COMMUTATION NEAR-RINGS OF A GROUP

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1. Introduction

Let G be a group and let $\Omega(G)$ denote the semigroup of all mappings of G into G with the usual composition of mappings as multiplication, namely $g(\theta_1\theta_2) = (g\theta_1)\theta_2$.

With each element $x \in G$ we associate two mappings $\rho(x)$ and $\lambda(x)$ of G into G defined as follows:

$$\rho(x) : g \to g^{-1}x^{-1}gx, \quad \lambda(x) : g \to x^{-1}g^{-1}xg.$$

Let $P(G)$ denote the subsemigroup of $\Omega(G)$ generated by all ρ's and let $\Lambda(G)$ denote the corresponding subsemigroup generated by all λ's. These semigroups are called the commutation semigroups of G. Clearly a word of length n in $P(G)[\Lambda(G)]$ maps an element $g \in G$ to a left-normed [right-normed] commutator of weight $n+1$ with first entry g.

The operations $\rho(x)$ and $\lambda(x)$ are important tools of group theory. The difference of behaviour of these two operations is probably widely known. But while certain aspect of it and specific examples can be found in the literature (e.g. Heineken [3]), no connected account seems to exist. It has been shown in [2] that the commutation semigroups of a group are not in general isomorphic; not even for groups which are both metabelian and nilpotent.

In addition to the multiplication of mappings of G into G, we also define an operation of "addition" by the rule $g(\theta_1+\theta_2) = g\theta_1 \cdot g\theta_2$, so that the algebraic system $\Omega(G)$ becomes a group (not necessarily abelian) under addition, in which zero and negatives are given by $g0 = 1$ and $g(-\theta) = (g\theta)^{-1}$ respectively. The multiplication is associative and the left-distributive law $\theta_1(\theta_2+\theta_3) = \theta_1\theta_2+\theta_1\theta_3$ holds. In virtue of these properties $\Omega(G)$ becomes a left-distributive near-ring which we denote by $\mathfrak{N}(G)$ (c.f. Blackett [1]).

Let $\mathfrak{M}(G)$ denote the sub near-ring of $\mathfrak{N}(G)$ generated by all ρ's and λ's. Then each element of $\mathfrak{M}(G)$ is a certain word in ρ's and λ's. Since $\lambda(x) = \rho(x^{-1})+\rho(x^{-1})\rho(x)$ by (9), each such word may be written in terms of ρ's only. Consequently the algebraic system $\mathfrak{R}(G)$, generated by all mappings $\rho(x)$ and closed with respect to the operations of addition and multiplication, is also closed with respect to subtraction and it coincides with the near-ring $\mathfrak{M}(G)$. In this connection we prove,

THEOREM 1. *There exists a group G such that $\mathfrak{L}(G)$ is not a sub near-ring of $\mathfrak{R}(G)$; and if for a particular group G, $\mathfrak{L}(G)$ is also a sub near-ring of $\mathfrak{R}(G)$, then $\mathfrak{L}(G) = \mathfrak{R}(G)$. [$\mathfrak{L}(G)$ is generated by all mappings $\lambda(x)$.]*

While a complete characterization of groups for which $\mathfrak{L}(G)$ is a near-ring is not known, we prove

THEOREM 2. *If either*

(i) *all elements of G are left-Engel elements, or*

(ii) *the derived group G' has finite exponent, then $\mathfrak{L}(G) = \mathfrak{R}(G)$.* [1]

Two other theorems proved in this paper are

THEOREM 3. *$\mathfrak{M}(G)$ is a ring if and only if G is 3-metabelian.* [2]

THEOREM 4. *Multiplication in $\mathfrak{L}(G)$ [or in $\mathfrak{R}(G)$] is commutative if and only if G is nilpotent of class 2.*

2. Proofs of the theorems

Some further notation is needed. As usual we write

$$x^y = y^{-1}xy, \quad [x, y) = x^{-1}y^{-1}xy, \quad [x, y, z] = [[x, y], z].$$

Thus

$$[x, y] = x\rho(y) = y\lambda(x).$$

The standard commutator identities

(1) $$[y, x][x, y] = 1,$$

(2) $$[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z],$$

(3) $$[z, xy] = [z, y][z, x]^y = [z, y][z, x][z, x, y],$$

(4) $$[x, y] = [y, x^{-1}]^x = [y, x^{-1}][y, x^{-1}, x], \text{ and}$$

(5) $$[y, x] = [x^{-1}, y]^x = [x^{-1}, y][x^{-1}, y, x]$$

give respectively

(6) $$\rho(x)+\lambda(x) = 0,$$

(7) $$\lambda(xy) = \lambda(x)+\lambda(x)\rho(y)+\lambda(y),$$

[1] x is a left-Engel element of G if $\rho^n(x) = 0$ for some integer n. The identities $[x, y] = [y^{-1}, x]^y = [y^{-1}, x^y]$ give $\rho(y) = \lambda(y^{-1})\omega(y) = \omega(y)\lambda(y^{-1})$, where $g\omega(y) = g^y$. Hence $\rho^n(x) = \lambda^n(x^{-1})\omega^n(x)$ and so $\rho^n(x) = 0$ is equivalent to $\lambda^n(x^{-1}) = 0$. Heineken [4] and some others call x left-Engel if $\lambda^n(x) = 0$ for some n.

[2] G is n-metabelian if every n-generator subgroup is metabelian. Macdonald [6] has shown that G is 3-metabelian if and only if $\rho(x)+\rho(y) = \rho(y)+\rho(x)$ holds for all $x, y \in G$.

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(8) $$\rho(xy) = \rho(y)+\rho(x)+\rho(x)\rho(y),$$

(9) $$\lambda(x) = \rho(x^{-1})+\rho(x^{-1})\rho(x), \text{ and}$$

(10) $$\rho(x) = \lambda(x^{-1})+\lambda(x^{-1})\rho(x).$$

Further if $\theta_1, \theta_2 \in \mathfrak{N}(G)$, then for all $g, x \in G$,

$$g(\theta_1+\theta_2)\rho(x) = [g\theta_1 \cdot g\theta_2, x] = (g\theta_2)^{-1}[g\theta_1, x]g\theta_2[g\theta_2, x]$$

(by (2)) which gives

(11) $$(\theta_1+\theta_2)\rho(x) = -\theta_2+\theta_1\rho(x)+\theta_2+\theta_2\rho(x),$$

and taking negatives

(12) $$(\theta_1+\theta_2)\lambda(x) = \theta_2\lambda(x)-\theta_2+\theta_1\lambda(x)+\theta_2.$$

Proof of theorem 1. Let G be the infinite dihedral group generated by a and b, b inducing the inverting automorphism on $\{a\}$. The proof of the first part consists in showing that $\lambda(b)$ has no additive inverse in $\mathfrak{L}(G)$. Let ξ be an arbitrary element of $\mathfrak{L}(G)$. Using

$$a\lambda(a^i) = a^0 \text{ and } a\lambda(a^ib) = a^2$$

we deduce easily that

$$a\xi = a^t, \text{ where } t \geqq 0.$$

Thus if ξ is an additive inverse of $\lambda(b)$, we have

$$1 = a\xi \cdot a\lambda(b) = a^t \cdot a^2 = a^{t+2}$$

which is a contradiction since $\{a\}$ is an infinite cycle.

To prove the second part we assume that $\mathfrak{L}(G)$ is a sub near-ring of $\mathfrak{N}(G)$ and we note by (9) that $\mathfrak{L}(G) \subseteqq \mathfrak{N}(G)$. The proof consists in showing that $\rho(a) \in \mathfrak{L}(G)$ for all $a \in G$. Since $\mathfrak{L}(G)$ is a near-ring we have

$$\xi_a+\lambda(a^{-1})\lambda(a) = 0$$

for some $\xi_a \in \mathfrak{L}(G)$. Thus we have

$$\begin{aligned} \rho(a) &= \lambda(a^{-1})+\lambda(a^{-1})\rho(a) \qquad \text{by (10)} \\ &= \lambda(a^{-1})+\xi_a+\lambda(a^{-1})\lambda(a)+\lambda(a^{-1})\rho(a) \\ &= \lambda(a^{-1})+\xi_a \in \mathfrak{L}(G). \end{aligned}$$

Since the choice of a is arbitrary we have the required result.

Proof of theorem 2. It suffices in each case to prove $\lambda(a) \in \mathfrak{L}(G)$ for all $a \in G$.

(i) Since a is a left-Engel element of G we have $\lambda^n(a^{-1}) = 0$ for some n. By (10),

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$$\begin{aligned}\rho(a) &= \lambda(a^{-1})+\lambda(a^{-1})\rho(a)\\ &= \lambda(a^{-1})+\lambda(a^{-1})(\lambda(a^{-1})+\lambda(a^{-1})\rho(a))\\ &= \cdots\\ &= \lambda(a^{-1})+\lambda^2(a^{-1})+\cdots+\lambda^{n-1}(a^{-1}) \in \mathfrak{L}(G).\end{aligned}$$

(ii) If n is the exponent of G', then

$$\begin{aligned}n\lambda(a) &= 0 \text{ and we have}\\ \rho(a) &= -\lambda(a)\\ &= (n-1)\lambda(a) \in \mathfrak{L}(G).\end{aligned}$$

Proof of theorem 3. If $\mathfrak{M}(G)$ is a ring then $\mathfrak{R}(G)$ is a ring and

$$\rho(a)+\rho(b) = \rho(b)+\rho(a)$$

holds for all $a, b \in G$ and it follows that G is 3-metabelian. Therefore it remains to prove, conversely, that if G is 3-metabelian, then $\mathfrak{M}(G)$ is a ring. From the hypothesis it follows that

$$\alpha(a)+\alpha(b) = \alpha(b)+\alpha(a) \tag{13}$$

holds for all $a, b \in G$ where $\alpha(a) = \rho(a)$ or $\lambda(a)$ and $\alpha(b) = \rho(b)$ or $\lambda(b)$. Let $\mathfrak{S}(G)$ denote the set of all sums $\Sigma\alpha(a_i)$, where $\alpha(a_i) = \rho(a_i)$ or $\lambda(a_i)$. By (13) $\mathfrak{S}(G)$ is an abelian group under addition.

For all $b \in G$, we have by (7) and (8) that $\alpha(a_i)\rho(b)$ and hence $\alpha(a_i)\lambda(b)$ are in $\mathfrak{S}(G)$. Therefore, by (11),

$$\begin{aligned}\left(\sum_1^r \alpha(a_i)\right)\rho(b) &= -\sum_2^r \alpha(a_i)+\alpha(a_1)\rho(b)+\sum_2^r \alpha(a_i)+\left(\sum_2^r \alpha(a_i)\right)\rho(b)\\ &= \alpha(a_1)\rho(b)+\left(\sum_2^r \alpha(a_i)\right)\rho(b)\\ &= \cdots\\ &= \sum_1^r \alpha(a_i)\rho(b) \in \mathfrak{S}(G).\end{aligned}$$

By above and by left-distributivity we get

$$\left(\sum_1^r \alpha(a_i)\right)\left(\sum_1^s \alpha(b_i)\right) \in \mathfrak{S}(G).$$

Hence $\mathfrak{S}(G)$ is closed under multiplication and the right-distributive law holds; so $\mathfrak{S}(G)$ is a ring. Since $\mathfrak{S}(G)$ contains all mappings $\rho(a)$, $\mathfrak{S}(G) = \mathfrak{R}(G) = \mathfrak{M}(G)$ and thus $\mathfrak{M}(G)$ is a ring.

Proof of theorem 4. Levin [5] has proved:

If z is an element of a group G such that $[z, x, y] = [z, y, x]$ for all $x, y \in G$, then $[z, [x, y]] = 1$ for all $x, y \in G$.

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Under these conditions z is a 2nd left-Engel element of $G : [z, x, z] = [z, z, x] = 1$ for all $x \in G$. Thus the following result generalizes Levin's:

LEMMA. *Let x, y be elements of a group G and let z be a 2nd left-Engel element of G. Then each of the conditions*

$$[z, x, y] = [z, y, x] \tag{14}$$

$$[y, [x, z]] = [x, [y, z]] \tag{15}$$

implies

$$[x^{-1}, y^{-1}, z] = 1. \tag{16}$$

PROOF. Since z is 2nd left-Engel element, its normal closure $\langle {}^{G}_{z} \rangle$ is abelian. If (14) holds,

$$\begin{aligned}
[z, yx] &= [z, x][z, y][z, y, x] && \text{by (3)} \\
&= [z, y][z, x][z, y, x] \text{ as } \langle {}^{G}_{z} \rangle \text{ is abelian} \\
&= [z, y][z, x][z, x, y] && \text{by (14)} \\
&= [z, xy] && \text{by (3)} \\
&= [z, [x^{-1}, y^{-1}]yx] \\
&= [z, yx][z, [x^{-1}, y^{-1}]]^{yx} && \text{by (3)}
\end{aligned}$$

and so $[x^{-1}, y^{-1}, z] = 1$. The other part is similarly proved.

To complete the proof of Theorem 4, we first observe that if G is nilpotent of class 2, $\rho(a)\rho(b) = 0 = \lambda(a)\lambda(b)$ for all $a, b \in G$; hence multiplication in $\mathfrak{R}(G)$ and in $\mathfrak{L}(G)$ is commutative.

Conversely, if multiplication in $\mathfrak{L}(G)$ is commutative, then

$$\lambda(a)\lambda(b) = \lambda(b)\lambda(a) \text{ for all } a, b \in G. \tag{17}$$

This implies that every element a of G is 2nd left-Engel element: $[a, [b, a] = [b, [a, a]] = 1$ for all $b \in G$. It follows from the second part of the above lemma that G is nilpotent of class 2. The other part is similarly proved by using the first part of the above lemma.

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References

[1] Blackett, D. W., 'Simple and semisimple near-rings', *Proc. Amer. Math. Soc.* 4 (1953), 772–785.

[2] Gupta, N. D., 'On commutation semigroups of a group', *J. Austral. Math. Soc.* 6 (1966), 36–45.

[3] Heineken, Hermann, 'Commutator closed groups', *Illinois J. Math.* 9 (1965), 242–255.
[4] Heineken, Hermann, 'Engelsche Elemente der Länge drei', *Illinois J. Math.* 5 (1961), 681–707.
[5] Levin, Frank, 'On some varieties of soluble groups', *Math. Zeitschr.* 85 (1964), 369–372.
[6] Macdonald, I. D., 'On certain varieties of groups', *Math. Zeitschr.* 76 (1961), 270–282.

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