THE RIEMANN SURFACE OF A RING

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Introduction

The contravariant functor F from the category of Riemann surfaces and analytic mappings to the category of complex algebras and homomorphisms which takes each surface Ω to the algebra of analytic functions on Ω does not have an adjoint on the right; but it nearly does. To each algebra A there is associated a surface $\Sigma_1(A)$ and a homomorphism σ_A from A into $F\Sigma_1(A)$, indeed onto an algebra of functions not all of which are constant on any component of $\Sigma_1(A)$, such that every such non-trivial representation $A \to F(\Omega)$ is induced by a unique analytic mapping $\Omega \to \Sigma_1(A)$.

An actual adjunction arises if $\Sigma_1(A)$ is united with a discrete set $\Sigma_0(A)$ of all homomorphisms of A into the complex field. Though Σ_1 is not functorial, $\Sigma^1 = \Sigma_1 \cup \Sigma_0$ is, from algebras to at most one-dimensional analytic manifolds and open analytic mappings. Its adjoint on the right still takes each manifold to the algebra of all analytic functions on it, so may still be called F.

For these manifolds and mappings, the algebra of meromorphic functions is also functorial, and that functor has an adjoint too. The existence of the adjoints is a routine inference from the basic result, that the category of manifolds is complete.

The natural extension to at most n-dimensional complex analytic manifolds would involve those analytic mappings such that the image of every non-empty open set has an interior point. The completeness and adjointness questions are open for these categories. There is a standard completion procedure which seems best, if any is needed. Consider the real analytic manifolds. No precise analogue of Σ_1 exists; the algebra of even real analytic functions on a line has no best representation. Rather, the best representation is on a *closed* half-line. In fact completeness and the adjoints carry over to real analytic at most 1-dimensional manifolds with boundary, with mappings as indicated above. Here too the questions are open in higher dimensions. However, the left regular completion of any of these categories is at worst a category of manifolds with nowhere dense sets of singularities. The proof will be omitted, though the left regular completion is described. The point is that this presents a less abstract problem than completeness: what sort of singularities? In complex dimension 1, none; in real dimension 1, boundaries.

The complex algebra $\sigma_A(A)$ need not separate points on $\Sigma_1(A)$, but the ordered pairs of distinct points which it does not separate form a closed discrete set in $\Sigma^1(A) \times \Sigma^1(A)$. The ordinary Riemann surface Ω of a function f and the covering mapping from Ω into the plane appear in a natural way when Σ^1 is applied to the algebras generated by the identity function i on the plane, and by i and f together.

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1. Completeness

Here, a *complex* (*real*) *n-manifold* will be a Hausdorff space M provided with a family of homeomorphisms from open sets of complex (real) k-spaces, $0 \leqq k \leqq n$, to open sets of M, covering M and overlapping analytically. A *morphism* of manifolds is an analytic mapping such that the image of every non-empty open set has an interior point.

The purposes of the following discussion are to reduce the completeness question to a question of quotients (readily solved for complex 1-manifolds since images of small open sets are open in the quotient) and to reduce it to a question of products, which seems better for real manifolds with boundary and will be a remaining question in the complex case if the category of genuine complex n-manifolds is incomplete. I do not know a clearly best definition of a real manifold with boundary. The 1-dimensional real case will be fitted in by simple ad hoc arguments at the end (not all given; it is an easy exercise to prove completeness of that category).

The category $C^n(\mathscr{R}^n)$ of complex (real) n-manifolds is a category of algebras in an almost classical sense. For ground set of M one must not take the set of points of M, but the set M' of morphisms from the open unit ball of (scalar) n-space to M. Then they are classical algebras. M' has infinitely many operations, all of which are unary, indexed by the morphisms e from the standard ball B into itself; for $f : B \to M$ in M', $e(f)$ is the composite $fe : B \to M$. Routine work (of some length) shows that if $h : M' \to N'$ is a homomorphism, then every value $h(f)$ is $h^f f$ for some partially defined function h^f, and that these h^f are consistent and make up a morphism $\bar{h} : M \to N$. Since every morphism $\bar{h}$ induces a homomorphism h, we have a full faithful representation of manifolds by algebras.

We use the terminology of Isbell [1] generally, since we want a result and some notions from that paper. In that terminology, we have just noted that the ball B is left adequate in the category of n-manifolds. We shall not need all the degrees of completeness from [1]. We simply call a category *left complete* if every small diagram has an inverse limit, *right complete* if the dual holds, *complete* if it is left and right complete. It is well known that a category is right complete if every family of objects has a coproduct and every coterminal pair of morphisms has a coequalizer.

Note that C^n (and $\mathscr{R}^n$, and any reasonable category of real analytic manifolds with boundary) certainly has coproducts, which are simply disjoint unions. It also has equalizers, which are open-closed subsets. Two morphisms not agreeing on any open subset have an empty equalizer, since they cannot have the same composition with any (nearly open) morphisms; and where they do agree on an open set, they agree on a whole component.

LEMMA. *For each n, the following statements are all true or all false.*

(i) *C^n is complete.*

(ii) *C^n is left complete.*

(iii) *Every coinitial family of morphisms $f_\alpha : M \to N_\alpha$ of C^n factors across a surjective morphism $g : M \to Q$ such that for coterminal morphisms $h, k : L \to M$, if $f_\alpha h = f_\alpha k$ for all α then $gh = gk$.*

(iv) *C^n is right complete.*

PROOF. Statement (i) contains (ii). From (ii) an immediate step towards (iii) yields a factorization across a non-surjective morphism $g_0 : M \to P$, where P is a product of all N_α and g_0 satisfies the concluding condition of (iii). For every point β of P not in $g_0(M)$, g_0 gives us a morphism $f'_\beta : M \to P - \{\beta\}$. Going into a product again, the image is open and closed, hence a submanifold. To deduce (iv) from (iii) we must find a coequalizer of arbitrary $h, k : L \to M$. If $\{f_\alpha\}$ consists of all morphisms $M \to N_\alpha$ in C^n such that $f_\alpha h = f_\alpha k$ and the set of points of N_α is a subset of M, the g given by (iii) is a coequalizer. Finally, (iv) implies (i) because the category has a left adequate object [**1**; **4.7**].

THEOREM 1. *C^1 is complete.*

PROOF. To verify (iii) for morphisms $f_\alpha : M \to N_\alpha$, construct the discrete part of Q from the components of M on which all f_α are constant in the obvious way. On the rest of M, M_1, form a quotient space by the equivalence relation consisting of all pairs $(h(0), k(0))$ for h, k in M' such that $f_\alpha h = f_\alpha k$ for all α. Since the f_α are open mappings, the projection to the quotient space is an open mapping $\pi : M \to Q$. Near any point x of M_1, some of the f_α are non-constant mappings locally like z^n at 0 for various n, and π is like z^m where m is a minimum over analytic functions of finitely many f_α. (This is precise in terms of a schlicht mapping $B \to M_1$ taking

0 to x.) Hence Q is at least a non-Hausdorff manifold. But the f_α induce analytic mappings $\bar{f}_\alpha : Q \to N_\alpha$ such that near any point $\bar{x}$ of Q, finitely many f_α define a schlicht mapping. If $\bar{x}_1, \bar{x}_2$ had no disjoint neighborhood, we should have for any $x_i \in \pi^{-1}(\bar{x}_i)$ two mappings h, k in M', $h(0) = x_1$, h like z^{m_2}, $k(0) = x_2$, k like z^{m_1}, such that all $f_\alpha h = \bar{f}_\alpha \pi h$ and $f_\alpha k$ would agree on a non-empty open set, and so everywhere.

As for the real analytic 1-manifolds with boundary, the completeness of the category is easily checked. It reduces to left completeness and thus to products in several ways, of which we want the following. First, these manifolds are perfectly good algebras of the same species as the algebras described above for $\mathscr{R}^1$. Morphisms $B \to M$ are necessarily two-to-one near a boundary point. Near any boundary point, such morphisms exist. Now $\mathscr{R}^1$ is a full subcategory of a variety of algebras, containing the free algebras. The *left regular completion* of such a category is simply its closure under inverse limits in the algebras [1] (and is left and right complete). Every real analytic 1-manifold with boundary, M, can be embedded in a manifold without boundary, N; and M is an open-closed set in the product of all $N - \{x\}$, $x \in N - M$.

2. Representation

It happens that the universal representation $\Sigma^1(A)$ can be constructed without difficulty for a mere ring A. It can equally well be constructed for a set A, (more easily, or) by the same proof, since the forgetful functors algebra $\to$ ring $\to$ set have adjoints. We use the term *algebra* for an algebra over the integers (i.e., a ring) or a complex algebra. It happens also, and this is more interesting, that a universal representation by meromorphic functions can be constructed in almost exactly the same way.

THEOREM 2. *For every algebra A there exist a complex 1-manifold $\Sigma^1(A)$ (resp. $M^1(A)$) and a homomorphism σ from A into the algebra of all analytic (meromorphic) complex-valued functions on $\Sigma^1(A)$ ($M^1(A)$) such that every such representation r of A by functions on a 1-manifold T has the form $r(a) = \varphi \circ \sigma(a)$ for a unique morphism $\varphi : T \to \Sigma^1(A)$ ($\varphi : T \to M^1(A)$).*

PROOF. The conclusion simply evaluates at A an adjoint and adjunction (on the right) to the algebra-of-analytic-functions functor (resp. meromorphic) F; $F(M)$ is the algebra of all such functions on M, and for $f : M \to M'$, $F(f)$ takes $F(M')$ to $F(M)$ by composition with f. A dual of Proposition 7.1 of [2] assures that these must exist, since C^1 is complete and has a left adequate object, provided F takes coproduct manifolds to product algebras and coequalizers to equalizers. The former condition is evident, and the latter would be also if we had algebras of morphisms of

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C^1. But in effect, we do. Analytic mappings from M to a plane Π (or sphere) correspond one-to-one with morphisms to Π (or a sphere) with a discrete copy of Π adjoined, since just the components of constancy must go into the discrete part. As the (products and) equalizers are the same in algebras as in sets, the theorem follows.

The same theorem and proof hold for real 1-manifolds with boundary.

For the next proposition, some remarks. A *monomorphism* in a category is a morphism f such that the relation $fh = fk$ implies $h = k$. A monomorphism of C^1 must be locally schlicht; for the other local possibilities, constancy and z^n, are evidently not monomorphic. Note that the uniqueness in Theorem 2 secures that all the functions $\sigma(a)$ suffice to define a monomorphism from $\Sigma^1(A)$ to a product of planes or spheres, thus a locally schlicht mapping.

PROPOSITION. *The set of all ordered pairs of distinct points of $\Sigma^1(A)$ not separated by functions $\sigma(a)$ is closed and discrete.*

PROOF. The set of ordered pairs of points not separated by any $\sigma(a)$ is obviously closed; and for a diagonal point (p, p), the functions $\sigma(a)$ separate distinct points near p. If no $\sigma(a)$ separates p from q, p isolated in $\Sigma^1(A)$, the functions $\sigma(a)$ separate points near q from q and therefore from p. Suppose p and q are non-isolated points and the functions $\sigma(a)$ fail to separate points p_n arbitrarily near p from points q_n arbitrarily near q. A schlicht map $\varphi : U \to V$ of a neighborhood of p upon a neighborhood of q can be constructed by means of analytic functions of functions $\sigma(a)$. Necessarily $\varphi(p_n) = q_n$ and $\varphi(p) = q$. Then the embeddings $i : U \subset \Sigma^1(A)$ and $\varphi : U \to \Sigma^1(A)$ have the property that for all $\sigma(a)$, $\sigma(a)i$ and $\sigma(a)\varphi$ agree on all p_n and thus coincide. By universality, $i = \varphi$, $p = q$.

PROPOSITION. *If B is an algebra of analytic functions on $\Sigma^1(A)$ containing $\sigma(A)$, this representation of B induces an open-closed embedding of $\Sigma^1(A)$ in $\Sigma^1(B)$.*

PROOF. Obviously it induces an embedding. For a connected open set U in $\Sigma^1(B)$ meeting $\Sigma^1(A)$, the representation of $A \subset B$ on U must be induced by a unique morphism $U \to \Sigma^1(A)$. This is true also for $V = U \cap \Sigma^1(A)$; so the morphism fixes every point of V, hence every point of U, and $U \subset \Sigma^1(A)$.

These propositions evidently hold in the real and meromorphic cases too.

Obviously specializing to connected Riemann surfaces would complicate universality statements, but it would not destroy them. For instance, the complex plane Π is the 1-dimensional part $\Sigma_1(A)$ of $\Sigma^1(A)$ if A is the algebra of complex polynomials. If f is an analytic function on an open set $U \subset \Pi$,

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the algebra B generated by A and f is represented on U, inducing an embedding $U \subset \Sigma^1(B)$. The same holds for the classical Riemann surface Ω of f. $\Omega \subset \Sigma^1(B)$ is a whole component, for if there were more it could be constructed by the classical procedure. The algebra embedding $A \subset B$ induces the representation of A by polynomials 'on the plane under Ω', and thus induces the classical mapping $\Omega \to \Pi$. There is no substantial change in the argument for M^1.

The construction of Riemann surfaces from algebras began with the work of L. Bers, extended by W. Rudin to show that any Riemann surface Ω without compact components is reconstructible from its algebra $F(\Omega)$. (Similar results for meromorphic functions are due to H. Royden and M. Heins.) I. Richards has given a generalization [3] which is easily generalized further to arbitrary algebras, to yield the interior of the subset of $\Sigma_1(A)$ consisting of those points p such that the ideal $\{a \in A : \sigma(a)(p) = 0\}$ is principal. Richards showed that for $A = F(\Omega)$ this is just Ω. His lemmas show, for any A, that functions $\sigma(a)$ separate any two distinct points one of which is such a 'principal' point p. The question is open whether $\Sigma_1 F(\Omega)$ can have non-principal points.

Added in proof. For connected Ω, it cannot, by Royden's Theorem 1 (Trans. Amer. Math. Soc. 83 (1956), 272). Hence it cannot, unless the cardinal number of Ω is measurable.

References

[1] J. Isbell, 'Structure of categories', *Bull. Amer. Math. Soc.* 72 (1966), 619–655.
[2] J. Lambek, *Completions of Categories* (Springer Lecture Notes 24, Berlin, 1966).
[3] I. Richards, 'A criterion for rings of analytic functions', *Trans. Amer. Math. Soc.* 128 (1967), 523–530.

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