A GENERALIZATION OF DIVISIBILITY AND INJECTIVITY IN MODULES

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1. Classically, there has been, for obvious reasons, an intimate relation between the concepts "rings of quotients" and "divisible modules". Recently, however, their generalizations have appeared to diverge.

For example, Hattori ([9]) and Levy ([15]) have generalized the concept of "divisibility" as follows: Hattori (respectively Levy) defines a left R-module M over a ring R to be divisible if and only if $\mathrm{Ext}_R^1(R/I, M) = 0$ for every principal left ideal $I \subset R$ (respectively, every principal left ideal $I \subset R$ which is generated by a regular element of R).

On the other hand, a series of results by Johnson ([12]), Utumi ([16]), and Gabriel ([2]), which culminate in the beautiful paper of Lambek ([14]), have generalized the concept of "ring of quotients" in terms of the injective envelope, as developed by Eckman and Schopf ([5]), and suitable inverse limits.

This paper may be considered to be a preliminary step toward the unification of these ideas.

2. Let Σ be a set of left ideals in a ring* R. We define:

(i) A left R-module M is Σ-divisible if and only if $\mathrm{Ext}_R^1(R/I, M) = 0$, for each $I \in \Sigma$;

* We shall assume all rings have a multiplicative identity.

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(ii) If M is a submodule of a left R-module N, then M is a Σ-essential submodule of N (N is a Σ-essential extension of M) if and only if for every $0 \neq x \in N$, the left ideal $I_M(x) = \{r \in R \mid rx \in M\} \in \Sigma$ and $I_M(x)x \neq 0$;

(iii) If M is a submodule of a left R-module N, then M is a Σ-pure submodule of N (N is a Σ-pure extension of M) if and only if whenever $I \in \Sigma$ and f is a homomorphism from I into M, if f is extendable to a homomorphism from R into N, it is extendable to a homomorphism from R into M, (Kertesz [13] and Butler and Horrocks [3], page 210-211).

One may easily verify that these definitions generalize the customary definitions for abelian groups. In view of the properties of Ext_R^1, one may also verify that many classical results, as well as those of Hattori, are easily generalized in this context. However, we shall concentrate on applications of these ideas to generalized rings of quotients.

Note that $\mathrm{Ext}_R^1(R/I, M) = 0$ if and only if every homomorphism from I to M may be extended to a homomorphism from R to M, (Cartan-Eilenberg [4]). Thus, by Baer's theorem ([1]), the term "Σ-injectivity" might have been substituted for "Σ-divisibility".

We only remark that the above is dualizable: If Σ is a set of right ideals in R, a left R-module M is Σ-torsion free if $\mathrm{Tor}_1^R(R/I, M) = 0$ for all $I \in \Sigma$. This has been noted for principal right ideals by Hattori ([9]). Also, some global dimension theoretic potential exists in these concepts and will be studied in a later paper (cf. Butler and Horrocks [3]).

3. Let us now list three properties which Σ may satisfy:

(P_1) If $I \in \Sigma$ and J is a left ideal which contains I, then $J \in \Sigma$;

(P_2) If $I \in \Sigma$ and $r \in R$, then $Ir^{-1} = \{x \in R \mid xr \in I\} \in \Sigma$;

(P_3) If I is a left ideal in R, $J \in \Sigma$, and $Ij^{-1} \in \Sigma$, for each $j \in J$, then* $I \in \Sigma$.

We note that if E is an essential extension of the module M and $\Sigma = \{I_M(x) \mid x \in E\}$, then Σ satisfies P_2, since $I_M(rx) = I_M(x)r^{-1}$.

THEOREM 1. Assume Σ satisfies property P_1. A left R-module M is Σ-divisible if and only if $\mathrm{Ext}^1_R(R/I, M) = 0$, for every large left ideal $I \in \Sigma$. (I is large if it is an essential sub-module of ${}_R R$.)

COROLLARY. M is injective if and only if $\mathrm{Ext}^1_R(R/I, M) = 0$, for every large left ideal I in R.

This result was, in essence, noted by Johnson ([12]).

Proof. (Only if) This is clear.

(If) Suppose that $I \in \Sigma$, and f is a homomorphism from I to M. Let S be the set $\{(J, g) \mid J$ is a left ideal in R which contains I and g is a homomorphism from J into M which extends $f.\}$. By a simple use of Zorn's lemma we determine that S has a maximal element, say (J, g). If J is not large, then there is a left ideal $0 \neq K \subset R$ such that $K \cap J = \{0\}$. Define $\bar{g}$ from $J + K$ to M by $\bar{g}(j + K) = g(j)$. $\bar{g}$ clearly extends g, and hence f, and we contradict the maximality of (J, g). Thus J is large and the result follows.

4. LEMMA 1. If Σ satisfies property P_3, if P is a Σ-essential extension of N, and N is a Σ-essential extension of M, then P is a Σ-essential extension of M.

* As has been indicated in a private communication, K. L. Chew has proved that Gabriel's fourth property ([2]; see also [14]). "If $I, J \in \Sigma$, then $I \cap J \in \Sigma$", is a consequence of the other three.

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Proof. If $x \in P$, then $I_N(x) \in \Sigma$ and $I_N(x)x \neq 0$. If $i \in I_N(x)$, then $ix \in N$ and thus $I_M(ix) = I_M(x)i^{-1} \in \Sigma$. Since this is true for all $i \in I_N(x)$, $I_M(x) \in \Sigma$. By the definition of $I_M(x)$, if $I_M(x)x = 0$ then $I_M(x)(rx) = 0$ for all $r \in R$. But there is an $i \in I_N(x)$ and a $j \in I_M(ix)$ such that $ij \in I_M(x)$ and $ijx \neq 0$ and we have a contradiction.

THEOREM 2. Suppose Σ satisfies properties P_1, P_2, and P_3, and let M be an arbitrary left R-module. Then there is a unique, up to isomorphism, extension E to M which satisfies the following equivalent conditions:

(i) E is a maximal Σ-essential extension of M;

(ii) E is a minimal Σ-divisible extension of M;

(iii) E is a Σ-essential, Σ-divisible extension of M.

This generalizes Eckmann and Schopf's injective envelope. Also, Maranda ([18], page 121) has proved a similar, although less explicit, result.

Proof. Clearly every module has both Σ-essential and Σ-divisible extensions, since it is a Σ-essential extension of itself and every injective module is Σ-divisible.

Let N be a Σ-essential extension of the R-module M and let P be a Σ-divisible module. Suppose that g is an R-homomorphism from M to P. Define $S = \{(\bar{N}, \bar{g}) \mid \bar{N}$ is a sub-module of N which contains M, and $\bar{g}$ extends g from $\bar{N}$ into $P.\}$. Let $(\bar{N}, \bar{g})$ be a maximal element in S, which exists by Zorn's lemma, assume $\bar{N} \neq N$, and let $x \in N - \bar{N}$. Since N is a Σ-essential extension of M, $I_M(x) \in \Sigma$, and since $I_M(x) \subset I_{\bar{N}}(x)$, we have $I_{\bar{N}}(x) \in \Sigma$. Define f from $I_{\bar{N}}(x)$ to P by $f(y) = \bar{g}(yx)$ for $y \in I_{\bar{N}}(x)$. Clearly f is a homomorphism, and since P is Σ-divisible, f is extendable to $\bar{f}$ from R into P. Define h from $\bar{N} + Rx$ to P by $h(n + rs) = \bar{g}(n) + \bar{f}(r)$. h is an extension of $\bar{g}$ and we contra-

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dict maximality unless $\bar{N} = N$.

Note that if the kernel of g is $\{0\}$, then, since N is an essential extension of M, necessarily the kernel of $\bar{g}$ must be $\{0\}$. In particular, every Σ-essential extension of M is embeddable in every Σ-divisible extension of M.

Now let T be the set of all Σ-essential extensions of M which are contained in the injective envelope* E of M, i.e., by Zorn's lemma again, we find a maximal Σ-essential extension in E, say N.

Now suppose $I \in \Sigma$, and f is a non-zero homomorphism from I to N. f may be extended to a homomorphism $\bar{f}$ from R to E. Let $y = \bar{f}(1)$. Since $I \subset I_N(y)$, we have $I_N(y)r^{-1} \in \Sigma$ for each $r \in R$. Further, if $x \in N$ and $r \in R$, then $I_N(x + ry) = I_N(y)r^{-1} \in \Sigma$. If $I_N(x + ry)(x + ry) = 0$, then $R(x + ry) \cap N = \{0\}$ and $x + ry = 0$, since $Ry + N \subset P$ and is thus an essential extension of M. Hence $N + Ry$ is a Σ-essential extension of N. But this contradicts the maximality of N unless $y \in N$. But then N is a Σ-pure submodule of the Σ-divisible module and thus must be Σ-divisible. The equivalence of the three properties and the uniqueness of this maximum Σ-essential extension now follow easily from the first part of the proof.

Perhaps it is appropriate to call this extension the Σ-divisible envelope of M, - in symbols $E_\Sigma(M)$.

5. Now let Σ be a set of left ideals in R, $E = E_\Sigma(R)$, $H = \mathrm{Hom}\ (E, E)$, and $Q = Q_\Sigma = \mathrm{Hom}\ (E, E)$. We embed R in Q in the customary way. Q_Σ may be called the ring of quotients with respect to Σ.

In view of the injective envelope-like properties of E,

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* It is unfortunate that we must use the injective envelope. However, the author sees no way to remove this difficulty at the present time.

one may easily verify many of Lambek's results, particularly those of sections 2, 6, 7, and 8 of [14], in this context, if we suitably generalize the definition of dense submodule: A submodule N of M is Σ-dense in a module M if for each $h \in \mathrm{Hom}_R(E_\Sigma(M), E_\Sigma(M))$, $h(N) = \{0\}$ implies $h(M) = \{0\}$. The proofs are identical to Lambek's if we reinterpret the symbols. Note that as the size of Σ increases the set of dense ideals decreases in size.

Following Gabriel ([2]), we suppose S is a multiplicatively closed subset of the ring R and define $\Sigma(S) = \{I$ a left ideal in $R \mid S \cap I$ is non-empty$\}$. We leave it to the reader to determine that $\Sigma(S)$ satisfies P_1, P_2, and P_3 if the condition of Ore is satisfied.* In fact, Ore's condition is equivalent in this case to P_2.

Let us suppose subsequently that S is the set of regular elements in R and that Ore's condition is valid for S. As in Lambek's appendix I ([14]), we see that every regular element in R has an inverse in $Q_{\Sigma(R)}$.

Now assume $q \in Q$. Then q is completely determined by $q(1)$, and $q(1) \neq 0$ if $q \neq 0$. $q(1) \in E$ and thus $I_R(q(1)) \in \Sigma$. Now there is a $b \in S \cap I_R(q(1))$, and we have $bq(1) \in R$. Further, since b has an inverse in Q, $bq(1) \neq 0$. Let $bq(1) = a$ i.e., $q(1) = b^{-1}a$. Clearly $H(1) = E$, so if $e \in E$, there is an $h \in H$ such that $h(1) = e$. We have

$$q(e) = q(h(1)) = h(q(1)) = h(b^{-1}a) = b^{-1}ah(1) = b^{-1}ae \quad \text{for all } e \in E$$

and $q = b^{-1}a$. We state our result in

THEOREM 3. Let the set of regular elements in R satisfy Ore's condition. If Σ is the set of left ideals in R which contain regular elements, then Σ satisfies properties P_1, P_2 and P_3. In this case Q is isomorphic to the classical ring of quotients of R([11]). (Note this is the left classical ring of quotients.)

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* Condition of Ore: If $a, b \in R$, $b \in S$, there exist $\bar{a}, \bar{b} \in R$, with $\bar{b} \in S$, such that $\bar{b}a = \bar{a}b$. (Jacobson [14], page 118.)

COROLLARY. If every large left ideal in R contains a regular element and if the regular elements in R satisfy Ore's condition, then Utumi's ring of quotients ([14]) corresponds to the classical ring of quotients.

Proof. By theorem 1, the injective envelope and the Σ-divisible envelope determined by the set of left ideals Σ which intersect the set of regular elements correspond, and thus so must the rings of quotients.

By a theorem of Goldie's ([7], theorem 4.8), every semi-prime ring with maximum condition on left ideals satisfies the above criterion. We thus have an alternate proof of Goldie's theorem ([8]) which states that the two rings of quotients coincide in this case.

Clearly every commutative integral domain satisfies the conditions of the corollary. However, as an example, let G be the free semi-group generated by the symbols x and y, and define R to be the semi-group ring of G over the integers with the identity adjoined. Clearly R is a (non-commutative) integral domain and every large left ideal in R contains regular elements. But $Rx \cap Ry = \{0\}$ and thus Ore's condition is not satisfied. Hence, Utumi's ring of quotients of R is not the classical ring of quotients, since the latter doesn't even exist.

Conjecture: The hypotheses of the corollary are necessary as well as sufficient.

6. Remark: It would appear that many of these ideas could be carried over to the case where our ring does not have an identity. Vehicles could be suitable generalizations of either Kertesz's algebraically closed modules ([13]) or Faith and Utumi's Baer modules ([6]. The definition of Σ-essential extension would however necessarily have to be revised in the more general context, since a module would need not be a Σ-essential extension of itself (cf. Herstein and Small [10], lemma 2).

As Lambek has noted in his case ([14]), the set of Σ-dense left ideals, associated with a particular Σ which satisfies P_1,

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P_2 and P_3, is a suitable system for Gabriel's construction procedure ([2]). However, the exact relationships between the two constructions still require clarification.

Also, note a possible generalization of Johnson's singular submodule: Let M be a left R-module and suppose Σ is a set of left ideals which satisfies P_1, P_2 and further, if I, J $\in \Sigma$, then $I \cap J \in \Sigma$. The set $s_\Sigma(M) = \{m \in M \mid \mathrm{Ann}_R(m) \in \Sigma.\}$ is then a submodule of M, which we might call the Σ-singular submodule of M. Its possible usefulness is indicated by the following suggestive example: If R is a commutative integral domain and Σ is the set of all non-zero ideals in R, then $s_\Sigma(M)$ is the torsion submodule of M (cf. Gentile [17] and Maranda [18]). The implications of this definition will be discussed in a subsequent paper.

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