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SUBORDINACY ANALYSIS AND ABSOLUTELY CONTINUOUS SPECTRA FOR STURM-LIOUVILLE EQUATIONS WITH TWO SINGULAR ENDPOINTS

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ABSTRACT. The Gilbert-Pearson characterization of the spectrum is established for a generalized Sturm-Liouville equation with two singular endpoints. It is also shown that strong absolute continuity for the one singular endpoint problem guarantees absolute continuity for the two singular endpoint problem. As a consequence, we obtain the result that strong nonsubordinacy, at one singular endpoint, of a particular solution guarantees the nonexistence of subordinate solutions at both singular endpoints.

1. **Introduction.** We consider a generalized Sturm-Liouville system

$$u^{[1]}(t) = u^{[1]}(a) + \int_a^t \Big[dQ(s) + z\,dW(s)\Big]u(s), \tag{1}$$

$$u^{[1]}(t) \equiv R(t)u'(t), -\infty \leq a < t < b \leq \infty, a < 0 < b \tag{2}$$

where R, Q and W are real, $R(t) \geq 0$ for all t, $[R(t)]^{-1}$ is locally Lebesgue integrable, Q and W are locally of bounded variation with W non-decreasing and $z = \lambda + i\varepsilon$ is a complex spectral parameter with $\operatorname{Im} z = \varepsilon \geq 0$. Our main assumption throughout this paper will be that (1) - (2) is in the limit point case both at $x = a$ and $x = b$. Hence a self-adjoint operator H is associated to (1)–(2) on (a, b) in the usual way [3]. We will also consider the operator H^k_θ, $k = b$ or a, which we associate to (1)–(2) on $[0, b)$ or $(a, 0]$ respectively by the boundary condition

$$u(0)\cos\theta + u^{[1]}(0)\sin\theta = 0, \theta \in [0, 2\pi).$$

Denote by $u_1 = u_1(x, z, \alpha)$ the solutions of (1)–(2) defined by the conditions

$$\begin{pmatrix} u_1 & u_2 \\ u_1^{[1]} & u_2^{[1]} \end{pmatrix}(0, z, \alpha) = \begin{pmatrix} -\sin\alpha & \cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}, \alpha \in [0, 2\pi).$$

The purpose of this paper is threefold. First, we extend the Gilbert-Pearson subordinacy theory [7] to (1)–(2) on (a, b) subject to the limit point hypothesis. Our main result in this respect is Theorem 2.1. The theory is well known for one as well as two singular endpoints Schrödinger operators [7], [6], and has been extended to (1)–(2) with one singular endpoint [4]. Extensions and applications of the theory to Dirac systems

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and perturbed periodic Schrödinger operators are also well known [1], [2], [8]. Next, we establish for (1)–(2) results analogous to those of [5]. In particular, we give sufficient conditions on the absolute continuity of the spectrum of H_θ^k to guarantee the absolute continuity of that of H. In this respect, our main result is Theorem 2.2. Our third main result, Theorem 2.3, gives sufficient conditions on the nonsubordinacy of the solution u_1 to guarantee the nonexistence of any subordinate solutions of (1)–(2).

As is observed in [4], the proofs used for the Schrödinger operators in [7] carry over to our case with only minor modifications. We have thus omitted much detail and refer the reader to the relevant Schrödinger case literature as we do not see much to be gained by publishing them here. From this viewpoint our results may be viewed as an assertion that what one knows to hold in the Schrödinger case does in fact hold for (1)–(2) as well, in respect of Theorems 2.1–2.3.

We shall proceed as follows. In Section 2 we give the definitions pertinent to our analysis and state our main results. We give proofs of the results in Section 3.

2. **Definitions and statement of results.** Let k denote either $x = a$ or $x = b$ and set $\|u\|_N^2 = \int_0^N |u(x)|^2 dW(x)$. Then for $z = \lambda \in \Re$ we say a solution $u(x, \lambda)$ of (1)–(2) is *subordinate* at $x = b$ if for every linearly independent solution $v(x)$

$$\lim_{N \to k} \frac{\|u(\cdot, \lambda)\|}{\|v(\cdot, \lambda)\|}_N = 0;$$

and we have a similar definition for the subordinacy of $u(x, \lambda, \theta)$, where we note that solutions will be written as $u(x, z)$ and $u(x, z, \theta)$ to respectively refer to the full-line and half-line problems. We will say a solution $u(x, \lambda, \theta)$ of the half-line problem is *strongly nonsubordinate* at k provided there exists a constant $c > 0$, independent of θ and λ, such that

$$\limsup_{N \to k} \frac{\|v(\cdot, \lambda, \theta)\|}{\|u(\cdot, \lambda, \theta)\|}_N \leq c$$

for every linearly independent solution $v(\cdot, \lambda, \theta)$.

Let us recall that if μ is a positive measure on $\Re$, then a measurable subset S of $\Re$ is said to be a minimal support of μ provided (i) $\mu(\Re)\backslash S = 0$ and (ii) if $S_0 \subset S$ with $\mu(S_0) = 0$, then $\ell(S_0) = 0$, where ℓ denotes Lebesgue measure. We also recall the equivalence relation $\overset{\mu}{\sim}$, which is defined by [6]

$$S \overset{\mu}{\sim} S' \iff \mu(S \triangle S') = \ell(S \triangle S') = 0$$

for measurable subsets S and S' of $\Re$, where $S \triangle S' = (S \backslash S') \cup (S \backslash S')$.

Associated to (1)–(2) are the (half-line) Titchmarsh-Weyl m-functions [3] $m_a(z)$ and $m_b(z)$, which are (uniquely) defined such that

$$u_a(x, z) \equiv u_2(x, z) + m_z(z) u_1(x, z) \in L_W^2(a, 0]$$

and

$$u_b(x, z) \equiv u_2(x, z) + m_b(z) u_1(x, z) \in L_W^2[0, b).$$

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The full-line m-function is then given by

$$M(z) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}(z), \tag{3}$$

where $m_{11} = \frac{1}{m_a - m_b}$, $m_{22} = \frac{m_a m_b}{m_a - m_b}$ and $m_{12} = m_{21} = \frac{1}{2}\frac{m_a + m_b}{m_a - m_b}$.

It is related to the spectral function $\rho(z)$ of H by the Titchmarsh-Kodaira formula

$$\rho(\lambda_2) - \rho(\lambda_1) = -\frac{1}{\pi}\lim_{\epsilon\longrightarrow 0}\int_{\lambda_1}^{\lambda_2} \operatorname{Im} M(\lambda + i\varepsilon)\, d\lambda \tag{4}$$

at points of continuity λ_1, λ_2 of ρ. The formula (4) also holds in respect of the spectral function $\rho^k_\theta(z)$ of H^k_θ, with M replaced by m_k. We will say the spectrum of H^k_θ is *strongly absolutely continuous* on an interval I provided that for some α, β, $\alpha \neq \beta$, ρ^k_α and ρ^k_β are absolutely continuous on I and there exist constants M, N and N' such that

$$0 < N < \frac{d\rho^k_\alpha}{d\lambda}\Big|_{\lambda\in I} < M < \infty \tag{5}$$

and

$$0 < N' < \frac{d\rho^k_\beta}{d\lambda}\Big|_{\lambda\in I}. \tag{6}$$

We are now in a position to state our results.

THEOREM 1. *Let ρ, ρ_{ac}, ρ_s, ρ_{sc} and ρ_p denote the spectral measure for H together with its usual decompositions. Then the respective minimal supports $M(H)$, $M_{ac}(H)$, $M_s(H)$, $M_{sc}(H)$ and $M_p(H)$ are as follows:*

(a) $M(H) = \Re\backslash\{\lambda \in \Re :$ a solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate at a but not at b, and a solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate at b but not at $a\}$.

(b) $M_{ac}(H) = \{\lambda \in \Re :$ either no solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate at a, or no solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate at b, or both$\}$.

(c) $M_s(H) = \{\lambda \in \Re :$ a solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate both at a and at $b\}$.

(d) $M_{sc}(H) = \{\lambda \in \Re :$ a solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate both at a and at b but is not $L^2_W(a,b)\}$.

(e) $M_p(H) = \{\lambda \in \Re :$ a solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate both at a and at b and is in $L^2_W(a,b)\}$.

THEOREM 2. *Suppose ρ^k_θ to be strongly absolutely continuous on some interval $I \subset \Re$, where $k = a$ or $k = b$. Then ρ is absolutely continuous on I.*

THEOREM 3. *Suppose $u_1(x,\lambda,\theta)$ to be strongly nonsubordinate either at a or at b. Then either no solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate at b, or no solution $u(x,\lambda)$ of (1)–(2) exists which is subordinate at a, or both.*

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3. **Proofs.** We demonstrate our results by establishing a series of propositions which although well-known in some cases, have hitherto not been obtained for the present case. However, due to the similarity of proofs with the known cases, we leave out the details and only give the reader appropriate references to the literature.

PROOF OF THEOREM 2.1. It suffices to prove only (b) and (c) since (e) is well known, (c) and (e) imply (d), and (b) and (c) imply (a).

PROOF OF (B). As in [4] or [6], one establishes the following result.

PROPOSITION 4. *Let $m_k^+(\lambda)$ denote $\lim_{\varepsilon\downarrow 0} m_k(\lambda + i\varepsilon)$. The minimal supports $M_{ac}(H_\theta^a)$ and $M_{ac}(H_\theta^b)$ of ρ_θ^a and ρ_θ^b are as follows: (i) $M_{ac}(H_\theta^a) = \{\lambda \in \Re : -\infty < \operatorname{Im} m_a^+(\lambda) < 0\}$; (ii) $M_{ac}(H_\theta^b) = \{\lambda \in \Re : 0 < \operatorname{Im} m_b^+(\lambda) < \infty\}$.*

Then letting $m(z) \equiv m_{11}(z) + m_{22}(z)$ generate a measure $\tilde{\rho}(z)$, via the Titchmarsh-Kodaira formula (4), which is equivalent to $\rho(z)$ since $(\rho_{ij}) \ll \rho_{11} + \rho_{22} \ll (\rho_{ij})$, one proceeds as in [6] to obtain the next result.

PROPOSITION 5. *The minimal support $\tilde{M}_{ac}(H)$ of $\tilde{\rho}_{ac}$ is $\tilde{M}_{ac}(H) = \{\lambda \in \Re : 0 < \operatorname{Im} M^+(\lambda) < \infty\}$, where $M^+(\lambda)$ denotes $\lim_{\varepsilon\downarrow 0} M(\lambda + i\varepsilon)$.*

Combining Propositions 4 and 5, and recalling that minimal supports for a measure μ are unique up to sets of μ and ℓ measure zero, we arrive at the following result.

PROPOSITION 6. $M_{ac}(H) \overset{\mu_{ac}}{\sim} M_{ac}(H_\theta^a) \cup M_{ac}(H_\theta^b)$.

Combining Proposition 6 with Theorems 2.1 and 2.2 of [4], and recalling the definition of the equivalence relation $\overset{\mu}{\sim}$ then completes the demonstration of (b).

We note the following well known consequence of Proposition 6.

COROLLARY 7. $\sigma_{ac}(H) = \sigma_{ac}(H_\theta^a) \cup \sigma_{ac}(H_\theta^b)$.

PROOF OF (C). Letting $m_k^+(\lambda)$ have the same meaning as in Proposition 4 we obtain, as in [6], the following result.

PROPOSITION 8. *Let $S_0 = \{\lambda \in \Re : m_a^+(\lambda)$ and $m_b^+(\lambda)$ exist finitely and are equal$\}$ and $S_\infty = \{\lambda \in \Re : |m_a(z)| \longrightarrow \infty, |m_b(z)| \longrightarrow \infty$ as $z \longrightarrow \lambda\}$. Then $M_s(H) = S_0 \cup S_\infty$ is a minimal support of ρ_s.*

Proposition 8 combines with Theorems 2.1 and 2.2 of [4] to yield (c), and hence completes the proof of Theorem 2.1.

PROOF OF THEOREM 2.2. Proceeding exactly as in the proof of Theorem 2 of [5], we obtain the following result.

PROPOSITION 9. *Let ρ_θ^k be strongly absolutely continuous on an interval I with constants M, N and N' as in (5) and (6). Then there exists $\delta > 0$ such that we have $|m_k(\lambda + i\varepsilon)| < N$ and $|\operatorname{Im} m_k(\lambda + i\varepsilon)| > M$ uniformly for all $0 < \varepsilon < \delta$ on closed subintervals of I.*

Again proceeding as in the proof of Theorem 1 of [5], but using $M(z)$ as given in (3) instead of the characteristic matrices of [5], one obtains the following.

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PROPOSITION 10. *Suppose there exist constants* $\delta > 0$, $N > 0$, $M > 0$ *such that* $|m_k(\lambda + i\varepsilon)| < N$ *and* $|\operatorname{Im} m_k(\lambda + i\varepsilon)| > M$ *for all* λ *in a closed interval* $I \in \Re$ *and* $0 < \varepsilon < \delta$, *then* ρ *is absolutely continuous on* I.

Combining Propositions 9 and 10 and using the fact that absolute continuity on all closed subintervals of I implies absolute continuity on I hence completes the proof of Theorem 2.2.

PROOF OF THEOREM 2.3. Theorem 3.1 of [4] immediately yields the following result.

PROPOSITION 11. *Let* $\lambda \in I$ *and suppose* $u_1(x, \lambda, \theta)$ *is strongly nonsubordinate at* k. *Then* ρ^k_θ *is strongly absolutely continuous on* I.

Hence Theorem 2.2 gives the following result.

PROPOSITION 12. *Let* $\lambda \in I$ *and suppose* $u_1(x, \lambda, \theta)$ *is strongly nonsubordinate at* a *or at* b. *Then* ρ *is absolutely continuous on* I.

Theorem 2.3 then follows from Proposition 12 together with part (b) of Theorem 2.1.

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