FURTHER DUALS OF A VERBAL SUBGROUP

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1. In a previous paper [**3**] we gave two methods for constructing subgroups which in certain senses may be considered to be dual to a verbal subgroup $V_f(G)$ of an arbitrary group G. Associated with a word $h(u, v)$ in the two symbols u and v, we have (i) the first dual subgroup $H_f^1(G)$, which is defined as the minimal subgroup of G containing all elements ξ of G for which

$$f(h(x_1, \xi), x_2, \ldots, x_n) = \ldots = f(x_1, x_2, \ldots, h(x_n, \xi)) = f(x_1, x_2, \ldots, x_n)$$

for all values of $x_1, x_2, \ldots, x_n$ in G, and (ii), the second dual subgroup $H_f^2(G)$ which is defined as the minimal subgroup of G containing all elements z of G for which

$$h(f(x_1, x_2, \ldots, x_n), z) = 1$$

for all values of $x_1, x_2, \ldots, x_n$ in G. Below we introduce slight variations to these definitions, which give rise to the concepts of the third and the fourth dual subgroups respectively. For certain values of $h(u, v)$ we obtain concepts which also arise from $H_f^1(G)$ and $H_f^2(G)$, namely, the marginal subgroup, the invariable subgroup and the centralizer of a verbal subgroup. We also obtain the new concepts of elemental subgroups and commutal subgroups and briefly sketch some of their properties. Finally we conclude by showing that MacLane's dual for the centralizer of a verbal subgroup is a closely related verbal subgroup.

2. The third and fourth dual subgroups. Let $h(u, v)$ be a word in the symbols u, v and their inverses and let $f(x_1, x_2, \ldots, x_n)$ denote a typical word of a set of words. To obtain the various duals of $V_f(G)$ — the subgroup generated by all values of the words f in G — we shall consider the set of words f to be fixed and shall vary $h(u, v)$.

The third dual subgroup $H_f^3(G)$ of a group G is defined to be the subgroup of G generated by all elements ξ of G for which

$$f(h(x_1, \xi), x_2, \ldots, x_n) = f(h(x_1, 1), x_2, \ldots, x_n),$$

$$f(x_1, h(x_2, \xi), \ldots, x_n) = f(x_1, h(x_2, 1), \ldots, x_n),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(x_1, x_2, \ldots, h(x_n, \xi)) = f(x_1, x_2, \ldots, h(x_n, 1)),$$

for all values of $x_1, x_2, \ldots, x_n$ in G.

The following four particular cases are of interest :

(*a*) $h(u, v) = uv, vu, u^{-1}v, vu^{-1}, uv^{-1}, v^{-1}u, u^{-1}v^{-1}, v^{-1}u^{-1}$
give the *marginal subgroup* $M_f(G)$.†

(*b*) $h(u, v) = v^{-1}uv, vuv^{-1}, v^{-1}u^{-1}v, vu^{-1}v^{-1}$
give the *invariable subgroup* $I_f(G)$.†

† In these cases the set of all ξ not only generates but actually forms the dual subgroup $H_f^3(G)$.

(*c*) $h(u, v) = v$

gives the *elemental subgroup* $E_f(G)$.

(*d*) $h(u, v) = [u, v], [v^{-1}, u]$

give the *commutal subgroup* $C_f(G)$.

The fourth dual subgroup $H_f^4(G)$ of a group G is defined to be the subgroup of G generated by all elements z of G for which

$$h(f(x_1, x_2, \ldots, x_n), z) = h(f(x_1, x_2, \ldots, x_n), 1)$$

for all values of $x_1, x_2, \ldots, x_n$ in G.

We now have the following particular cases corresponding to those given above :

(a') $h(u, v) = uv, vu, u^{-1}v, vu^{-1}, uv^{-1}, v^{-1}u, u^{-1}v^{-1}, v^{-1}u^{-1}$

give the trivial subgroup.

(b') $h(u, v) = v^{-1}uv, vuv^{-1}, v^{-1}u^{-1}v, vu^{-1}v^{-1}$

give the *centralizer of the verbal subgroup* $V_f(G)$, *namely*, $Z_f(G)$.

(c') $h(u, v) = v, v^{-1}$

give the trivial subgroup.

(d') $h(u, v) = [u, v], [v, u], [u^{-1}, v], [v, u^{-1}], [u, v^{-1}], [v^{-1}, u], [u^{-1}, v^{-1}], [v^{-1}, u^{-1}]$

give the centralizer of the verbal subgroup $V_f(G)$, namely, $Z_f(G)$.

The properties of the third dual subgroup are similar to those of the first dual subgroup as exhibited in [**3**]. In particular, we have the following two important results :

(1) *The intersection of third dual subgroups (with the same $h(u, v)$) of a group is a third dual subgroup of the group, namely, that associated with all the words by which the original third dual subgroups were defined.*

For two fixed words f and g we define the subgroups A and B of G (depending on f and g) by the relations

$$H_f^3(G/Z_g(G)) = A/Z_g(G) \quad \text{and} \quad H_g^3(G/Z_f(G)) = B/Z_f(G)$$

(with the same $h(u, v)$) respectively. Then we have

(2) *If f and g are typical words of two sets of words on distinct symbols, then the third dual subgroup associated with all commutators of the form $[f, g]$ is the intersection of all subgroups A and B obtained by varying f and g over their distinct sets.*

Marginal subgroups, invariable subgroups and the centralizer of a verbal subgroup have already been studied in [**3**]. We now proceed to outline a number of properties of elemental subgroups and commutal subgroups.

We note the following special cases :

(i) *If* $f(x_1, x_2, \ldots, x_n) = [[\ldots[[x_1, x_2], x_3], \ldots], x_n]$, *then* $E_f(G)$ *and* $C_f(G)$ *are the* $(n-1)$*th and the nth members of the upper central series of G respectively.*

(ii) *If* $f = x^n$, *then* $E_f(G)$ *is the subgroup generated by all elements* ξ *of G which satisfy the equation* $\xi^n = 1$. *On the other hand,* $C_f(G)$ *is the subgroup generated by all elements* ξ *of G which satisfy the relation* $[G, \xi]^n = 1$.

As in the case of marginal subgroups and invariable subgroups, the union of elemental subgroups of a group is not, in general, an elemental subgroup of the group. The example given for marginal subgroups in [**3**] holds also for elemental subgroups. This is because the

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concepts of elemental subgroup and marginal subgroup coincide for abelian groups.

For abelian groups, the concept of a commutal subgroup is of no importance. However, the commutal subgroups of nilpotent groups of class two are of particular interest. They are either equal to the whole group or of the type given in (ii) above, namely, the set of all elements ξ satisfying the relation $[G, \xi]^n = 1$. In the latter case we have that

$$C_n(G) \geqslant E_n(G) \,.\, Z_n(G),$$

where $Z_n(G)$, $E_n(G)$ and $C_n(G)$ are the centralizer of the verbal subgroup, the elemental subgroup and the commutal subgroup corresponding to x^n respectively.

We can now state that *the union of commutal subgroups of a group is not, in general, a commutal subgroup of the group.*

Example. Let G_n denote the nilpotent group of class two having order p_n^3 and exponent p_n for each prime number $p_n > 2$ $(n = 1, 2, \ldots)$. With the set of words

$$x_1^{p_1}, x_2^{p_2}, \ldots, x_n^{p_n}, \ldots$$

we associate the commutal subgroups

$$C_{p_1}(G), C_{p_2}(G), \ldots, C_{p_n}(G), \ldots$$

respectively. Let $H = \prod_{n=1}^{\infty}{}^{\times}\, G_n$ and $G = H \times (F/{}^2F)$, where $F/{}^2F$ is a non-abelian free nilpotent group of class two having finite rank. Now

$$C_{p_m}(G) = C_{p_m}(H) \times C_{p_m}(F/{}^2F) = C_{p_m}(H) \times ({}^1F/{}^2F),$$

since, by Witt [**5**], $F/{}^2F$ is locally infinite and ${}^1F/{}^2F$ is its centre, and

$$C_{p_m}(H) = \prod_{n=1}^{\infty}{}^{\times}\, C_{p_m}(G_n).$$

However, $C_{p_m}(G_m) = G_m$. Hence

$$\{C_{p_m}(G)\,;\ m = 1, 2, \ldots\} = H \times ({}^1F/{}^2F).$$

If the union of commutal subgroups of G is a commutal subgroup of G, then, for some set of words f,

$$H \times ({}^1F/{}^2F) = C_f(G) = C_f(H) \times C_f(F/{}^2F).$$

Thus $C_f(F/{}^2F) = {}^1F/{}^2F$ and $C_f(H) = H$. However, there exist no words f such that both these latter two results hold. This gives the required contradiction and thus the union of commutal subgroups is not, in general, a commutal subgroup.

In contrast to marginal subgroups and invariable subgroups, the elemental subgroup of a free product need not, in general, be trivial. In fact, if $f = x^n$, it is easy to verify that the elemental subgroup of a free product F of groups G_α $(\alpha \in M)$ is given by

$$E_f(F) = \Big(\prod_{\alpha \in M}{}^{*}\, E_f(G_\alpha)\Big)^F \dagger.$$

$C_f(G)$ always contains the centre of G, but this does not, in general, hold for $E_f(G)$.

$M_f(G)$ is the smallest of the subgroups $M_f(G)$, $I_f(G)$, $E_f(G)$ and $C_f(G)$, as it is always contained in each of the others.

† The superscript F denotes normal closure in F. Cf. Moran [**4**].

Now with the set of words f we associate the set of words $g(x_1, x_2, \ldots, x_n, x) = [f, x]$, where x is distinct from all the other symbols. It is easy to see, from the result (2) stated above for the third dual subgroup, that

$$H_g^3(G) \geqslant H_f^3(G) \cap Z_f(G).$$

In contrast to marginal subgroups and invariable subgroups, the following example shows that, in general, $E_f(G)$ is neither contained in $E_g(G)$ nor in $Z_f(G)$.

Example. Let G be the infinite dihedral group generated by a and b which are of order two. If f is the word y^2, then g is the word $[y^2, x]$. It is easy to see that $E_f(G) = G$, while $E_g(G) = 1$ and $Z_f(G) = \{ab\}$.

Finally we have that, if $\xi \in C_f(G)$, then $[G, \xi]$ is contained in $E_f(G)$. On the other hand, if $\xi \in C_g(G)$, then $[G, \xi]$ is contained in $Z_f(G)$.

3. MacLane's dual for the centralizer of a verbal subgroup. A process due to S. MacLane [**1, 2**] assigns a dual statement to every statement concerning groups and homomorphisms. Thus it is possible to show that the upper central series and the lower central series are dual concepts. In the case of the centralizer of a verbal subgroup we proceed as follows.

Let $I(f)$ denote the set of all inner automorphisms of a group G associated with the values of all the words f. For example, if $f = x^2$, then $I(f)$ will be the set of all inner automorphisms ϕ_x of G which map an arbitrary element y of G onto $x^{-2}yx^2$. Associated with the set of words f we have, as above, another set of words of which $g(x_1, x_2, \ldots, x_n, x) = [f, x]$ is a typical word. We are now in a position to give the required duality between the centralizer of a verbal subgroup $Z_f(G)$ and the verbal subgroup $V_g(G)$.

If $I(f)$ is the set of inner automorphisms of G associated with all the words f, then $Z_f(G)$ is the maximal subgroup N of G such that, for each ϕ of $I(f)$, ϕ induces the identity automorphism on N.

If $I(f)$ is the set of inner automorphisms of G associated with all the words f, then $G/V_g(G)$ is the maximal quotient group Q of G such that, for each ϕ of $I(f)$, ϕ induces the identity automorphism on Q.

REFERENCES

1. S. MacLane, Groups, categories and duality, *Proc. Nat. Acad. Sci.*, **34** (1948), 263–267.
2. S. MacLane, Duality for groups, *Bull. Amer. Math. Soc.*, **56** (1950), 485–516.
3. S. Moran, Duals of a verbal subgroup, *J. London Math. Soc.*, **33** (1958), 220–236; Corrigenda, **34** (1959), 250.
4. S. Moran, Associative operations on groups III, *Proc. London Math. Soc.* (3) **9** (1959), 287–317.
5. E. Witt, Treue Darstellung Liesche Ringe, *J. reine angew. Math.* **177** (1937), 152–160.

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