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A CONJECTURE OF LENNOX AND WIEGOLD CONCERNING SUPERSOLUBLE GROUPS

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Abstract

We prove a conjecture of Lennox and Wiegold that a finitely generated soluble group, in which every infinite subset contains two elements generating a supersoluble group, is finite-by-supersoluble.

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Lennox and Wiegold [1] consider the following general problem: given a class X of groups, describe the class $X^{\#}$ of groups G such that every infinite subset of G contains two elements generating an X-group. Their considerations were motivated by a solution to this problem, by Neumann [2], when X is the class of abelian groups; we refer the interested reader to [1] for a fuller discussion of the motivation. After observing that the problem is likely to be intractible as it stands, even for relatively simple classes X, Lennox and Wiegold consider analogous problems where G is restricted to be finitely generated and soluble. They leave, however, an open conjecture for the case when X is the class of supersoluble groups. This paper settles that conjecture.

THEOREM. *Let G be a finitely generated soluble group. Then G is finite-by-supersoluble if and only if within every infinite subset of G there are two distinct elements generating a supersoluble subgroup.*

The 'only if' implication (proved already by Lennox and Wiegold) is relatively straightforward. We include it, however, for completeness. Suppose that G has a finite normal subgroup F with G/F supersoluble. As G is residually finite, we can

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find a normal subgroup H of G with $F \cap H = \{1\}$ and G/H finite. Let S be an infinite subset of G. As G/H is finite, there exist distinct elements $x, y \in S$ with $xH = yH$. Let K be the subgroup generated by x and y. Then, as $xH = yH$, the derived subgroup K' of K lies in H. But $[K \cap F, K] \leqslant K' \cap F \leqslant H \cap F = \{1\}$ and so $K \cap F$ is central in K. As $K/K \cap F \cong KF/F \leqslant G/F$, $K/K \cap F$ is supersoluble and so K is also supersoluble. Thus S contains two elements generating a supersoluble subgroup of G.

Suppose now that G is a finitely generated soluble group in which every infinite subset contains two elements generating a supersoluble group. We begin by claiming the following.

(a) G is polycyclic.

(b) We may assume that G has no non-trivial finite normal subgroups.

(c) It suffices to prove that G has a non-trivial cyclic normal subgroup.

The first statement follows from the fact that supersoluble groups are polycyclic and the proof, by Lennox and Wiegold in [1], of the theorem analogous to ours but with "polycyclic" replacing "supersoluble". Thus G has the maximum condition on subgroups and so, as quotients of supersoluble groups are again supersoluble, it clearly suffices to assume that every proper quotient of G is finite-by-supersoluble but G itself is not. Then (b) is clear.

For (c), suppose that C is an infinite cyclic normal subgroup of G. By assumption, G/C has a finite normal subgroup D/C with G/D supersoluble. Let H be the centraliser of C in D. Then $H \trianglelefteq G$, $|D : H| \leqslant 2$ and H is central-by-finite. Thus H', the derived group of H, is finite and so the torsion elements of H form a finite G-normal subgroup of H containing H'. By assumption, this is trivial and so H is abelian and torsion-free and so cyclic. Thus H and D complete a cyclic normal series for G and G is supersoluble.

We will henceforth assume (a) and (b) and aim to prove the statement of (c). Suppose, firstly, that G is abelian-by-cyclic, say $G = \langle A, g \rangle$ with A abelian and normal. We claim that the theorem is true in this case. Let a be non-trivial in A, and so of infinite order. Then the set $\{a^i g : i \in \mathbf{Z}\}$ is infinite and so contains two elements which generate a supersoluble subgroup H. If $A \cap H = \{1\}$, then H is cyclic and g centralises some power of a, giving a non-trivial cyclic normal subgroup. If $A \cap H > \{1\}$, then it is a non-trivial normal subgroup of H and so contains a non-trivial cyclic H-normal subgroup C. Since both H and A normalise C, G normalises C, as required. We observe that we have used the fact here that A is torsion-free rather than the full force of (b).

We come, finally, to the general proof. As G is polycyclic, it is nilpotent-by-abelian-by-finite. Let A be a normal subgroup of G, lying in the centre of the Fitting subgroup, which is a non-trivial rationally irreducible G-module. Let C be the centraliser of A in G and let $\bar{G}$ denote G/C. Then $\bar{G}$ is abelian-by-finite. By

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(b), A is torsion-free and so, as $\mathbf{Z}$-module, embeds in $\bar{A} = A \otimes_{\mathbf{Z}} \mathbf{Q}$ (where $\mathbf{Z}$ denotes the integers and $\mathbf{Q}$ the rationals). Thus $\bar{G}$ is an irreducible linear group over $\mathbf{Q}$. From the previous paragraph, it follows easily that each $g \in \bar{G}$ has all its eigenvalues in this action equal to ± 1 and so g^2 is unipotent. Thus, if B is the abelian normal subgroup of $\bar{G}$ with $\bar{G}/B$ finite, B^2 is a unipotent normal subgroup of an irreducible group and so is trivial (for example from 1.21 of Wehrfritz [3]). Thus $\bar{G}$ is finite and, for each $g \in \bar{G}$, g^2 is both unipotent and of finite order and so again is trivial. Hence $\bar{G}$ is a finite abelian irreducible group and so is cyclic—necessarily of order 2. Hence $\bar{A}$ has $\mathbf{Q}$-dimension one and so A, as a finitely generated subgroup of the rationals, is cyclic. Thus, having shown that G contains a non-trivial normal cyclic subgroup, we have completed the proof.

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