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THE CHARACTERISATION OF MODULAR GROUP ALGEBRAS HAVING UNIT GROUPS OF NILPOTENCY CLASS 3

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ABSTRACT. The unit group of the modular group algebra of a finite p-group in characteristic p is nilpotent. The p-groups for which it is of nilpotency class 3 were determined in work of Coleman, Passman, Shalev and Mann when $p \geq 3$. We resolve the $p = 2$ case here which completes the classification.

If G is a finite p-group and F is a field of characteristic p, then the group $U(FG)$ of units of the modular group algebra FG is a direct product $U(F) \times V$, where $V = V(FG)$ is the subgroup of normalised units. As $V = 1 + I$ where $I = I(FG)$ is the augmentation ideal, V is also a p-group, finite if F is. Since I is nilpotent as an ideal, V is a nilpotent group and the class of $U(FG)$ is equal to that of V. Again the nilpotency of I shows that I, and so FG, are nilpotent as Lie algebras and of the same class. A recent theorem of Du [4] shows that the nilpotency class of V as a group is equal to that of I as a Lie algebra (here denoted $\mathrm{cl}\, V$, $\mathrm{cl}\, I$ respectively.)

Recent studies, particularly those of Shalev, have indicated and established many patterns in the dependence of the class of V on aspects of G. The classification of those G for which $\mathrm{cl}\, V = 3$ has been completed except for $p = 2$. The present paper resolves this case, which gives the following theorem.

THEOREM. *Let G be a finite p-group and F a field of characteristic p. Then the following are equivalent:*

(i) $\mathrm{cl}\, U(FG) = 3$;

(ii) $\mathrm{cl}\, FG = 3$;

(iii) $\mathrm{cl}\, G = 2$ *and* $G' \approx C_3$ *or* C_2^2.

One of the earliest results on $\mathrm{cl}\, V$ is that of Coleman and Passman [3] which shows that $\mathrm{cl}\, V \geq p$ if G is a non-abelian p-group. The theorem, then, is of significance only for $p = 2$ or 3. That it holds for $p = 3$ is a consequence of the fact that $\mathrm{cl}\, V = p$ if and only if $G' \approx C_p$. As mentioned in [10], this was conjectured by the second author (in work which was advanced through discussions with Frank Levin who kindly made available the material of [5] before its publication.) The sufficiency of the condition is not difficult; it was also noticed by Baginski [1]. The necessity was established in [8].

Adapting the techniques of [3] and introducing new ones, Shalev, together with Mann, obtained many results on $\mathrm{cl}\, V$, mainly upper bounds [11, 8, 12]. The most far-reaching

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results are presented in [12] and were deduced on the basis of the recent breakthrough of Du [4] generalising the work of H. Laue [7]. The main theorem in [4] implies that the n-th term $\zeta_n(V)$ of the upper central series of V coincides with $1 + \zeta_n(I)$, where $\zeta_n(I)$ denotes the n-th term of the upper central series of I considered as Lie algebra.

Du's results permit the replacement of group commutator calculations by ones involving Lie commutators. This simplification underlies the proof given here. It is not, however, how the result was arrived at. The first author [9] has developed a suite of programs written in the Cayley language [2] which construct power-commutator presentations for the groups $V(FG)$ where $|G|$ divides 32 and $|F| = 2$. Once available, such a presentation facilitates the analysis of V using built-in Cayley functions. We studied explicit examples of 2-groups G for which $\operatorname{cl} V(FG) = 4$ in order to gain insight into how nontrivial 4-fold commutators arose. These calculations guided the elaboration of the proof given here.

PROOF OF THE THEOREM. That (i) and (ii) are equivalent has been noted; we may thus use $\operatorname{cl} V$ and $\operatorname{cl} I$ interchangably. We will show that (ii) and (iii) are equivalent. The introductory remarks have shown that only the case $p = 2$ is at issue. The statement in this case reduces to: $\operatorname{cl} I = 3$ if and only if $\operatorname{cl} G = 2$ and $G' \approx C_2^2$. The sufficiency of the conditions follows from Corollary C of [8] for $p = 2$.

Their necessity is established by showing that a group G which fails to satisfy them has $\operatorname{cl} I \geq 4$, contrary to the hypothesis. This is accomplished by the detailed study of a number of cases. The usual pattern specifies a minimal counterexample sufficiently closely to allow the explicit construction of a nontrivial Lie commutator of length 4.

The hypothesis that $\operatorname{cl} I = 3$ leads to a dichotomy: either $\operatorname{cl} G = 2$ or $\operatorname{cl} G = 3$. In the first case we want to conclude that $G' \approx C_2^2$. In the second, we want to derive a contradiction. We begin with the first case, and so assume that

(I) $\operatorname{cl} G = 2$ and $\operatorname{cl} I = 3$.

We show that G' is elementary abelian of order 4 in two steps: that its exponent is 2; that its order is 4.

(Ia) $\exp G' = 2$.

PROOF. Assume that $\exp G' \geq 4$. It suffices to show that $I_4 \neq 0$, where I_4 denotes the 4-th term of the lower central series of I as Lie algebra, or, equivalently, that $V_4 \neq 1$, where V_4 denotes the 4-th term of the lower central series of the group V. The latter follows from a remark of Shalev [11, p. 265]. As the proof of this remark is omitted, we provide here an independent proof that $I_4 \neq 0$.

Since G' is central, we may assume that $G' \approx C_4$. Write $G' = \langle a \rangle$ where $a = [x, y]$ for some $x, y \in G$. Then, with the convention that multiple commutators be left-normed, $(x, y, y, y) = y^3x(a-1)^3 \neq 0$.

(Ib) $d(G') = 2$.

PROOF. Assume that $d(G') \geq 3$. Again we show that $I_4 \neq 0$. In a minimal counterexample G, $d(G') = 3$ and any proper subgroup is of class ≤ 2 with commutator subgroup elementary abelian of rank ≤ 2. Note that (Ia) implies that $G^2 \leq \zeta(G)$.

If a, b, c are independent elements of the elementary abelian group G', then it is easy to see that $(c-1)(b-1)(a-1) \neq 0$. The next item shows how to obtain such a product in a 4-fold Lie commutator.

LEMMA. *Let $g, h, k, \ell \in G$. Write $a = [g,h]$, $b = [h,k]$, $c = [k,\ell]$. Then the commutator (g, gh, hk, ℓ) is equal to $m(c-1)(b-1)(a-1)$ for some $m \in G$.*

PROOF. $(g, gh) = hz(a-1)$, where $z = [h, g^{-1}]g^2 \in \zeta(G)$. Thus, $(g, gh, hk) = kz'(b-1)(a-1)$ for some $z' \in \zeta(G)$ and $(g, gh, hk, \ell) = \ell kz'(c-1)(b-1)(a-1)$.

This lemma then gives us the following criterion for showing that $I_4 \neq 0$ and so obtaining a contradiction:

$I_4 \neq 0$ if and only if there are elements g, h, k, ℓ in G such that $[g,h]$, $[h,k]$ and $[k,\ell]$ are independent elements of the elementary abelian group G'.

From this we can conclude that the minimal counterexample G has $d(G) \geq 4$, for, if $G = \langle x, y, z\rangle$, then $G' = \langle [x,y], [y,z], [z,x]\rangle$ and these commutators must be independent as $G' \approx C_2^3$. In particular, if H is a 3-generator subgroup of G, then H is proper. As $d(H') \leq 2$, there is an element $z \in H$ such that $H' = [H, \langle z\rangle]$. It follows that there are commuting elements $x, y \in H$ such that $H = \langle x, y, z\rangle$. To prove this, choose $x, y \in H$ such that $H = \langle x, y, z\rangle$. But $[x,y] \in \{1, [x,z], [y,z], [xy,z]\}$ by hypothesis. Each possibility is then examined and new generators chosen as required. For example, if $[x,y] = [x,z]$, then $[x, zy] = 1$ while $H = \langle x, zy, z\rangle$.

We next turn our attention to the possibilities for centralisers. Since, for any g in G, $|G : C_G(g)| = |g^G| = |\{[g,x] : x \in G\}|$, the index of a centraliser divides 8. Also centralisers are normal in a class 2 group.

Suppose that there is an element g whose centraliser is of index 8. Then $G' = [G, \langle g\rangle]$ so that $G' = \langle [x,g], [y,g], [z,g]\rangle$ for some $x, y, z \in G$. As above, we may assume that $[x,y] = 1$ from which it follows that $[x, yg]$, $[yg, g]$ and $[g,z]$ are independent. By our criterion, $I_4 \neq 0$, a contradiction.

Suppose that there is an element g whose centraliser is of index 4. Then there are elements $x, y \in G$ such that $G = C\langle x, y\rangle$, where $C = C_G(g)$. Let $H = \langle x, y, g\rangle$ so that $H' = [H, \langle g\rangle]$ and H' has order 4. As above, we may assume that $[x,y] = 1$ so that the subgroup $K = \langle x, y\rangle$ is abelian. Thus $G' = C'[C,K]$ and $H' \leq [C,K]$. If $G' = [C,K]$, then there must be an element $u \in C$ such that $[u,K] \not\leq H'$. We may assume that $[u,x] \notin H'$ whence $[u,x]$, $[x,g]$ and $[g,y]$ are independent, a contradiction. If $G' \neq [C,K]$, then $G' = C'H'$ and there are elements $u, v \in C$ such that $[u,v] \notin H'$. But then $[u, vx]$, $[vx, g]$ and $[g,y]$ are independent, again a contradiction.

We have now seen that all proper centralisers are of index 2. Choose $x, y \in G$ such that $[x,y] \neq 1$. Their centralisers are proper and distinct; let N denote the intersection of these centralisers so that $N \triangleleft G$. Putting $K = \langle x, y\rangle$, we have $G = NK$ and $G' = N'\langle [x,y]\rangle$ whence $N' = \langle [u,v], [s,t]\rangle$ for some $u, v, s, t \in N$. But then $[u, vx]$, $[vx, sy]$ and $[sy, t]$ are independent, providing the final contradiction in this case.

REMARKS. The arguments above establish more than required. They justify the observation, prompted by Coleman, that a 2-group G which is minimal with respect to its commutator subgroup being central and elementary abelian of rank 3, has $d(G) = 3$ or 4. An example of the latter is the semi-direct product $G = VE$ in which V is a vector space of dimension 4 over the field of 2 elements and E is the elementary abelian subgroup of GL(4, 2) generated by the matrices $I + E_{1j}, j = 2, 3, 4$, where I is the identity matrix and E_{ij} the standard matrix units, in its natural action.

Alternatives to these arguments are possible: all such groups can be shown to have an element with centraliser of index 8; Mann notes the relevance of a theorem of Knoche [6, p. 309] which states that a p-group G all of whose centralisers are of index $\leq p$, has $|G'| \leq p$.

We turn now to the second case. Its proof is also an examination of possible minimal counterexamples; it appeals to the result of case (I). Our assumption is

(II) $\operatorname{cl} G = 3$ and $\operatorname{cl} I = 3$.

We show that this case is impossible.

Let G be a group of minimal order satisfying (II). If S is a section of G having $\operatorname{cl} S = 3$, then $\operatorname{cl} I(FS) = 3$. It follows that every proper section S of G is abelian or of class 2; if $\operatorname{cl} S = 2$, then S' is elementary abelian of rank ≤ 2. By the minimality of G, $|G_3| = 2$.

As Lie algebra $I(FG/G_3)$ has class 2 or 3. If it is 3, then case (I) shows that $G'/G_3 \approx C_2^2$. If it is 2, then $G'/G_3 \approx C_2$ by Lemma 8(a) of [8]. (For $p = 2$, there is a simple alternative way of deducing that, for E a field of characteristic 2 and X a 2-group, $\operatorname{cl} EX = 2$ implies that $X' \approx C_2$: $\operatorname{cl} EX = 2$ if and only if $I(X')EX$, the 2-nd term of the lower central series of EX, is central in EX; as X' is central, $\big(I(X')EX, EX\big) = I(X')^2EX$ which is 0 if and only if $I(X')^2 = 0$ and this happens precisely when $X' \approx C_2$.)

Suppose first that $G'/G_3 \approx C_2^2$. Then $d(G) \geq 3$ so that every 2-generator subgroup is proper and so of class ≤ 2 with an elementary abelian commutator subgroup. It follows as in the proof of case (I) that $G^2 \leq \zeta(G)$. But $G' \leq G^2$ so that $\operatorname{cl} G = 2$, a contradiction.

Secondly, suppose that $G'/G_3 \approx C_2$. Once again this leads us to two cases, $G' \approx C_4$ or C_2^2. In both, the subgroup $C = C_G(G')$ is used and has index 2.

In the case $G' \approx C_4$, C' is proper in G' since C' is elementary abelian. Thus, G' is generated by some commutator $g = [x, y]$ where we may assume that $x \notin C$. As $|G : C| = 2$, we may assume that $y \in C$. But then $(x, y, y, y) = y^3x(g-1)^3 \neq 0$ and so $I_4 \neq 0$, a contradiction.

In the case $G' \approx C_2^2$, we have $G' = \langle g \rangle G_3$, $g = [y, x]$ for some $x, y \in G$, where we may assume that $x \notin C$ since $C' \leq \zeta(G)$. But $C = C_G(g)$ so that $G_3 = \langle z \rangle$ for $z = [g, x] = [y, x, x]$. But then $(y, x, x, x) = x^3y(g-1)(z-1) \neq 0$. As before, this gives the contradiction $I_4 \neq 0$, which completes the proof.

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