbb{R}^m)$. Equation (14) determines an invariant manifold to system (16).

Recall that Proposition 1 implies $a_i(x_0)v = 0$.

The key point of our argument is as follows: systems (13) and (14) possesses the energy integral H, thus the function H is also the first integral for system (16).

Summarize the above argument as a lemma.

LEMMA 3.1. *For the constant $\tau > 0$ and for the initial conditions*

$$x_i(0) = x_0, \quad \dot{x}_i(0) = v,$$

system (16) has a solution $x_i(t) \in C^2(I_\tau)$ such that

$$H(x_i(t), \dot{x}_i(t)) = h' \tag{17}$$

and $a_i(x_i(t))\dot{x}_i(t) = 0, \quad \forall t \in I_\tau$.

COROLLARY 1. *The sequence $\{x_i(t)\}$ contains a subsequence that is convergent in $C^{1,\alpha}(I_\tau)$.*

For this subsequence, we use the same notation, that is

$$\|x_i - x\|_{C^{1,\alpha}(I_\tau)} \to 0.$$

Indeed, combining Proposition 1 and formulas (17), (15), (13), one has

$$\|\phi_i(x_i(s, v_i), \dot{x}_i(s, v_i))\| \le K. \tag{18}$$

The constant K is independent of $t \in (I_\tau)$ and i. Then, from Lemma 3.1 and formulas (16) and (18), one concludes that the sequence $\{\ddot{x}_i(t)\}$ is uniformly bounded in (I_τ).

LEMMA 3.2. *The function $x(t)$ from Corollary 1 satisfies (1).*

Proof. Follows directly from Proposition 1 and Corollary 1. □

Introduce matrices

$$A_i(x) = \left(a_{il}^k(x)\right)_{k,l=1,\dots,n}.$$

LEMMA 3.3. *The following estimate holds*

$$\sup_i \left\| \frac{\partial A_i^{-1}(x)}{\partial x} \right\|_{L^\infty(D_h)} < c_3.$$

Proof. Differentiate the $A_i^{-1}A_i = I$ by x^s:

$$\frac{\partial A_i^{-1}}{\partial x^s} = -A_i^{-1}\frac{\partial A_i}{\partial x^s}A_i^{-1}.$$

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Now, the assertion follows from Proposition 1 and assumption (7):

$$\left\| \frac{\partial A_i^{-1}}{\partial x^s} \right\|_{L^\infty(D_h)} \le \left\| A_i^{-1} \right\|^2_{L^\infty(D_h)} \cdot \left\| \frac{\partial A_i}{\partial x^s} \right\|_{L^\infty(D_h)}.$$

The Lemma is proved. □

Observe also that since A_i is uniformly closed to A and $\det A(x) \ge c_3 > 0, \quad x \in D_h$ one obtains $\|A_i^{-1}(x)\|_{L^\infty(D_h)} \le c_4$ for some $c_4 > 0$ if only i is sufficiently large.

Consider spaces

$$E_i = \{\psi \in H_0^1(I_\tau) \mid a_i(x_i(t))\psi(t) = 0\}.$$

LEMMA 3.4. *For any $\psi \in E$, there exist a sequence $\{\psi_i\}, \quad \psi_i \in E_i$ such that $\psi_i \to \psi$ weakly in $H_0^1(I_\tau)$ and strongly in $C(I_\tau)$.*

Proof. Introduce matrices

$$Q_i(x) = \big(a_{il}^k(x)\big), \quad l = n+1, \dots, m, \quad k = 1, \dots, n.$$

Fix an arbitrary function

$$\hat{\psi} = (\hat{\psi}^{n+1}, \dots, \hat{\psi}^m)^T \in H_0^1(I_\tau).$$

Consider a sequence

$$\tilde{\psi}_i(t) = -A_i^{-1}(x_i(t))Q_i(x_i(t))\hat{\psi}(t).$$

This sequence is bounded in $C[0, 1]$. By Lemma 3.3, the sequence

$$\begin{aligned}\frac{d}{dt}\tilde{\psi}_i(t) = &-\frac{\partial A_i^{-1}(x_i(t))}{\partial x^l}\dot{x}^l(t)Q_i(x_i(t))\hat{\psi}(t)\\ &- A_i^{-1}(x_i(t))\dot{x}^l(t)\frac{\partial Q_i(x_i(t))}{\partial x^l}\dot{x}^l(t)\hat{\psi}(t)\\ &- A_i^{-1}(x_i(t))Q_i(x_i(t))\frac{d}{dt}\hat{\psi}(t)\end{aligned}$$

is bounded in $L^2(I_\tau)$.

So, using the same notation for subsequences, we have $\tilde{\psi}_i(t) \to \tilde{\psi}(t)$ weakly in $H^1(I_\tau)$. Convergence in $C(I_\tau)$ follows from compact embedding $H^1(I_\tau) \subset C(I_\tau)$.

We want to pass to the limit as $i \to \infty$ in the equality

$$A_i(x_i(t))\tilde{\psi}_i(t) + Q_i(x_i(t))\hat{\psi}(t) = 0.$$

Since $H^1(I_\tau)$ is compactly embedded in $C[0, 1]$, the sequence $\tilde{\psi}_i$ converges to $\tilde{\psi}(t)$ in $C(I_\tau)$. Thus, we have

$$A(x(t))\tilde{\psi}(t) + Q(x(t))\hat{\psi}(t) = 0.$$

Thus, the sequence we are looking for is

$$\psi_i = (\tilde{\psi}_i^1, \dots, \tilde{\psi}_i^n, \hat{\psi}^{n+1}, \dots, \hat{\psi}^m)^T.$$

The Lemma is proved. □

Let us observe another evident fact.

LEMMA 3.5. *Suppose that a sequence* $\{u_i\} \in L^2(I_\tau)$ *converges weakly to* $u \in L^2(I_\tau)$. *We also have a sequence of functions* $\{f_i\} \subset C(I_\tau)$. *This sequence converges uniformly to* $f \in C(I_\tau)$.

Then,

$$(f_i, u_i)_{L^2(I_\tau)} \to (f, u)_{L^2(I_\tau)}.$$

Proof. Indeed, one has

$$(f_i, u_i)_{L^2(I_\tau)} = (f_i - f, u_i)_{L^2(I_\tau)} + (f, u_i)_{L^2(I_\tau)}$$

and since the sequence $\{u_i\}$ is bounded in $L^2(I_\tau)$ **[10]**, it follows that

$$|(f_i - f, u_i)_{L^2(I_\tau)}| \le \|f_i - f\|_{C(I_\tau)} \|u_i\|_{L^2(I_\tau)} \to 0.$$

The Lemma is proved. □

LEMMA 3.6. *Take any* $\psi \in E$ *and choose the sequence* ψ_i *in accordance with Lemma 3.4. Then,*

$$\int_{I_\tau} \frac{\partial L}{\partial \dot{x}}\big(x_i(t), \dot{x}_i(t)\big)\dot{\psi}_i(t)\,dt \to \int_{I_\tau} \frac{\partial L}{\partial \dot{x}}\big(x(t), \dot{x}(t)\big)\dot{\psi}(t)\,dt, \tag{19}$$

$$\int_{I_\tau} \frac{\partial L}{\partial x}\big(x_i(t), \dot{x}_i(t)\big)\psi_i(t)dt \to \int_{I_\tau} \frac{\partial L}{\partial x}\big(x(t), \dot{x}(t)\big)\psi(t)\,dt. \tag{20}$$

Proof. Limit (20) is trivial. Let us prove formula (19). Since

$$\int_{I_\tau} \frac{\partial L}{\partial \dot{x}}\big(x(t), \dot{x}(t)\big)\dot{\psi}(t)\,dt = \big(\dot{x}^T(\cdot)G(x(\cdot)), \dot{\psi}(\cdot)\big)_{L^2(I_\tau)},$$

$$\int_{I_\tau} \frac{\partial L}{\partial \dot{x}}\big(x_i(t), \dot{x}_i(t)\big)\dot{\psi}_i(t)\,dt = \big(\dot{x}_i^T(\cdot)G(x_i(\cdot)), \dot{\psi}_i(\cdot)\big)_{L^2(I_\tau)},$$

the assertion of the Lemma follows from Lemma 3.5.

The Lemma is proved. □

It remains to observe that the existence in Theorem 2.1 follows directly from Lemma 3.6.

The Theorem is proved.

4. Proof of the Theorem 2.2.

Introduce the following spaces:

$$X = \{\psi = (\psi^1, \dots, \psi^m)^T \mid \psi^k \in H_0^1(I_\tau)\}, \quad \|\cdot\|_X = \|\cdot\|_{L^2(I_\tau)},$$
$$Y = \{\varphi = (\varphi^1, \dots, \varphi^n)^T \mid \varphi^k \in L^2(I_\tau)\}, \quad \|\cdot\|_Y = \|\cdot\|_{L^2(I_\tau)}.$$

Note that the space X is not a Banach space.

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Let $S : X \to Y$ stand for the operator $\psi(t) \mapsto a(x(t))\psi(t)$. $F : X \to \mathbb{R}$ stands for the linear functional

$$\psi \mapsto \int_{I_\tau} \Big(\frac{\partial L}{\partial x}(x(t), \dot{x}(t))\psi(t) + \frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t))\dot{\psi}(t)\Big)\, dt \\ - \frac{\partial L}{\partial \dot{x}}(x(\tau), \dot{x}(\tau))\psi(\tau).$$

We know that $\ker S \subseteq \ker F$, let us check inequality (21).

For a vector $y = (y^1, \dots, y^m)^T \in \mathbb{R}^m$ introduce operations

$$\tilde{y} = (y^1, \dots, y^n)^T, \quad \hat{y} = (y^{n+1}, \dots, y^m)^T.$$

Let $\psi \in X$, then put

$$\tilde{\psi}_o(t) = B(x(\tau))\hat{\psi}(t), \quad \psi_o(t) = (\tilde{\psi}_o^T(t), \hat{\psi}(t)^T)^T, \quad \psi_\dagger(t) = \psi(t) - \psi_o(t)$$

so as

$$\psi_o \in \ker S, \quad \psi = \psi_o + \psi_\dagger, \quad \hat{\psi}_\dagger = 0.$$

So, we have

$$\|S(x(\cdot))\psi(\cdot)\|^2_{L^2(I_\tau)} = \|A(x(\cdot))\tilde{\psi}_\dagger(\cdot)\|^2_{L^2(I_\tau)} \geq c_8\|\psi_\dagger(\cdot)\|^2_{L^2(I_\tau)}$$

for some $c_8 > 0$ and finally one yields

$$\|\psi_\dagger(\cdot)\|^2_{L^2(I_\tau)} \geq \inf_{v \in \ker S} \|\psi(\cdot) + v(\cdot)\|^2_{L^2(I_\tau)}.$$

By Lemma 5.1, we have a bounded functional

$$\Gamma : S(X) \to \mathbb{R}, \quad F = \Gamma S, \quad \|\Gamma\| \leq \frac{1}{c_8}\|F\|.$$

Using the Hahn–Banach theorem, we extend this functional to a bounded functional $\Gamma_1 : Y \to \mathbb{R}$.

By the Riesz Representation Theorem, one can find a function $\gamma(\tau) = (\gamma_1, \dots, \gamma_n)(\tau), \quad \gamma_k \in L^2(I_\tau)$ such that

$$\Gamma_1\varphi = (\gamma, \varphi)_{L^2(I_\tau)}, \quad \|\gamma\|_{L^2(I_\tau)} \leq \frac{1}{c_8}\|F\|.$$

The Theorem is proved.

5. A lemma from functional analysis.

The following lemma is well known. We bring its proof just for completeness of exposition.

Let X, Y, Z be normed spaces and linear operators

$$F : X \to Z, \quad S : X \to Y$$

be bounded.

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LEMMA 5.1. *Suppose that* $\ker S \subseteq \ker F$ *and*

$$\inf_{u\in\ker S} \|z+u\|_X \leq C\|Sz\|_Y, \tag{21}$$

for some $C > 0$.

Then, there is a bounded operator $\Gamma : S(X) \to Z$ *such that*

$$F = \Gamma S, \quad \|\Gamma\| \leq C\|F\|.$$

Proof. Let

$$\pi_S : X \to V = X/\ker S, \quad \pi_F : X \to U = X/\ker F, \quad \pi : V \to U$$

be natural projections.

The spaces U, V are normed spaces with norms

$$\|u\|_U = \inf_{w\in\ker F} \|[u]+w\|_X, \quad \|v\|_V = \inf_{w\in\ker S} \|[v]+w\|_X,$$

where $[u] \in X$ is the element that generates corresponding class u that is $u = \ker F + [u]$.

From [**9**], we know that $F = F_1\pi_F$, $\quad S = S_1\pi_S$ and the bounded operators $F_1 : U \to F(X)$, $\quad S_1 : V \to S(X)$ are one-to-one.

Hence, we have $\Gamma = F_1\pi S_1^{-1}$. By formula (21), the operator S_1^{-1} is bounded.

The Lemma is proved. □

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