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Singular Moduli of Shimura Curves

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Abstract. The j-function acts as a parametrization of the classical modular curve. Its values at complex multiplication (CM) points are called singular moduli and are algebraic integers. A Shimura curve is a generalization of the modular curve and, if the Shimura curve has genus 0, a rational parameterizing function exists and when evaluated at a CM point is again algebraic over **Q**. This paper shows that the coordinate maps given by N. Elkies for the Shimura curves associated to the quaternion algebras with discriminants 6 and 10 are Borcherds lifts of vector-valued modular forms. This property is then used to explicitly compute the rational norms of singular moduli on these curves. This method not only verifies conjectural values for the rational CM points, but also provides a way of algebraically calculating the norms of CM points with arbitrarily large negative discriminant.

1 Introduction

The classical modular curve $\mathcal{X}_1^*$ is given as the one-point compactification of the Riemann surface $\mathrm{GL}_2(\mathbb{Z})\backslash\mathfrak{h}^\pm$ where $h^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. Since $\mathcal{X}_1^*$ is a genus-0 surface, there exists an isomorphism $\mathcal{X}_1^* \xrightarrow{\sim} \mathbb{P}^1$. The classical choice of such a map has Fourier expansion

$$j(\tau) = \frac{1}{\mathbf{q}} + 744 + 196884\mathbf{q} + \cdots \in \frac{1}{\mathbf{q}}\mathbb{Z}[[\mathbf{q}]],$$

(where $\mathbf{q} = e^{2\pi i\tau}$) at the cusp at ∞. The j-function also provides an identification of points on the modular curve with isomorphism classes of elliptic curves. When the associated elliptic curve has an extra endomorphism called complex multiplication (CM), τ is an irrational quadratic imaginary point of $\mathfrak{h}^\pm$ and is called a CM point. A singular modulus is a value of the j-function at a CM point and is an algebraic integer. In 1984, Gross and Zagier [8] gave an explicit formula to compute the norms of singular moduli.

A Shimura curve is a generalization of the modular curve. Let B be the quaternion algebra over $\mathbb{Q}$ with discriminant $D = D(B) > 1$ and let $\Gamma^* = N_{B^\times}(\mathcal{O}) \subset B^\times$ be the normalizer of a maximal order $\mathcal{O} \subset B$. Since there is an algebra embedding $B \hookrightarrow \mathrm{M}_2(\mathbb{R})$, the discrete group Γ^* embeds into $\mathrm{GL}_2(\mathbb{R})$ and hence acts on $\mathfrak{h}^\pm$. The Shimura curve $\mathcal{X}_D^*$ is then given as

$$\mathcal{X}_D^* = \Gamma^*\backslash\mathfrak{h}^\pm.$$

When B is a division algebra, $\mathcal{X}_D^*$ is a compact Riemann surface without cusps.

Points on a Shimura curve can also be identified with certain 2-dimensional abelian varieties, and again there is the notion of CM points. As before, there will be a

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generator of the function field $t_D\colon \mathcal{X}_D^* \to \mathbb{P}^1$, and, if properly normalized, the image of a CM point under t_D will be algebraic over $\mathbb{Q}$. However, since $\mathcal{X}_D^*$ has no cusps, such a map does not have a $\mathbf{q}$-expansion, and example calculations are more difficult than in the classical case.

In [7], Elkies considered the cases of $D = 6$ and $D = 10$. First, by identifying which quadratic imaginary fields have class group $(\mathbb{Z}/2\mathbb{Z})^r$ for $r \leq 2$, he determined which CM points have rational coordinates on $\mathcal{X}_D^*$. Then with $\Gamma^*(l) = \{\gamma \in \Gamma^* \mid \gamma \equiv 1 \bmod l\}$, Elkies used explicit calculations of the geometric involution on $\mathcal{X}_D^*(l) = \Gamma^*(l)\backslash\mathfrak{h}^{\pm}$ for small primes l to compute the coordinates for about half of the rational CM points on $\mathcal{X}_6^*$ and $\mathcal{X}_{10}^*$. The involutions on $\mathcal{X}_D^*(l)$ for higher l are unknown and are needed to explicitly find the coordinates of the remaining half of the CM points using this method. Elkies does, however, provide a table of conjectural values for the remaining CM points obtained via numerical approximations and their behavior under standard transformations.

In this paper, we use an alternate method that arises out of the theory of Borcherds forms to calculate the norms of singular moduli on the Shimura curves $\mathcal{X}_6^*$ and $\mathcal{X}_{10}^*$ and, as a special case, algebraically prove the conjectural values listed in [7]. Although the methods are only demonstrated here for $D = 6$ and $D = 10$, the techniques should extend to a larger class of functions $\mathcal{X}_D^* \to \mathbb{P}^1$ for arbitrary indefinite discriminants D.

Let L be a lattice in a rational inner product space $V \subset B$ with signature $(n, 2)$ and let $L^\vee$ be its integral dual. Then a meromorphic modular form F valued in $\mathbb{C}[L^\vee/L]$ can be given by its Fourier expansion

$$F(\tau) = \sum_{\eta \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\eta(m)\mathbf{q}^m e_\eta, \tag{1.1}$$

where e_η is the basis element of $\mathbb{C}[L^\vee/L]$ corresponding to η. When $c_\eta(m) \in \mathbb{Z}$ for $m < 0$, $c_0(0) = 0$, and F has weight $1 - \frac{n}{2}$, Borcherds [4] constructs a form $\Psi(F)\colon \mathcal{X}_D^* \to \mathbb{P}^1$ and gives its divisor in terms of rational quadratic divisors weighted by the coefficients $c_\eta(m)$ for $m < 0$. In this more general setting $\mathcal{X}_D^*$ is formed by $B^\times$ acting on the product of the adeles of B (viewed as an algebraic group) modulo a compact open set and a space of oriented negative 2-planes arising from the inner product.

Recently, Schofer [15] provided an explicit formula in terms of the coefficients of Eisenstein series for the norm

$$\prod_{z \in Z(\Delta)} \|\Psi(z, F)\|^2 \tag{1.2}$$

where $Z(\Delta)$ is the set of CM points of discriminant Δ on $\mathcal{X}_D^*$. As a corollary, he showed that since the j-function was in fact a Borcherds form, the Gross–Zagier factorization of singular moduli was a specific case of his main theorem.

In the cases of $D = 6$ and $D = 10$, the coordinate map $t_D\colon \mathcal{X}_D^* \xrightarrow{\sim} \mathbb{P}^1$ given in [7] is defined by its divisor and normalized by its value at a chosen point. We show how this divisor can be expressed in terms of rational quadratic divisors. We then find a

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meromorphic modular form F_D as in (1.1) that satisfies $\operatorname{div}\big(\Psi(F_D)^2\big) = \operatorname{div}(t_D)$. In the cases analyzed here, $n = 1$ and the lattice L arises as the trace-zero elements of $\mathcal{O}$. Then the proper vector-valued form F_D is lifted from a scalar-valued modular form that is a linear combination of Dedekind-η products. Next we compute a normalization constant, c_D, by applying (1.2) to a base case. Since the divisors are equal and the two functions agree on the chosen base point, we conclude

$$\Psi(F_D)^2 = c_D t_D.$$

Finally, (1.2) is used to calculate the norm of any CM point on $\mathcal{X}_D^*$. Since this method is a general calculation of norms, the tables of rational CM points found in [7] arise as specific cases. For example, we can recompute known values, *e.g.*,

$$t_6(\mathcal{P}_{-147}) = -\frac{11^4 23^4}{2^{10}3^3 5^6 7},$$

but can also explicitly verify the conjectural values such as

$$t_6(\mathcal{P}_{-163}) = \frac{3^{11}7^4 19^4 23^4}{2^{10}5^6 11^6 17^6}.$$

Moreover, we can algebraically compute the norm of CM points with arbitrarily large discriminants. For example $t_6(\mathcal{P}_{-996})$ is an algebraic number of degree 6 over $\mathbb{Q}$ and the method of this paper provides its norm:

$$|t_6(\mathcal{P}_{-996})| = \frac{2^{16}7^{12}71^4 83^2}{17^6 29^6 41^6}.$$

In addition, this method should generalize even further to computing norms of singular moduli on higher genus Shimura curves.

2 Shimura Curves

2.1 Quaternion Algebras

Quaternion algebras have a long history of study so we will only provide a brief summary of the important facts. For a more thorough exploration of quaternion algebras see [1], [9], and [16].

A rational quaternion algebra B is a central simple algebra of dimension 4 over $\mathbb{Q}$ and is either isomorphic to $M_2(\mathbb{Q})$ or is a skew field. In the latter case, B is called a division algebra. For each prime p, $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a $\mathbb{Q}_p$-algebra. If B_p is a division algebra, then B is said to be ramified at p. If B_p is not a division algebra, then $B_p \simeq M_2(\mathbb{Q}_p)$. A quaternion algebra is called definite (indefinite) if it ramifies (is not ramified) at the infinite prime.

The (reduced) discriminant $D = D(B)$ of a quaternion algebra B is given as the product of all finite ramified primes of B. Given an even number of finite or infinite primes, there exists a quaternion algebra over $\mathbb{Q}$ ramified exactly at those places. Further, two quaternion algebras are isomorphic if and only if they have the same discriminant.

Proposition 2.1 ([9, Proposition 3.1]) *Let B be an indefinite quaternion algebra over $\mathbb{Q}$ with $D = p_1 \cdots p_{2r}$. Choose q to be a prime such that $q \equiv 5 \bmod 8$ and $(\frac{q}{p_i}) = -1$ for every $p_i > 2$. Then $B \simeq \mathbb{Q}(\alpha, \beta)$ where $\alpha\beta = -\beta\alpha$ and $\alpha^2 = q$, $\beta^2 = D$. We denote this by $B = (\frac{q,D}{\mathbb{Q}})$.*

There are many ways to embed $B = (\frac{a,b}{\mathbb{Q}})$ into a matrix algebra over an extension of $\mathbb{Q}$. The one that we use in this paper is

$$\phi_b \colon B \hookrightarrow \mathrm{M}_2\big(\mathbb{Q}(\sqrt{b})\big)$$

given by

$$\phi_b(\alpha) = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, \quad \phi_b(\beta) = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}.$$

There is a natural involution on $x = x_0 + x_1\alpha + x_2\beta + x_3\alpha\beta$ given by

$$\overline{x} = x_0 - x_1\alpha - x_2\beta - x_3\alpha\beta.$$

This involution allows one to define the (reduced) trace and (reduced) norm as

$$\mathrm{tr}(x) = x + \overline{x} = 2x_0,$$
$$\mathrm{n}(x) = x\overline{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2.$$

Under the above embedding, these are just the usual matrix trace and determinant.

2.2 Maximal Orders

Definition 2.1 Let $\mathcal{K}$ be either $\mathbb{Q}$ or $\mathbb{Q}_p$ and $\mathcal{R}$ its ring of integers. An $\mathcal{R}$-order $\mathcal{O}$ in a quaternion algebra B over $\mathcal{K}$ is an $\mathcal{R}$-ideal that is a ring. Equivalently, an $\mathcal{R}$-order $\mathcal{O}$ is a ring whose elements have trace and norm in $\mathcal{R} \subset \mathcal{O}$, and $\mathcal{O} \otimes_{\mathcal{R}} \mathcal{K} = B$. A maximal order is an order that can not be properly contained in another order.

In general, B does not have a unique maximal order. In fact, if $\omega \in B^\times$ and $\mathcal{O}$ is a maximal order, then $\omega\mathcal{O}\omega^{-1}$ is also a maximal order. However, when B is indefinite, the conjugacy class of maximal orders is unique.

Proposition 2.2 ([9, Proposition 3.2]) *For B as in Proposition 2.1 with $\alpha^2 = q$ and $\beta^2 = D$, every maximal order is conjugate to*

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_1e_2$$

where

$$e_1 = \frac{1+\alpha}{2},$$
$$e_2 = \frac{m\alpha + \alpha\beta}{q},$$
$$D \equiv m^2 \bmod q.$$

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When p is a ramified prime, there is a unique maximal order $\mathcal{O}_p \subset B_p$, and it is given by

$$\mathcal{O}_p = \{\omega \in B \mid (\mathrm{ord}_p \circ \mathrm{n})(\omega) \geq 0\}.$$

Hence its group of units is given by

$$\mathcal{O}_p^\times = \{\omega \in B^\times \mid (\mathrm{ord}_p \circ \mathrm{n})(\omega) = 0\}.$$

Moreover, one can choose a uniformizer $\pi_p \in B_p^\times$ such that $B_p^\times = \mathcal{O}_p^\times \rtimes \pi_p^{\mathbb{Z}}$ with $(\mathrm{ord}_p \circ \mathrm{n})(\pi_p) = 1$ and $\pi_p^2 = p$.

Define the normalizer of an order as

$$\mathrm{N}_{B^\times}(\mathcal{O}) = \{\omega \in B^\times \mid \omega\mathcal{O}\omega^{-1} \subset \mathcal{O}\}.$$

The units of an order $\mathcal{O}$ are a subgroup of $\mathrm{N}_{B^\times}(\mathcal{O})$ and are related by the following lemma.

Lemma 2.3 ([16]) *Let $d(B)$ denote the number of ramified primes of B. Then*

$$\mathrm{N}_{B^\times}(\mathcal{O})/(\mathbb{Q}^\times\mathcal{O}^\times) \simeq (\mathbb{Z}/2\mathbb{Z})^{d(B)}.$$

2.3 Shimura Curves and CM Points

From now on let $B = (\frac{q,D}{\mathbb{Q}})$ with $\alpha^2 = q$ and $\beta^2 = D$ as in Proposition 2.1. Fix the embedding of $B \hookrightarrow \mathrm{M}_2(\mathbb{R})$ given by ϕ_D and the maximal order $\mathcal{O}$ as in Proposition 2.2. Define the following subgroups of $B^\times$,

$$\Gamma = \mathcal{O}^\times, \quad \Gamma^* = \mathrm{N}_{B^\times}(\mathcal{O}).$$

Their images under ϕ_D are discrete subgroups of $B^\times \subset \mathrm{GL}_2(\mathbb{R})$, and they act on $\mathfrak{h}^\pm = \mathbb{P}(\mathbb{C}) - \mathbb{P}(\mathbb{R})$ via fractional linear transformations. Define $\mathcal{X}$ and $\mathcal{X}^*$ to be the Shimura curves

$$\mathcal{X} = \mathcal{X}_D = \Gamma\backslash\mathfrak{h}^\pm, \quad \mathcal{X}^* = \mathcal{X}_D^* = \Gamma^*\backslash\mathfrak{h}^\pm.$$

When B is an indefinite division algebra, $\mathcal{X}$ and $\mathcal{X}^*$ are compact Riemann surfaces with no cusps. Also, Lemma 2.3 implies that $\mathcal{X}$ is a covering space of $\mathcal{X}^*$ of degree $2^{d(B)}$.

Fix a quadratic imaginary field $\mathbf{k}$ such that if $p \mid D$ then p does not split in $\mathbf{k}$. Then there are many embeddings $\iota\colon \mathbf{k} \hookrightarrow B$. However, all of the embeddings are conjugate to each other [16].

Definition 2.2 The image $\iota(\mathbf{k}^\times) \to B^\times/\mathbb{Q}^\times \subset \mathrm{PGL}_2(\mathbb{R})$ has a unique fixed point on $\mathfrak{h}^+$. A complex-multiplication (CM) point of $\mathcal{X}$ (resp., $\mathcal{X}^*$) is the Γ-orbit (Γ^*-orbit) of such a point. It is said to have discriminant equal to the field discriminant of $\mathbf{k}$.

Since all embeddings are conjugate, a CM point is independent of the embedding. In the classical case of $B = \mathrm{M}_2(\mathbb{Q})$, the CM points are irrational imaginary solutions to integral quadratic equations with the corresponding discriminant.

2.4 Involutions on $\mathcal{X}_D^*(l)$

In this section, we summarize the method used in [7] to calculate the coordinates of rational CM points on $\mathcal{X}^*$. Let $\mathcal{P}_\Delta$ be the CM point with discriminant $\Delta < 0$ and let $R \subset \mathbf{k}$ be the maximal order in the quadratic imaginary field of discriminant Δ.

Proposition 2.4 ([7]) *$\mathcal{P}_\Delta$ is a rational point on $\mathcal{X}_D^*$ if and only if the class group of $\mathbf{k}$ is generated by ideals $I \subset R$ such that $I^2 = (p)$ for some $p \mid D$.*

This implies that for a rational CM point, the class group of $\mathbf{k}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ where $r \le d(B)$. In the case of $d(B) = 2$, all such fields are known, and thus the rational CM points can be identified. (See Table 2 for $D = 6$ and Table 4 for $D = 10$.)

Now let l be a prime not dividing D, so that $B \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq \mathrm{M}_2(\mathbb{Q}_l)$. Define

$$\Gamma^*(l) = \{\gamma \in \Gamma^* \mid \gamma \equiv \pm 1 \bmod l\}$$

and the congruence subgroup $\Gamma_0^*(l)$ in the same fashion as its classical counterpart. Then the curves

$$\mathcal{X}_D^*(l) = \Gamma^*(l)\backslash\mathfrak{h}^{\pm}, \quad \mathcal{X}_{D,0}^*(l) = \Gamma_0^*(l)\backslash\mathfrak{h}^{\pm}$$

are coverings of $\mathcal{X}_D^*$ whose points are also associated to abelian varieties. From the geometric structure, $\mathcal{X}_{D,0}^*(l)$ inherits an involution $w_l\colon \mathcal{X}_{D,0}^*(l) \to \mathcal{X}_{D,0}^*(l)$ which preserves the set of rational CM points.

In the case of $D = 6$, the image of $\Gamma^* \hookrightarrow \mathrm{PGL}_2(\mathbb{R})$ is generated by three elements and is a triangle group. An area calculation [7] shows that $\mathcal{X}_6^*$ has genus 0. Any coordinate map $t_6\colon \mathcal{X}_6^* \to \mathbb{P}^1$ is defined up to a $\mathrm{PGL}_2(\mathbb{R})$ action, so such a map is only well-defined once its values at three points have been given. Since there are three distinguished elements of Γ^*, the coordinate map is defined to take the values of 0, 1, ∞ at $\mathcal{P}_{-4}$, $\mathcal{P}_{-24}$, $\mathcal{P}_{-3}$, the CM points associated to the three generators.

The covering curves $\mathcal{X}_{6,0}^*(l)$, for $l = 5, 7, 13$ have genus 0 and w_l can be expressed explicitly as a rational function. Then by examining the fixed points of w_l and the w_l-orbits of 0, 1, and ∞, Elkies was able to compute the coordinates of 17 of the 27 rational CM points (see Table 2).

In order to compute the remaining ten rational CM points using this method, involutions on $\mathcal{X}_{6,0}^*(l)$ for higher l are needed. However, these curves have genus greater than 0 and explicit expressions for w_l are unknown. Instead, Elkies used numerical techniques to calculate the coordinates to an arbitrary precision. He then recognized them as fractional values through continued fractions and their behavior under standard transformations. For example, one expects that the factorizations of both $t_6(\mathcal{P}_\Delta)$ and $t_6(\mathcal{P}_\Delta) - 1$ should only contain small primes to large powers.

From the point of view of arithmetic intersection theory, the minimal model for $\mathcal{X}_6^*$ is the one given in [2]. The relations between their uniformization j and Elkies' t are $j = 16(t-1)/27$ for $\mathcal{X}_6^*$ and $j = t/8$ for $\mathcal{X}_{10}^*$.

3 Quadratic Spaces and Lattices

For a given indefinite quaternion algebra B, define the $\mathbb{Q}$-vector space

$$V = \{x \in B \mid \mathrm{tr}(x) = 0\}.$$

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There is a natural quadratic form on V given by $Q(x) = \mathrm{n}(x) = -x^2$. Let $(x, y) = \mathrm{tr}(x\overline{y})$ denote the associated inner product which has signature $(1, 2)$.

3.1 The Lattice $\mathcal{O} \cap V$

Define the lattice $L = \mathcal{O} \cap V$. Let $L^\vee$ be the $\mathbb{Z}$-dual of L and consider $L_p^\vee / L_p$ where $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

For $p \nmid D$ and p odd, there is an isomorphism $B_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$ such that $\mathcal{O}_p \simeq \mathrm{M}_2(\mathbb{Z}_p)$. Then L_p is the set of trace zero elements of $\mathrm{M}_2(\mathbb{Z}_p)$ and $L_p^\vee / L_p$ is trivial. Thus

$$L^\vee / L \simeq \prod_{p \mid 2D} L_p^\vee / L_p.$$

Now consider $p \mid D$ and p odd. Let $\delta \notin \mathbb{Z}_p^\times$, $\delta^2 \in \mathbb{Z}_p^\times$ and $\mathbb{Z}_{p^2} = \mathbb{Z}_p + \mathbb{Z}_p \delta$ be the ring of integers in the unramified quadratic extension of $\mathbb{Q}_p$ with Galois automorphism σ. Then

$$L_p = \mathbb{Z}_p \delta + \mathbb{Z}_p \pi_p + \mathbb{Z}_p \delta \pi_p, \quad L_p^\vee = \mathbb{Z}_p \delta + p^{-1} \mathbb{Z}_{p^2} \pi_p. \tag{3.1}$$

Since $\frac{1}{p}\mathbb{Z}_{p^2} / \mathbb{Z}_{p^2} \simeq \mathbb{F}_{p^2}$, the field of p^2 elements, there is an isomorphism

$$\mathbb{F}_{p^2} \xrightarrow{\sim} L_p^\vee / L_p, \quad \tilde{v} \mapsto v \pi_p^{-1} + L_p.$$

Under this isomorphism, the quadratic form Q induces the function

$$Q(\tilde{v}) = v v^\sigma p^{-1} \bmod \mathbb{Z}_p,$$

which is equivalent to the norm map $\mathrm{n}\colon \mathbb{F}_{p^2} \to \mathbb{F}_p$ via $\mathbb{F}_p \xrightarrow{\sim} \frac{1}{p}\mathbb{Z}_p / \mathbb{Z}_p$.

The case of $p = 2$ has $L_2^\vee = \frac{1}{2} L_2$. This time the isomorphism is

$$\mathbb{F}_2 \oplus \mathbb{F}_4 \xrightarrow{\sim} L_2^\vee / L_2, \quad (\tilde{w}, \tilde{v}) \mapsto w \frac{\sqrt{5}}{2} + v \pi_2^{-1} + L_2, \tag{3.2}$$

and Q induces the function

$$Q(\tilde{w}, \tilde{v}) = -\frac{1}{4} w^2 - \frac{1}{2}\, \mathrm{n}(v) \bmod \mathbb{Z}_2.$$

This surjects onto $\frac{1}{4}\mathbb{Z}/\mathbb{Z}$, given by whether or not each of the components is nonzero.

Proposition 3.1 *Let D_0 be the odd part of D. Then*

$$|L^\vee / L| = 8 D_0^2.$$

Proposition 3.2 *Let $B_p^\times$ act on $L_p^\vee / L_p$ via conjugation. Then the $B_p^\times$ orbits of $L_p^\vee / L_p$ for odd $p \mid D$ (resp., $p = 2$) are indexed by elements of $\mathbb{F}_p$ ($\mathbb{F}_4$).*

Proof For odd p, write $B_p^\times$ as

$$B_p^\times = (\mathcal{O}_p^\times \cup \mathcal{O}_p^\times \pi_p) p^{\mathbb{Z}}.$$

First, the powers of p are central and hence act trivially. Then by (3.1)

$$L_p^\vee / L_p \xrightarrow{\sim} \mathcal{O}_p / \pi_p \mathcal{O}_p.$$

Thus the elements of $\mathcal{O}_p^\times$ act through their image under the reduction map $\mathcal{O}_p \to \mathbb{F}_{p^2}$. More explicitly, $\tilde{\nu} \in \mathbb{F}_{p^2}^\times$ acts via left multiplication by ν/ν^σ. However, this is just the action of $\mathbb{F}_{p^2}^1 = \ker(\mathrm{n}\colon \mathbb{F}_{p^2}^\times \to \mathbb{F}_p^\times)$. Lastly, π_p acts by σ, and so there is a surjection

$$B_p^\times \twoheadrightarrow \mathbb{F}_{p^2}^1 \rtimes \langle \sigma \rangle.$$

Hence the orbits of $B_p^\times$ are indexed by the elements of $\mathbb{F}_p$.

For $p = 2$, the action of $B_2^\times$ preserves the first component of (3.2) and acts on the second component the same way it did in the odd p case. So again the orbits are indexed by the four values of Q. ∎

3.2 The Order of the Orbits of Γ^*

Define the set

$$V(t) = \{x \in V \mid Q(x) = t\}$$

and $L(t) = L \cap V(t)$. The discrete groups Γ and Γ^* both act on L by conjugation, and the order of Γ^*-orbits in $L(t)$ will play an important role in Section 7.

Let $0 > \Delta \in \mathbb{Z}$ be the field discriminant of $\mathbf{k} = \mathbb{Q}(\sqrt{-t})$, and set $-4t = n^2\Delta$. Then the order $\mathbb{Z}[\sqrt{-t}]$ has discriminant $-4t$. Hence, its conductor is n, and if any other order R in $\mathbf{k}$ contains $\mathbb{Z}[\sqrt{-t}]$, then the conductor of R divides n.

Set

$$\mathcal{E} = \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(\mathbf{k}, B).$$

Assume that for every prime $p \mid D$, p is nonsplit in $\mathbf{k}$ so that $\mathcal{E}$ is nontrivial. For every $x \in L(t)$, define $\iota_x \in \mathcal{E}$ by $\iota_x(\sqrt{-t}) = x$. For $\iota \in \mathcal{E}$, $\iota^{-1}\big(\mathcal{O} \cap \iota(\mathbf{k})\big)$ is an order in $\mathbf{k}$. Let $\mathrm{cond}(\iota)$ denote the conductor of this order and define

$$\mathcal{E}(c) = \{\iota \in \mathcal{E} \mid \mathrm{cond}(\iota) = c\}.$$

For $x \in L$, define $\mathrm{cond}(x) = \mathrm{cond}(\iota_x)$ and let

$$L(t,c) = \{x \in L(t) \mid \mathrm{cond}(x) = c\}.$$

Then for a fixed t and c, there is a bijection $L(t,c) \xrightarrow{\sim} \mathcal{E}(c)$ given by $x \mapsto \iota_x$ and Γ^* acts on $L(t,c)$ via conjugation. This action is compatible with the action on $\mathcal{E}(c)$, therefore

$$\Gamma^* \backslash L(t,c) \xrightarrow{\sim} \Gamma^* \backslash \mathcal{E}(c). \tag{3.3}$$

To determine the set of Γ^*-orbits in $L(t,c)$, we examine the right-hand side of (3.3). Let R be the ring of integers of an imaginary quadratic field $\mathbf{k}$. Fix an embedding $\iota_0\colon \mathbf{k} \hookrightarrow B$ with $\operatorname{cond}(\iota_0) = 1$, *i.e.*, $\iota_0(R) \subset \mathcal{O}$. Since all embeddings of $\mathbf{k}$ into B are conjugate, there is a bijection

$$B^\times/\mathbf{k}^\times \xrightarrow{\sim} \mathcal{E},$$
$$\omega \mapsto \operatorname{Ad}(\omega) \circ \iota_0.$$

Then

$$\Gamma^*\backslash B^\times/\mathbf{k}^\times \xrightarrow{\sim} \Gamma^*\backslash\mathcal{E},$$

where the action of Γ^* on $B^\times/\mathbf{k}^\times$ is left multiplication. Define

$$B^\times(c) = \bigl\{\omega \in B^\times \mid \operatorname{cond}\bigl(\operatorname{Ad}(\omega)\circ\iota_0\bigr) = c\bigr\}$$

so that

$$\Gamma^*\backslash B^\times(c)/\mathbf{k}^\times \xrightarrow{\sim} \Gamma^*\backslash\mathcal{E}(c). \tag{3.4}$$

Let $\operatorname{Ord} = \operatorname{Ord}(B)$ be the set of all maximal orders of B. For any $\mathcal{O} \in \operatorname{Ord}$, define the conductor of $\mathcal{O}$ to be the conductor of $\iota_0^{-1}\bigl(\mathcal{O} \cap \iota_0(\mathbf{k})\bigr)$. Define for $\omega \in B^\times$, $\mathcal{O}_\omega = \omega^{-1}\mathcal{O}\omega \in \operatorname{Ord}$. Then the conductor of $\mathcal{O}_\omega$ is $\operatorname{cond}(\omega)$.

The action of $B^\times_{\mathbb{A}_f} = (B \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$ on Ord via

$$\xi \cdot \mathcal{O}_\omega = \xi^{-1}\widehat{\mathcal{O}_\omega}\xi \cap B$$

where $\widehat{\mathcal{O}} = \mathcal{O} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is transitive, thus

$$\mathrm{N}_{B^\times_{\mathbb{A}_f}}(\widehat{\mathcal{O}})\backslash B^\times_{\mathbb{A}_f} \xrightarrow{\sim} \operatorname{Ord},$$
$$\xi \mapsto \xi^{-1}\widehat{\mathcal{O}}\xi \cap B.$$

Furthermore, the double cosets

$$\mathrm{N}_{B^\times_{\mathbb{A}_f}}(\widehat{\mathcal{O}})\backslash B^\times_{\mathbb{A}_f}/B^\times$$

correspond to the $B^\times$-conjugacy classes of the maximal orders in B. Since B is an indefinite quaternion algebra, all maximal orders of B are conjugate. Thus

$$\mathrm{N}_{B^\times_{\mathbb{A}_f}}(\widehat{\mathcal{O}})\backslash B^\times_{\mathbb{A}_f} \simeq \mathrm{N}_{B^\times}(\mathcal{O})\backslash B^\times.$$

Let $\operatorname{Ord}(c) \subset \operatorname{Ord}$ be the subset of orders with conductor c. Then, with notations as before,

$$\mathrm{N}_{B^\times}(\mathcal{O})\backslash B^\times(c) \xrightarrow{\sim} \operatorname{Ord}(c), \tag{3.5}$$

and the $\mathbf{k}_{\mathbb{A}_f}^\times = (\mathbf{k} \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$ action on Ord given by $\xi \cdot \mathcal{O} = \xi^{-1}\widehat{\mathcal{O}}\xi \cap B$ preserves $\mathrm{Ord}(c)$.

From the Chevalley–Hasse–Noether theorem, for a given $\mathcal{O}_c \in \mathrm{Ord}(c)$ there is a bijection

$$\mathrm{N}_{B_{\mathbb{A}_f}^\times}(\mathcal{O}_c) \cap \mathbf{k}_{\mathbb{A}_f}^\times \backslash \mathbf{k}_{\mathbb{A}_f}^\times \xrightarrow{\sim} \mathrm{Ord}(c) \tag{3.6}$$

given by the orbit of $\mathcal{O}_c$ under the transitive action of $\mathbf{k}_{\mathbb{A}_f}^\times$. Then the composition of the bijections in (3.3), (3.4), (3.5), and (3.6) yield

$$\Gamma^* \backslash L(t,c) \overset{\sim}{\leftrightarrow} \mathrm{N}_{B_{\mathbb{A}_f}^\times}(\mathcal{O}_c) \cap \mathbf{k}_{\mathbb{A}_f}^\times \backslash \mathbf{k}_{\mathbb{A}_f}^\times / \mathbf{k}^\times .$$

Let Δ_0 be the product of all the primes that ramify in $\mathbf{k}$ and define

$$\delta(\Delta_0, D) = \#\{p \text{ prime} \mid p | \gcd(\Delta_0, D)\} - \begin{cases} 1 & \text{if } \Delta_0 \mid D, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.3 *Let $R_c \in \mathbf{k}$ be the order of conductor c. Then*

$$[\widehat{R_c}^\times \backslash \mathbf{k}_{\mathbb{A}_f}^\times / \mathbf{k}^\times : \mathrm{N}_{B_{\mathbb{A}_f}^\times}(\mathcal{O}_c) \cap \mathbf{k}_{\mathbb{A}_f}^\times \backslash \mathbf{k}_{\mathbb{A}_f}^\times / \mathbf{k}^\times] = 2^{\delta(\Delta_0, D)}.$$

Proof For a prime $p \nmid D$,

$$\mathrm{N}_{B_p^\times}(\mathcal{O}_c) = \mathcal{O}_{c,p}^\times \mathbb{Q}_p^\times .$$

Thus

$$\mathrm{N}_{B_p^\times}(\mathcal{O}_c) \cap \mathbf{k}_p^\times = R_{c,p}^\times \mathbb{Q}_p^\times . \tag{3.7}$$

For primes $p | D$, $\mathrm{N}_{B_p^\times}(\mathcal{O}_c) = B_p^\times$. When p is inert in $\mathbf{k}$, (3.7) still holds. However, when p is ramified in $\mathbf{k}$,

$$\mathrm{N}_{B_p^\times}(\mathcal{O}_c) \cap \mathbf{k}_p^\times = R_{1,p}^\times \mathbb{Q}_p^\times \cup R_{1,p}^\times \mathbb{Q}_p^\times \pi_p$$

where $\pi_p^2 = p$.

Altogether, then, there is a surjection

$$\widehat{R_c}^\times \backslash \mathbf{k}_{\mathbb{A}_f}^\times / \mathbf{k}^\times \twoheadrightarrow \mathrm{N}_{B_{\mathbb{A}_f}^\times}(\mathcal{O}_c) \cap \mathbf{k}_{\mathbb{A}_f}^\times \backslash \mathbf{k}_{\mathbb{A}_f}^\times / \mathbf{k}^\times$$

given by factoring out the subgroup generated by the elements $(1, \dots, 1, \pi_p, 1, \dots)$ for p ramified in both B and $\mathbf{k}$. The size of this subgroup is $2^{\delta(\Delta_0, D)}$. ∎

Corollary 3.4 *Let $h(c^2\Delta)$ be the ideal class number of the order of conductor c in the quadratic field of discriminant Δ and $-4t = n^2\Delta$ as before. Then*

$$|\Gamma^* \backslash L(t)| = 2^{-\delta(\Delta_0, D)} \sum_{c|n} h(c^2\Delta),$$

where $h(c^2\Delta)$ is the class number of R_c, the order of conductor c in $\mathbf{k}$.

Proof This follows from recognizing $\widehat{R_c}^\times \backslash \mathbf{k}_{\mathbb{A}_f}^\times / \mathbf{k}^\times$ as the desired ideal class group and noting that $L(t) = \coprod_{c|n} L(t,c)$. ∎

4 Borcherds Forms

4.1 Rational Quadratic Divisors

Let $\mathfrak{D}$ be the space of oriented negative 2-planes in V. Call $[z_1, z_2] \in \mathfrak{D}$ a proper basis if $(z_1, z_1) = (z_2, z_2) = -1$ and $(z_1, z_2) = 0$. In addition, define

$$\mathcal{Q} = \{v \in V(\mathbb{C}) \mid (v, v) = 0, (v, \bar{v}) < 0\}/\mathbb{C}^{\times}.$$

This is an open subset of a quadric in $\mathbb{P}\big(V(\mathbb{C})\big)$. Recall that $B = (\frac{q,D}{\mathbb{Q}})$ with $\alpha^2 = q$ and $\beta^2 = D$ and let V have the canonical basis $\{\alpha, \beta, \alpha\beta\}$. Then there is a pair of bijections

$$\mathfrak{h}^{\pm} \xrightarrow{w} \mathcal{Q} \xleftarrow{\sigma} \mathfrak{D}(\mathbb{R})$$

where the maps are given by

$$w(z) = \Big(\frac{q - z^2}{2q}\Big)\alpha + \Big(\frac{z}{\sqrt{D}}\Big)\beta + \Big(\frac{q + z^2}{2q\sqrt{D}}\Big)\alpha\beta, \tag{4.1}$$

$$\sigma([z_1, z_2]) = z_1 - iz_2. \tag{4.2}$$

Write $\mathfrak{D} = \mathfrak{D}^+ \cup \mathfrak{D}^-$ where $\mathfrak{D}^+$ (resp., $\mathfrak{D}^-$) are the planes with positive (negative) orientation. For $x \in V(\mathbb{Q})$ define

$$\mathfrak{D}_x = \big\{z \in \mathfrak{h}^{\pm} \mid \big(x, w(z)\big) = 0\big\}.$$

By (4.1), for $x = x_1\alpha + x_2\beta + x_3\alpha\beta$,

$$\big(x, w(z)\big) = \Big(\frac{x_1 + x_3\sqrt{D}}{2}\Big)z^2 - (x_2\sqrt{D})z - \frac{q(x_1 - x_3\sqrt{D})}{2}. \tag{4.3}$$

Hence

$$\mathfrak{D}_x = \Big\{\frac{x_2\sqrt{D} \pm \sqrt{-Q(x)}}{x_1 + x_3\sqrt{D}}\Big\}.$$

Let $\mathfrak{D}_x^{\pm} = \mathfrak{D}_x \cap \mathfrak{D}^{\pm}$.

Proposition 4.1 *For $x \in V$ with $Q(x) > 0$, $\mathfrak{D}_x$ is the set of fixed points of the image of x in $\mathrm{PGL}_2(\mathbb{R})$ under the embedding ϕ_D.*

Proof Let $x = x_1\alpha + x_2\beta + x_3\alpha\beta$. Then

$$\phi_D(x) = \begin{pmatrix} x_2\sqrt{D} & q(x_1 - x_3\sqrt{D}) \\ x_1 + x_3\sqrt{D} & -x_2\sqrt{D} \end{pmatrix}.$$

A fixed point, z, of this matrix satisfies

$$zx_2\sqrt{D} + q(x_1 - x_3\sqrt{D}) = z^2(x_1 + x_3\sqrt{D}) - zx_2\sqrt{D}.$$

This is equivalent to (4.3). ■

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Definition 4.1 Let $G = \Gamma$ or Γ^* and let $G\eta$ denote the G-orbit of $\eta \in L^\vee/L$. The rational quadratic divisor $Z(d, \eta; G)$ is given by

$$Z(d, \eta; G) = \sum_{\substack{x \in L^\vee \cap V(d) \\ x+L \in G\eta \\ \bmod G}} \mathrm{pr}_G(\mathfrak{D}_x^+),$$

where $\mathrm{pr}_G \colon \mathfrak{D}^+ \to G\backslash\mathfrak{D}^+$ and each point is counted with weight $|\operatorname{Stab}(x)|^{-1}$.

For more details on this definition in the case of $G = \Gamma$, see [11, Appendix].

4.2 Borcherds Forms

Let $H = \mathrm{GSpin}(V)$. Viewed as an algebraic group, $H(\mathcal{A}) \simeq (B \otimes_{\mathbb{Q}} \mathcal{A})^\times$ for any $\mathbb{Q}$-algebra $\mathcal{A}$. Let $K \subset H(\mathbb{A}_f)$ be a compact open set such that $H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+K$, where $H(\mathbb{R})^+$ is the component of $H(\mathbb{R})$ that contains the identity.

Definition 4.2 A modular form of weight $k \in \mathbb{Z}$ on $\mathfrak{D} \times H(\mathbb{A}_f)/K$ is a function $\Psi \colon \mathfrak{D} \times H(\mathbb{A}_f)/K \to \mathbb{C}$ such that

$$\Psi(\gamma z, \gamma h) = j(\gamma, z)^k \Psi(z, h)$$

for all $\gamma \in H(\mathbb{Q})$, where $j(\gamma, z)$ is the automorphy factor given in [10].

The cases we will focus on have $k = 0$ and thus the automorphy factor will be inconsequential.

Let L be a lattice and F be a modular form valued in $\mathbb{C}[L^\vee/L]$ with Fourier expansion given by

$$F(\tau) = \sum_{\eta \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\eta(m) \mathbf{q}^m e_\eta \tag{4.4}$$

where $\{e_\eta\}_{\eta \in L^\vee/L}$ form the basis of $\mathbb{C}[L^\vee/L]$. Since Γ and Γ^* act on $L^\vee/L$, they also act via linearity on the algebra $\mathbb{C}[L^\vee/L]$ and the function F.

Definition 4.3 For a lattice L with signature $(n, 2)$, a Borcherds form $\Psi(F)$ is a meromorphic modular form on $\mathfrak{D} \times H(\mathbb{A}_f)/K$ arising from the regularized theta lift of a weight $1 - \frac{n}{2}$ meromorphic modular form F as in (4.4) with $c_\eta(m) \in \mathbb{Z}$ for $m \leq 0$. See [15], [10], [4].

Borcherds forms have the following key properties.

Theorem 4.2 ([10, Theorem 1.3]) *Assume F is given as in* (4.4) *and is Γ^* invariant.*

(i) *The weight of $\Psi(F)$ is $c_0(0)$.*

(ii) $\operatorname{div}\big(\Psi(F)^2\big) = \sum_{\eta \in L^\vee/L} \sum_{m>0} c_\eta(-m) Z(m, \eta; \Gamma^*)$.

4.3 Adelic View

We can rephrase some of the definitions from Section 2.3 from an adelic point of view. This will allow the machinary of Borcherds forms to apply to the computation of singular moduli on $\mathcal{X}_D$ and $\mathcal{X}_D^*$.

Let K_Γ be the compact open set $\widehat{\mathcal{O}}^\times \subset H(\mathbb{A}_f)$. Then $\Gamma = H(\mathbb{Q}) \cap H(\mathbb{R})^+ K_\Gamma$. Let K_{Γ^*} be defined analagously. Then $\mathcal{X}_D$ and $\mathcal{X}_D^*$ are given by

$$\mathcal{X}_D \simeq \Gamma \backslash \mathfrak{D} \simeq H(\mathbb{Q}) \backslash \big(\mathfrak{D} \times H(\mathbb{A}_f)/K_\Gamma \big),$$
$$\mathcal{X}_D^* \simeq \Gamma^* \backslash \mathfrak{D} \simeq H(\mathbb{Q}) \backslash \big(\mathfrak{D} \times H(\mathbb{A}_f)/K_{\Gamma^*} \big).$$

Notice that $\mathcal{X}_D$ and $\mathcal{X}_D^*$ are natural domains for weight-0 Borcherds forms.

The CM points can be viewed adelically as well. An element $x \in V(\mathbb{Q})$ with positive norm gives rise to the decomposition of V as $V = \mathbb{Q}x \oplus U$ where $U = x^\perp$ is a negative plane. This splitting corresponds to a two-point set $\mathfrak{D}_x$. As a rational inner product space $U \simeq \mathbf{k}$ for some quadratic imaginary field $\mathbf{k}$ with quadratic form given by a constant times the norm on $\mathbf{k}$. Set $T \simeq \mathrm{GSpin}(U)$. Then, with ι_x as in Section 3.2, $T(\mathbb{Q}) \simeq \iota_x(\mathbf{k}^\times) \subset H(\mathbb{Q})$ and the CM points are the image of

$$Z_{\Gamma^*}(U) = T(\mathbb{Q}) \backslash \big(\mathfrak{D}_x \times T(\mathbb{A}_f)/K_{\Gamma^*} \big) \hookrightarrow \mathcal{X}_D^*. \tag{4.5}$$

The degree of this 0-cycle is given in [13, Chapter 3] as

$$|Z_{\Gamma^*}(U)| = 2 \cdot \sum_{c|n} \frac{h(c^2\Delta)}{w(c^2\Delta)} \cdot \prod_{p|D} \big(1 - \chi_\Delta(p)\big)$$

where $w(c^2\Delta)$ is the number of units in R_c and χ_Δ is the associated Dirichlet character for $\mathbf{k}$ given by the Kronecker symbol, $\chi_\Delta(n) = (\frac{\Delta}{n})$.

4.4 Borcherds Forms at CM Points

Recall that $L = \mathcal{O} \cap V$ is a lattice in V corresponding to a fixed maximal order $\mathcal{O}$. Then there are sublattices

$$L_+ = \mathbb{Q}x \cap L, \quad L_- = U \cap L.$$

In general, $L \neq L_- + L_+$, and

$$L_- + L_+ \subseteq L \subseteq L^\vee \subseteq L_-^\vee + L_+^\vee.$$

Hence an element $\eta \in L^\vee$ decomposes as $\eta = \eta_- + \eta_+$ for $\eta_\pm \in L_\pm^\vee$.

Definition 4.4 ([15]) For $\mu \in L_-^\vee/L_-$ and $\psi_\mu = \mathrm{char}(\mu + L_-)$, let $E(\tau, s; \psi_\mu, +1)$ be the incoherent Eisenstein series of weight 1 with Fourier expansion

$$E(\tau, s; \psi_\mu, +1) = \sum_m A_\mu(s, m, v)\mathbf{q}^m$$

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where the Fourier coefficients have Laurent expansions

$$A_\mu(s,m,v) = b_\mu(m,v)s + O(s^2)$$

at $s = 0$. Then for $\eta \in L^\vee/L$ and $m \in \mathbb{Q}$ define

$$\kappa_\eta(m) = \sum_{\lambda \in L/(L_+ + L_-)} \sum_{x \in \eta_+ + \lambda_+ + L_+} \kappa^-_{\eta_- + \lambda_-}\big(m - Q(x)\big) \tag{4.6}$$

where

$$\kappa^-_\mu(m') = \begin{cases} \lim_{v\to\infty} b_\mu(m',v) & \text{if } m' > 0 \\ k_0(0)\psi_\mu(0) & \text{if } m' = 0, \\ 0 & \text{if } m' < 0, \end{cases} \tag{4.7}$$

$$k_0(0) = \log(|\Delta|) + 2\frac{\Lambda'(1+\chi_\Delta)}{\Lambda(1,\chi_\Delta)}, \tag{4.8}$$

and $\Lambda(s,\chi_\Delta)$ is the normalized L-series $\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})L(s,\chi_\Delta)$.

Theorem 4.3 ([15, Corollary 3.4]) *Assume $c_\eta(m) \in \mathbb{Z}$ for $m \le 0$, $c_0(0) = 0$, and that the* 0*-cycle $Z_{\Gamma^*}(U)$ defined in* (4.5) *does not meet the divisor of $\Psi(F)$. Then*

$$\frac{1}{|Z_{\Gamma^*}(U)|} \sum_{z \in Z_{\Gamma^*}(U)} \log \|\Psi(z,f)\|^2 = \frac{-1}{2^{d(B)}} \sum_\eta \sum_{m \ge 0} c_\eta(m)\kappa_\eta(m) \tag{4.9}$$

where $h(\mathbf{k})$ is the ideal class number of the quadratic field $\mathbf{k} \simeq U$.

The power of this theorem lies in the explicit formulas for the right-hand side of (4.9). In Section 7 we will use this theorem to compute the norms of singular moduli. However, first a supply of appropriate vector-valued modular forms F is needed to serve as the input to the Borcherds construction of $\Psi(F)$.

5 Input Forms

This section is presented in general terms and follows [4] and [5]. However, rather than appearing redundant, the notation implies how the general theory applies to the set-up in Sections 2 through 4.

5.1 $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ and the Weil Representation

The Lie group $\mathrm{SL}_2(\mathbb{R})$ has a double cover $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ with elements of the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d}\right).$$

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The group structure is given by

$$\big(G_1, j_1(\cdot)\big)\big(G_2, j_2(\cdot)\big) = \Big(G_1G_2, j_1\big(G_2(\cdot)\big)\, j_2(\cdot)\Big).$$

The group $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ is defined as the inverse image in $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{Z})$ and is generated by the two elements

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right), \quad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right),$$

which satisfy

$$Z = S^2 = (ST)^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i\right).$$

The element Z generates the center of $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ and the quotient by Z^2 is $\mathrm{SL}_2(\mathbb{Z})$. Also, $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ acts on $\mathfrak{h}^{\pm}$ via its image in $\mathrm{SL}_2(\mathbb{Z})$. Throughout the following, let

$$\gamma = \gamma^{\pm} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d}\right) \in \widetilde{\mathrm{SL}_2}(\mathbb{Z}). \tag{5.1}$$

Let L be a lattice with quadratic form Q' and let $L^\vee$ be the dual lattice under the associated inner product. To ease notation, let $\Lambda_L = L^\vee/L$. Then Milgram's formula gives $\mathrm{sign}(L)$, the signature mod 8 of L, via

$$\sum_{\eta\in\Lambda_L} \mathbf{e}\big(Q'(\eta)\big) = \sqrt{|\Lambda_L|}\,\mathbf{e}\big(\mathrm{sign}(L)/8\big)$$

where $\mathbf{e}(x) = e^{2\pi i x}$. For $\eta \in \Lambda_L$, let e_η denote the corresponding basis element in the group ring $\mathbb{C}[\Lambda_L]$. In [4], Borcherds defines the Weil representation $\overline{\rho}_{\Lambda_L}$ on the generators of $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ in terms of Q'. However, we will use the dual representation $\rho_{\Lambda_L} = \overline{\rho}^{\vee}_{\Lambda_L}$ since the quadratic form in Sections 2 through 4 is actually given by $Q(x) = -Q'(x)$. On the generators ρ_{Λ_L} is given by

$$\rho_{\Lambda_L}(T)e_\eta = \mathbf{e}\big(-Q(\eta)\big)\, e_\eta,$$
$$\rho_{\Lambda_L}(S)e_\eta = C_L \sum_{\delta\in\Lambda_L} \mathbf{e}\big(-(\eta,\delta)\big)\, e_\delta$$

where

$$C_L = \frac{\mathbf{e}\big(\mathrm{sign}(L)/8\big)}{\sqrt{|\Lambda_L|}} = \frac{1}{|\Lambda_L|}\sum_{\eta\in\Lambda_L} \mathbf{e}\big(Q(\eta)\big).$$

(This approach follows [15] and [10]. However most of the results in this section are the dualized versions of those found in [5].) Define the *level* of Λ_L to be the smallest integer N such that $NQ(\eta) \in \mathbb{Z}$ for all $\eta \in L^\vee$. Then the representation ρ_{Λ_L} factors through $\widetilde{\mathrm{SL}_2}(\mathbb{Z}/N\mathbb{Z})$, the double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Define the *congruence subgroup* $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ as the preimage of the upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\widetilde{\Gamma_0}(N)$ as its inverse image in $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$.

Definition 5.1 ([5]) For $\gamma \in \widetilde{\Gamma_0}(N)$ define

$$\chi_n(\gamma) = \left(\frac{d}{n}\right), \tag{5.2}$$

$$\chi_\theta(\gamma^\pm) = \begin{cases} \pm\left(\frac{c}{d}\right) & d \equiv 1 \bmod 4, \\ \mp i\left(\frac{c}{d}\right) & d \equiv 3 \bmod 4, \end{cases}, \tag{5.3}$$

$$\chi_L(\gamma) = \begin{cases} (\chi_\theta^{-\operatorname{sign}(L)+(\frac{-1}{|\Lambda_L|})-1}\chi_{|\Lambda_L|2^{\operatorname{sign}(L)}})(\gamma) & 4|N, \\ \chi_{|\Lambda_L|}(\gamma) & 4\nmid N. \end{cases} \tag{5.4}$$

Theorem 5.1 ([5, Theorem 5.4]) *Suppose Λ_L has level N. If b and c are divisible by N then $\gamma \in \widetilde{\mathrm{SL}_2}(\mathbb{Z})$ acts on $\mathbb{C}[\Lambda_L]$ by*

$$\rho_{\Lambda_L}(\gamma)e_\eta = \chi_L(\gamma)e_{a\eta}.$$

Corollary 5.2 *Suppose Λ_L has level N and that $\eta \in \Lambda_L$ has norm 0. Then $\gamma \in \widetilde{\Gamma_0}(N)$ acts on the element e_η by*

$$\rho_{\Lambda_L}(\gamma)e_\eta = \chi_L(\gamma)e_{a\eta}.$$

Proof Any element $\gamma \in \widetilde{\Gamma_0}(N)$ can be written as

$$\gamma = T^n\left(\begin{pmatrix} a' & b' \\ c & d\end{pmatrix}, \pm\sqrt{c\tau + d}\right)$$

where N divides c and b'. Then χ_L is trivial on T. Since $a' \equiv a \bmod N$ and the order of η divides N, $a'\eta = a\eta$. ■

5.2 Vector-Valued Modular Forms

Define the slash operator of weight k for an element $\gamma \in \widetilde{\mathrm{SL}_2}(\mathbb{Z})$ by

$$f|^k_{\gamma^\pm}(\tau) = (\pm\sqrt{c\tau + d})^{2k} f(\gamma\tau).$$

Definition 5.2 Suppose ρ is a representation of $\Gamma \subset \widetilde{\mathrm{SL}_2}(\mathbb{Z})$ on a finite dimensional complex vector space $\mathcal{V}$. Then $F\colon \mathfrak{h}^\pm \to \mathcal{V}$ is a vector-valued modular form on Γ of weight $k \in \frac{1}{2}\mathbb{Z}$ and type ρ if it is meromorphic and satisfies

$$F(\gamma^\pm\tau) = (\pm\sqrt{c\tau + d})^{2k}\rho(\gamma^\pm)F(\tau)$$

for all $\gamma \in \Gamma$.

Definition 5.3 Suppose f is a scalar-valued weight k modular form on $\widetilde{\Gamma_0}(N)$ with character χ_L. Then define a weight k modular form $F_f(\tau)$ valued in $\mathbb{C}[L^\vee/L]$ via

$$F_f(\tau) = \sum_{\gamma \in \widetilde{\Gamma_0}(N)\backslash\widetilde{\mathrm{SL}_2}(\mathbb{Z})} f|^k_\gamma(\tau)\rho_{\Lambda_L}(\gamma^{-1})e_0. \tag{5.5}$$

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It can be shown [3] that $F_f(\tau)$ is well-defined and is a modular form of type ρ_{Λ_L} and weight k on $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$.

Proposition 5.3 *Let F_f have Fourier expansion as in (4.4). If $m + Q(\eta) \notin \mathbb{Z}$, then $c_\eta(m) = 0$.*

Proof Since F_f is a modular form,

$$F_f(\tau+1) = \rho_{\Lambda_L}(T)F_f(\tau)$$

$$\sum_{\eta\in\Lambda_L}\sum_{m\in\mathbb{Q}} c_\eta(m)\mathbf{q}^m\mathbf{e}(m)e_\eta = \sum_{\eta\in\Lambda_L}\sum_{m\in\mathbb{Q}} c_\eta(m)\mathbf{q}^m\rho_{\Lambda_L}(T)e_\eta$$

$$= \sum_{\eta\in\Lambda_L}\sum_{m\in\mathbb{Q}} c_\eta(m)\mathbf{q}^m\mathbf{e}\bigl(-Q(\eta)\bigr)\,e_\eta.$$

Thus $m + Q(\eta) \notin \mathbb{Z}$ implies $c_\eta(m) = 0$. ■

Proposition 5.4 *If f has no poles at finite cusps, then, for F_f as in (4.4), $c_\eta(m) = 0$ for $m < 0$ and $\eta \neq 0$.*

Proof If f does not have a pole at a finite cusp, then the coordinate function $f|^k_\gamma$ in (5.5) can have a pole only when $\gamma(\infty) = \infty$. However, this is satisfied only by the trivial coset representative which has $\rho_{\Lambda_L}(\gamma^{-1})e_0 = e_0$. ■

Now define, as in [5], $\Lambda_{L,n}$ to be the set of n-torsion points and define Λ_L^n via the exact sequence

$$0 \to \Lambda_{L,n} \to \Lambda_n \to \Lambda_L^n \to 0,$$

and

$$\Lambda_L^{n*} = \{\delta \in \Lambda_L^n \mid (\delta,\eta) = -nQ(\eta)\ \forall \eta \in \Lambda_{L,n}\}.$$

Lemma 5.5 *For a fixed n, either $\Lambda_L^{n*} = \varnothing$ or the membership of δ into Λ_L^{n*} is completely determined by $Q(\delta)$.*

Proof It suffices to examine the criteria locally at the primes that divide the level N. Recall from Section 3.1 that for an odd prime p, $\Lambda_{L,p} \simeq \mathbb{F}_{p^2}$ and $Q\colon \Lambda_{L,p} \to (1/p)\mathbb{Z}/\mathbb{Z}$. If $p|n$, then $(\Lambda_{L,p})_n = \Lambda_{L,p}$ and $(\Lambda_{L,p})^n = \{0\}$. Since $nQ(\delta) = 0 = (0,\delta)$ for all $\delta \in (\Lambda_{L,p})_n$, then $(\Lambda_{L,p})^{n*} = \{0\}$. If $p\nmid n$, then $(\Lambda_{L,p})_n = \{0\}$ and $(\Lambda_{L,p})^n = \Lambda_{L,p}$. Since $nQ(0) = 0 = (\delta, 0)$ for all $\delta \in (\Lambda_{L,p})^n = \Lambda_{L,p}$, then $(\Lambda_{L,p})^{n*} = \Lambda_{L,p}$. So for odd $p|N$,

$$(\Lambda_{L,p})^{n*} = \begin{cases} \{\delta \mid Q(\delta) \in (1/p)\mathbb{Z}_p/\mathbb{Z}_p\} & p\nmid n, \\ \{0\} & p|n. \end{cases} \tag{5.6}$$

Now consider $p = 2$ where $\Lambda_{L,2} \simeq \mathbb{F}_4 \oplus \mathbb{F}_2$ and $Q\colon \Lambda_{L,2} \to (1/4)\mathbb{Z}/\mathbb{Z}$. Suppose $2\nmid n$. Then $(\Lambda_{L,2})_n = \{0\}$, and $(\Lambda_{L,2})^n = \Lambda_{L,2}$. Since $nQ(0) = 0 = (\delta, 0)$ for all $\delta \in (\Lambda_{L,2})^n = \Lambda_{L,2}$, then $(\Lambda_{L,2})^{n*} = \Lambda_{L,2}$. Now suppose $2|n$. Then $(\Lambda_{L,2})_n = \Lambda_{L,2}$, and $(\Lambda_{L,2})^n = \{0\}$. However, $nQ(\delta) = 0 = (0, \delta)$ for all $\delta \in (\Lambda_{L,p})$ only when $4|n$. Thus

$$(\Lambda_{L,2})^{n*} = \begin{cases} \{\delta \mid Q(\delta) \in (1/4)\mathbb{Z}_2/\mathbb{Z}_2\} & 2\nmid n, \\ \varnothing & 2|n, \\ \{0\} & 4|n. \end{cases} \tag{5.7}$$

Combining (5.6) and (5.7) into one global statement yields

$$\Lambda_L^{n*} = \begin{cases} \{\delta \mid Q(\delta) \in (\frac{\gcd(n,N)}{N})\mathbb{Z}/\mathbb{Z}\} & 2\nmid n, \\ \varnothing & 2|n. \end{cases}$$

Thus the membership of an element is determined by its image under Q. ■

Lemma 5.6 ([5, Lemma 3.1]) *The sum*

$$\mathcal{S}_n(\delta) = \sum_{\eta \in \Lambda_L} \mathbf{e}\big(-(\eta, \delta) - nQ(\eta)\big)$$

is equal to 0 *when* $\delta \notin \Lambda_L^{n*}$ *and has magnitude* $\sqrt{|\Lambda_L|\,|\Lambda_{L,n}|}$ *otherwise.*

Lemma 5.7 ([5, Lemma 3.2]) *For* $\gamma \in \widetilde{\mathrm{SL}_2}(\mathbb{Z})$ *as in* (5.1), $\rho_{\Lambda_L}(\gamma)e_0$ *is a linear combination of the elements* e_δ *for* $\delta \in \Lambda_L^{c*}$.

Proof Since the coset representatives of $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ can all be chosen of the form $S^{-1}T^{-n}S^{-1}T^{-m}$, it is sufficient to prove this for γ of the form T^mST^nS for some $m, n \in \mathbb{Z}$ with $(N, n) = (N, c)$ since any γ is a product of an element of this form with an element of $\widetilde{\Gamma_0}(N)$ on the right, but e_0 is an eigenvector for $\widetilde{\Gamma_0}(N)$. Then

$$\begin{aligned} \rho_{\Lambda_L}(S)e_0 &= C_L \sum_{\delta \in \Lambda_L} e_\delta, \\ \rho_{\Lambda_L}(T^nS)e_0 &= C_L \sum_{\delta \in \Lambda_L} \mathbf{e}\big(-nQ(\delta)\big)\, e_\delta, \\ \rho_{\Lambda_L}(ST^nS)e_0 &= C_L^2 \sum_{\delta \in \Lambda_L} \sum_{\delta' \in \Lambda_L} \mathbf{e}\big(-nQ(\delta) - (\delta', \delta)\big)\, e_{\delta'} \\ &= C_L^2 \sum_{\delta \in \Lambda_L^{n*}} \mathcal{S}_n(\delta) e_\delta, \end{aligned}$$

$$\rho_{\Lambda_L}(T^mST^nS)e_0 = C_L^2 \sum_{\delta \in \Lambda_L^{n*}} \mathcal{S}_n(\delta)\mathbf{e}\big(-mQ(\delta)\big)\, e_\delta. \tag{5.8}$$

■

Theorem 5.8 *If* $Q(\delta) = Q(\delta')$, *then the* e_δ *and* $e_{\delta'}$ *components of* F_f *are equal.*

Proof This follows from the fact that the coefficient $\mathcal{S}_n(\delta)\mathbf{e}\big(-mQ(\delta)\big)$ in (5.8) depends only on $Q(\delta)$ which, by Proposition 5.5, is the same for all $\delta \in \Lambda_L^{n*}$. ■

Corollary 5.9 *The modular form* F_f *is* Γ^* *invariant.*

Proof This follows from the theorem and Proposition 3.2. ■

5.3 Dedekind-η Products

In this section we review a construction that produces scalar-valued modular forms over $\widetilde{\Gamma_0(N)}$. The Dedekind-η function is given by

$$\eta(\tau) = \mathbf{q}^{1/24} \prod_{k=1}^{\infty} (1 - \mathbf{q}^k)$$

and is a weight $\frac{1}{2}$ modular form on $\widetilde{\mathrm{SL}_2}(\mathbb{Z})$. It satisfies

$$\eta(\tau + 1) = \mathbf{e}(1/24)\eta(\tau), \quad \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau).$$

Let $\eta_m(\tau) = \eta(m\tau)$.

Theorem 5.10 ([5, Theorem 6.2]) *Given the following*
(1) *a lattice L with the level of Λ_L equal to N,*
(2) *r_δ for $\delta | N$ such that $|\Lambda_L| / \prod_{\delta|N} \delta^{r_\delta}$ is a rational square,*
(3) *$(1/24) \sum_{\delta|N} r_\delta \delta \in \mathbb{Z}$, and*
(4) *$(N/24) \sum_{\delta|N} r_\delta / \delta \in \mathbb{Z}$,*

then $\prod_{\delta|N} \eta_\delta^{r_\delta}$ is a modular form for $\widetilde{\Gamma_0(N)}$ of weight $k = \sum_\delta r_\delta / 2$ and of character $\chi_{|\Lambda_L|}$ if $4 \nmid N$ and $\chi_\theta^{2k + (\frac{-1}{|\Lambda_L|}) - 1} \chi_{2^{2k}|\Lambda_L|}$ if $4 | N$.

6 Calculating the $\kappa_\eta(m)$

The application of Theorem 4.3 requires computing $\kappa^-_{\eta_- + \lambda_-}(m)$. This section will cover the general techniques to complete this task using the notation and results of [12].

Recall from Definition 4.4 that for $\mu \in L_-^\vee / L_-$ and $\psi_\mu = \mathrm{char}(\mu + L_-)$,

$$E(\tau, s; \psi_\mu, +1) = \sum_{m \in \mathbb{Q}} E_m(\tau, s, \mu) = \sum_{m \in \mathbb{Q}} A_\mu(s, m, v) \mathbf{q}^m$$

where $\tau = u + iv$. The Fourier coefficients have Laurant expansions

$$A_\mu(s, m, v) = b_\mu(m, v)s + O(s^2)$$

at $s = 0$. Thus

$$\begin{aligned} b_\mu(m, v) &= \frac{\partial}{\partial s} \{A_\mu(s, m, v)\}_{s=0} \\ &= \mathbf{q}^{-m} \frac{\partial}{\partial s} \{E_m(\tau, s, \mu)\}_{s=0}. \end{aligned}$$

Let Δ denote the discriminant of $\mathbf{k} \simeq U$ and $h(\mathbf{k})$ its ideal class number. Following [12], there is a normalization E_m^* which satisfies

$$h(\mathbf{k}) \frac{\partial}{\partial s} \{E_m(\tau, s, \mu)\}_{s=0} = \frac{\partial}{\partial s} \{E_m^*(\tau, s, \mu)\}_{s=0}$$

and a factorization of $E^*_m(\tau, s, \mu)$ into Whittaker polynomials,

$$E^*_m(\tau, s, \mu) = v^{-\frac{1}{2}}|\Delta|^{\frac{s+1}{2}} W^*_{m,\infty}(\tau, s, \mu) \prod_p W^*_{m,p}(s, \mu). \tag{6.1}$$

Lemma 6.1 ([12, Lemmas 2.4 and 2.5]) *Suppose $U \simeq \mathbf{k}$ with discriminant Δ. If L is unimodular (self-dual) and $m \in Q(\mu) + \mathbb{Z}_p$, then*

$$W^*_{m,p}(s, \mu) = \sum_{r=0}^{\mathrm{ord}_p(m)} \Big(\frac{\Delta}{p}\Big)^r X^r.$$

Thus

$$W^*_{m,p}(0, \mu) = \rho_p(m) = \sum_{r=0}^{\mathrm{ord}_p(m)} \Big(\frac{\Delta}{p}\Big)^r.$$

If $\rho_p(m) = 0$, then

$$W^{*,\prime}_{m,p}(0, \mu) = \frac{1}{2}\log(p)\big(\mathrm{ord}_p(m) + 1\big)\,\rho_p(m/p).$$

Lemma 6.2 ([12, Proposition 2.6]) *The following values are obtained at $s = 0$.*
(1) $E^*_m(\tau, 0, \mu) = 0$.
(2) $W^*_{m,\infty}(\tau, 0, \mu) = -\gamma_\infty 2v^{\frac{1}{2}}\mathbf{q}^m$,
where γ_∞ is a local factor that will not affect later global calculations since $\prod_{p\le\infty}\gamma_p = 1$.

Note that $\rho_p(m) = \rho_p(p^{\mathrm{ord}_p(m)})$, and $\rho_p(1) = 1$. Hence $W^*_{m,p}(0, \mu) \neq 1$ for only a finite number of primes.

Theorem 6.3 *There exists a finite prime p' such that $W^*_{m,p'}(0, \mu) = 0$ and hence*

$$b_\mu(m, v) = \frac{-2\sqrt{|\Delta|}\gamma_\infty}{h(\mathbf{k})} \frac{\partial}{\partial s}\{W^*_{m,p'}(s, \mu)\}_{s=0} \prod_{p\neq p'} W^*_{m,p}(0, \mu). \tag{6.2}$$

Note that in this case $b_\mu(m, v)$ does not depend on v and thus (6.2) is equal to the limit in (4.7). Explicit formulas for $W_{m,p}(s, \mu) = \frac{W^*_{m,p}(s,\mu)}{L_p(s+1,\chi_\Delta)}$ are given in [14].

7 Examples

7.1 $D = 6$

First consider the quaternion algebra ramified at the primes 2 and 3. Let $B = (\frac{5,6}{\mathbb{Q}})$. By Proposition 2.2, B has a maximal order given by

$$\mathcal{O} = \mathbb{Z} + \Big(\frac{1+\alpha}{2}\Big)\mathbb{Z} + \Big(\frac{\alpha + \alpha\beta}{5}\Big)\mathbb{Z} + \Big(\frac{5 + \alpha + 5\beta + \alpha\beta}{10}\Big)\mathbb{Z}. \tag{7.1}$$

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Further, the image of Γ^* in $\mathrm{PGL}_2(\mathbb{R})$ is generated by three elements,

$$s_2 = -\frac{6}{5}\alpha + \beta + \frac{4}{5}\alpha\beta,$$
$$s_4 = 1 - \frac{1}{5}\alpha + \frac{1}{2}\beta + \frac{3}{10}\alpha\beta,$$
$$s_6 = \frac{3}{2} - \frac{3}{10}\alpha + \frac{1}{5}\alpha\beta,$$

which satisfy the group presentation

$$\langle s_2, s_4, s_6 | s_2^2 = s_4^4 = s_6^6 = s_2 s_4 s_6 = 1\rangle$$

(see [7, Section 3.1]). As mentioned previously, $\mathcal{X}_6^*$ has genus 0 and so there exists a parameterization $t_6\colon \mathcal{X}_6^* \xrightarrow{\sim} \mathbb{P}^1$ over $\mathbb{Q}$. Such a map giving the isomorphism is only well-defined up to a PGL_2 action on $\mathbb{P}^1$. However, the map is uniquely determined once the value at three points of $\mathcal{X}_6^*$ are chosen. Since there are three distinguished elements of Γ^*, namely s_2, s_4, s_6, it is only natural to fix the value of the isomorphism at their three fixed points, P_2, P_4, P_6. Thus, define the map $t_6\colon \mathcal{X}_6^* \xrightarrow{\sim} \mathbb{P}^1$ such that it takes on the values $0, 1, \infty$ at the points P_4, P_2, P_6, respectively. (Warning: In [7], the author chooses t_6 to have the values $0, 1, \infty$ at the points P_2, P_4, P_6.) This defining criteria can be expressed as

$$\begin{aligned} \operatorname{div}(t_6) &= P_4 - P_6, \\ t_6(P_2) &= 1. \end{aligned} \tag{7.2}$$

Let s_i^0 denote the trace-0 part of s_i. Since the action of $B^\times$ factors through $\mathrm{PGL}_2(\mathbb{R})$, the fixed point of s_i is the fixed point of all of $\mathbf{k}_i^\times \subset B^\times$ where $\mathbf{k}_i = \mathbb{Q}(s_i) = \mathbb{Q}(s_i^0)$. For the s_i as above,

$$\mathbf{k}_2 \simeq \mathbb{Q}(\sqrt{-6}), \quad \mathbf{k}_4 \simeq \mathbb{Q}(\sqrt{-1}), \quad \mathbf{k}_6 \simeq \mathbb{Q}(\sqrt{-3}). \tag{7.3}$$

Lemma 7.1 *The following equalities hold.*
(1) $Z(1,0;\Gamma^*) = \frac{1}{4}P_4$.
(2) $Z(3,0;\Gamma^*) = \frac{1}{6}P_6$.

Proof These identities follow from

$$|\Gamma^*\backslash L(1)| = |\Gamma^*\backslash L(3)| = 1$$

by Corollary 3.4 and that $|\operatorname{Stab}_{\Gamma^*}(s_4^0)| = 4$ and $|\operatorname{Stab}_{\Gamma^*}(s_6^0)| = 6$. ∎

Proposition 7.2

$$\operatorname{div}(t_6) = 4Z(1,0;\Gamma^*) - 6Z(3,0;\Gamma^*).$$

Hence, to use Theorem 4.2, the input vector-valued form must have, for $m < 0$,

$$c_0(m) = \begin{cases} 2 & m = -1, \\ -3 & m = -3, \\ 0 & \text{otherwise.} \end{cases}$$

7.1.1 The Input Form

By Corollary 3.1, $|L^\vee/L| = 72$ and $N = 12$. To vectorize properly, we need a form of weight $\frac{1}{2}$ and character $\chi_\theta\chi_{144}$.

Proposition 7.3 *Let $A_1, A_2, A_3, A_4, A_5 \in \mathbb{Z}$, and set*

$$r_1 = A_5, \tag{7.4}$$

$$r_2 = 16 - 12A_1 + 36A_2 - 9A_3 - 14A_4 - 6A_5, \tag{7.5}$$

$$r_3 = -30 + 24A_1 - 48A_2 + 16A_3 + 24A_4 + 5A_5, \tag{7.6}$$

$$r_4 = -17 + 12A_1 - 36A_2 + 9A_3 + 16A_4 + 5A_5, \tag{7.7}$$

$$r_6 = 43 - 36A_1 + 60A_2 - 21A_3 - 34A_4 - 6A_5, \tag{7.8}$$

$$r_{12} = -11 + 12A_1 - 12A_2 + 5A_3 + 8A_4 + A_5. \tag{7.9}$$

Then

$$\prod_{\delta|12} \eta_\delta^{r_\delta} \tag{7.10}$$

is a modular form for $\widetilde{\Gamma_0(12)}$ of weight $\frac{1}{2}$ and of character $\chi_\theta\chi_{144}$.

Proof One can check that the following hold.

$$72/\prod_{\delta|12} \delta^{r_\delta} = (2^{A_3}3^{A_4})^2,$$

$$(1/24)\sum_{\delta|12} r_\delta \delta = A_1,$$

$$(1/2)\sum_{\delta|12} r_\delta/\delta = A_2,$$

$$\sum_\delta r_\delta/2 = \frac{1}{2}.$$

Hence, by Theorem 5.10, (7.10) is a modular form for $\widetilde{\Gamma_0(12)}$ of weight $\frac{1}{2}$ and of character $\chi_\theta\chi_{144}$. ■

Now examine the structure of such a form at the various cusps of $\widetilde{\Gamma_0(12)}$. Table 1 gives the orders of the zeroes for a form defined by (7.4)–(7.10), where a negative value represents a pole. To construct a form defined by (7.4)–(7.10) such that it has neither a pole nor a zero at ∞ and no pole at any finite cusp, one simply solves the

Table 1: Order of the zero of a form defined by (7.4)–(7.10) at the cusps of $\widetilde{\Gamma_0(12)}$.

Cusp	Zero Order
$1 = 0$	$A_2/12$
$1/2$	$(15 - 12A_1 + 28A_2 - 8A_3 - 12A_4 - 4A_5)/12$
$1/3$	$(-5 + 4A_1 - 9A_2 + 3A_3 + 4A_4 + A_5)/4$
$1/4$	$(-4 + 3A_1 - 8A_2 + 2A_3 + 4A_4 + A_5)/3$
$1/6$	$(25 - 20A_1 + 36A_2 - 12A_3 - 20A_4 - 4A_5)/4$
$1/12 = \infty$	A_1

following system of inequalities over $\mathbb{Z}$.

$$0 \le A_2, \tag{7.11}$$

$$0 \le 15 - 12A_1 + 28A_2 - 8A_3 - 12A_4 - 4A_5, \tag{7.12}$$

$$0 \le -5 + 4A_1 - 9A_2 + 3A_3 + 4A_4 + A_5, \tag{7.13}$$

$$0 \le -4 + 3A_1 - 8A_2 + 2A_3 + 4A_4 + A_5, \tag{7.14}$$

$$0 \le 25 - 20A_1 + 36A_2 - 12A_3 - 20A_4 - 4A_5, \tag{7.15}$$

$$0 = A_1, \tag{7.16}$$

Doing so yields a unique solution

$$(A_1, A_2, A_3, A_4, A_5) = (0, 0, 1, 1, -2)$$

which produces

$$\psi_0 = \frac{\eta_2^5}{\eta_1^2\eta_4^2} = \sum_{n\in\mathbb{Z}} \mathbf{q}^{n^2} = \theta(\tau).$$

Similarly a form defined by (7.4)–(7.10) that has a pole of order k at ∞, but no pole at any other cusp can be found by solving the inequalities (7.11)–(7.15) with $A_1 = -k$ over $\mathbb{Z}$. For a simple pole at ∞, there are five such Dedekind-η products. They are

$$\psi_1 = \frac{\eta_2^{12}\eta_3}{\eta_1^5\eta_4^4\eta_6\eta_{12}^2} = \frac{1}{\mathbf{q}} + 5 + O[\mathbf{q}], \qquad \frac{\eta_2^3\eta_4^2\eta_6^2}{\eta_1^2\eta_{12}^4} = \frac{1}{\mathbf{q}} + 2 + O[\mathbf{q}],$$

$$\frac{\eta_2^2\eta_6^9}{\eta_1\eta_3^3\eta_{12}^6} = \frac{1}{\mathbf{q}} + 1 + O[\mathbf{q}], \qquad \frac{\eta_2^5\eta_3^3}{\eta_1^3\eta_4\eta_{12}^3} = \frac{1}{\mathbf{q}} + 3 + O[\mathbf{q}],$$

$$\frac{\eta_1\eta_2^3\eta_6^2}{\eta_3\eta_4\eta_{12}^3} = \frac{1}{\mathbf{q}} - 1 + O[\mathbf{q}].$$

For a triple pole, there are 35 such forms. One of them is

$$\psi_3 = \frac{\eta_2\eta_3^2\eta_4^4\eta_6^4}{\eta_{12}^{10}} = \frac{1}{\mathbf{q}^3} - \frac{1}{\mathbf{q}} - 2 + O[\mathbf{q}].$$

Thus the linear combination

$$f_6 = -6\psi_3 - 2\psi_1 - 2\psi_0 = -\frac{6}{\mathbf{q}^3} + \frac{4}{\mathbf{q}} + O[\mathbf{q}]$$

is a vectorizable modular form over $\widetilde{\Gamma_0(12)}$ for $\widetilde{\Gamma_0(12)}$ of weight $\frac{1}{2}$ of character $\chi_\theta\chi_{144}$ with no poles at finite cusps.

Theorem 7.4 *There exists a nonzero constant c_6 such that*

$$t_6 = c_6\Psi(F_{f_6})^2.$$

Proof There is an equality of divisors

$$\operatorname{div}(t_6) = 4Z(1,0;\Gamma^*) - 6Z(3,0;\Gamma^*) = \operatorname{div}\big(\Psi(F_{f_6})^2\big). \qquad \blacksquare$$

7.1.2 $\Delta = -24$

In this section we calculate $\Psi(F_{f_6})(P_2)$. The result of the calculation gives the value of c_6^{-1} in Theorem 7.4 since by definition $t_6(P_2) = 1$. Note that by (7.3), $P_2 = \mathcal{P}_{-24}$ the CM point with discriminant -24 on the Shimura curve $\mathcal{X}_6^*$.

Set $m = 1$ so that

$$L = \mathbb{Z}\ell_1 + \mathbb{Z}\ell_2 + \mathbb{Z}\ell_3$$

where

$$\ell_1 = \alpha, \quad \ell_2 = \frac{\alpha + \alpha\beta}{5}, \quad \ell_3 = \frac{\beta + \alpha\beta}{2}.$$

Take $z = \ell_3$ so that $Q(z) = 6$. Then the negative plane is spanned by

$$u_1 = 2\ell_2 - \ell_3, \quad u_2 = 2\ell_1 - 4\ell_2 + 2\ell_3,$$

and

$$Q(Xu_1 + Yu_2) = -2(X^2 + 6Y^2).$$

A basis of L_- is given by

$$\ell_1^- = 2\ell_2 - \ell_3, \quad \ell_2^- = \ell_1.$$

The group $L/(L_- + L_+)$ has order 2 and $\lambda = \ell_2 + (L_- + L_+)$ represents its nontrivial member. This has the decomposition

$$\lambda_+ = \frac{1}{2}z + L_+, \quad \lambda_- = \frac{1}{2}\ell_1^- + L_-.$$

By Theorem 4.3,

$$\sum_{z \in Z_{\Gamma^*}(\mathbb{Q}(\sqrt{-6}))} \log\|\Psi(z, F_{f_6})\|^2 = \left(\frac{-1}{4}\right)\big(-6\kappa_0(3) + 4\kappa_0(1)\big). \tag{7.17}$$

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Considering (4.6),

$$\begin{aligned} \kappa_0(1) &= \kappa_0^-(1), \\ \kappa_0(3) &= \kappa_0^-(3) + \kappa_{\lambda_-}^-(3/2) + \kappa_{\lambda_-}^-(3/2). \end{aligned} \tag{7.18}$$

The term $\kappa_{\lambda_-}^-(3/2)$ appears twice in (7.18), due to the two values $x = \pm z/2 \in \lambda_+ + L_+ = (\frac{1}{2} + \mathbb{Z})z$ that satisfy $3 - Q(x) \geq 0$.

The calculations via [14] and Section 6 yield

$$\kappa_0(1) = -6\log(2),$$
$$\kappa_0(3) = -8\log(2) - 4\log(3).$$

Thus

$$\sum_{z \in Z_{\Gamma^*}(\mathbb{Q}(\sqrt{-6}))} \log \|\Psi(z, F_{f_6})\|^2 = -6\log(3) - 6\log(2).$$

Corollary 7.5 $\|t_6\| = 6^6 \|\Psi(F_{f_6})^2\|$.

Note that we have only determined the value of c_6 in Theorem 7.4 up to sign. This can be resolved by repeating the above computations with a Borchards form corresponding to the function $t_6 - 1$.

7.1.3 $\Delta = -163$

We are now able to compute the coordinates of the other rational CM points listed in Table 2. We illustrate the calculations with the example of $\Delta = -163$.

Take $z = \ell_1 + 14\ell_2$ so that $Q(z) = 163$. Then the negative plane is spanned by

$$u_1 = 42\ell_2 - 13\ell_3, \quad u_2 = 166\ell_1 - 284\ell_2 + 163\ell_3,$$

and

$$Q(Xu_1 + Yu_2) = -498(X^2 + 163Y^2).$$

A basis of L_- is given by

$$\ell_1^- = 42\ell_2 - 13\ell_3,$$
$$\ell_2^- = \ell_1 - 5\ell_2 + 2\ell_3,$$

The group $L/(L_- + L_+)$ is cyclic of order 163 and $\lambda = \ell_3 + (L_- + L_+)$ represents a generator. This has the decomposition

$$\lambda_+ = \frac{42}{163} z + L_+,$$
$$\lambda_- = -\frac{19}{163}\ell_1^- - \frac{42}{163}\ell_2^- + L_-.$$

Then computations of Whittaker polynomials as before yield

$$\kappa_0(1) = -4\log(2) - 11\log(3) - 4\log(7) - 4\log(19) - 4\log(23),$$
$$\kappa_0(3) = -\frac{40}{3}\log(2) - 4\log(3) - 4\log(5) - 4\log(11) - 4\log(17).$$

(Due to the sheer number of Whittaker polynomials required, the calculations were implemented in Mathematica.) Thus by Theorem 4.3 the CM point $\mathcal{P}_{-163}$ with discriminant -163 has

$$\|t_6(\mathcal{P}_{-163})\| = \frac{3^{11}7^4 19^4 23^4}{2^{10}5^6 11^6 17^6}.$$

Note that this proves the conjectural value given in [7, Table 2]. In fact, all of the conjectural values can now be algebraically proven and are given in Table 2.

7.2 $D = 10$

7.2.1 The Input Form

Now consider the quaternion algebra ramified at the primes 2 and 5, $B = (\frac{13,10}{\mathbb{Q}})$. It contains the maximal order

$$\mathcal{O} = \mathbb{Z} + \Big(\frac{1+\alpha}{2}\Big)\mathbb{Z} + \Big(\frac{6\alpha+\alpha\beta}{13}\Big)\mathbb{Z} + \Big(\frac{78+6\alpha+13\beta+\alpha\beta}{26}\Big)\mathbb{Z}.$$

Then by [7, Section 4.1], the image of $\Gamma^* \subset \mathrm{PGL}_2(\mathbb{R})$ is presented as

$$\langle s_2, s_2', s_2'', s_3 \mid s_2^2 = s_2'^2 = s_2'^2 = s_3^3 = s_2 s_2' s_2'' s_3 = 1\rangle,$$

with

$$s_2 = -\frac{8}{13}\alpha - \frac{3}{13}\alpha\beta, \quad s_2' = -\frac{20}{13}\alpha - \frac{1}{2}\beta - \frac{15}{26}\alpha\beta,$$
$$s_2'' = -\frac{35}{13}\alpha - \frac{1}{2}\beta - \frac{23}{26}\alpha\beta, \quad s_3 = -\frac{1}{2} - \frac{31}{26}\alpha - \frac{5}{13}\alpha\beta,$$

and $\mathcal{X}_{10}^*$ has genus 0. Hence, there is a map $t_{10}\colon \mathcal{X}_{10}^* \xrightarrow{\sim} \mathbb{P}^1$ such that

$$\begin{aligned} \operatorname{div}(t_{10}) &= P_3 - P_2, \\ t_{10}(P_2'') &= 2, \end{aligned} \tag{7.19}$$

where P_2, P_2'', P_3 are the fixed points of s_2, s_2'', s_3, respectively. Again the fixed point of s_i is the fixed point of all of $\mathbf{k}_i^\times \subset B^\times$ where $\mathbf{k}_i = \mathbb{Q}(s_i^0)$. Now

$$\mathbf{k}_2 \simeq \mathbb{Q}(\sqrt{-2}), \quad \mathbf{k}_2' \simeq \mathbb{Q}(\sqrt{-10}), \quad \mathbf{k}_2'' \simeq \mathbb{Q}(\sqrt{-5}), \quad \mathbf{k}_3 \simeq \mathbb{Q}(\sqrt{-3}).$$

Lemma 7.6 *The following equalities hold.*

(1) $Z(2,0;\Gamma^*) = \frac{1}{2}P_2$.
(2) $Z(3,0;\Gamma^*) = \frac{1}{3}P_3$.

Proposition 7.7 *The following identity for t_{10} holds,*

$$\operatorname{div}(t_{10}) = 3Z(3,0;\Gamma^*) - 2Z(2,0;\Gamma^*).$$

Then the same line of reasoning as in Section 7.1.1 applied to the case $|L^\vee/L| = 200$ and $N = 20$ gives the following result.

Theorem 7.8 *Let*

$$\begin{aligned} f_{10} &= 3\Big(\frac{\eta_4^6\eta_{10}^8}{\eta_2^3\eta_5^2\eta_{20}^8}\Big) - 2\Big(\frac{\eta_2^3\eta_4^2\eta_{10}^2}{\eta_1^2\eta_{20}^4}\Big) - 5\Big(\frac{\eta_4^2\eta_{10}^6}{\eta_2\eta_5^2\eta_{20}^4}\Big) + 4\Big(\frac{\eta_2^5}{\eta_1^2\eta_4^2}\Big) \\ &= \frac{3}{\mathbf{q}^3} - \frac{2}{\mathbf{q}^2} + O[\mathbf{q}]. \end{aligned}$$

It is a vectorizable modular form over $\widetilde{\Gamma_0(20)}$ with no poles at finite cusps. Thus

$$\operatorname{div}(t_{10}) = 3Z(3,0;\Gamma^*) - 2Z(2,0;\Gamma^*) = \operatorname{div}\big(\Psi(F_{f_{10}})^2\big).$$

So again the two functions agree up to a nonzero constant,

$$t_{10} = c_{10}\Psi(F_{f_{10}})^2.$$

7.2.2 $\Delta = -20$

To compute the constant c_{10}, we now consider the case of $\Delta = -20$. Recall that $t_{10}(P_2') = 2$ by definition and by (7.2.1), $P_2'' = \mathcal{P}_{-20} \in \mathcal{X}_{10}^*$. Then

$$L = \mathbb{Z}\ell_1 + \mathbb{Z}\ell_2 + \mathbb{Z}\ell_3$$

where

$$\ell_1 = \alpha, \quad \ell_2 = \frac{6\alpha + \alpha\beta}{13}, \quad \ell_3 = \frac{\beta + \alpha\beta}{2}.$$

Take $z = \ell_1 - 3\ell_2$ so that $Q(z) = 5$. Then the negative plane is spanned by

$$u_1 = -\ell_2, \quad u_2 = 6\ell_1 - 13\ell_2 + 2\ell_3,$$

and

$$Q(Xu_1 + Yu_2) = -2(X^2 + 5Y^2).$$

A basis of L_- is given by

$$\ell_1^- = -\ell_2, \quad \ell_2^- = 3\ell_1 + \ell_3,$$

In this case the quotient $L/(L_- + L_+)$ is trivial. Theorem 4.3 yields

$$\sum_{z\in Z_{\Gamma^*}(\mathbb{Q}(\sqrt{-5}))} \log\|\Psi(z, F_{f_{10}})\|^2 = \Big(\frac{-1}{4}\Big)\big(3\kappa_0(3) - 2\kappa_0(2)\big) = 3\log(2).$$

Thus

$$\|\Psi(P_2'', F_{f_{10}})^2\| = 2^3.$$

Since $t_{10}(P_2'') = 2$,

$$\|t_{10}\| = 2^{-2}\|\Psi(F_{f_{10}})^2\|.$$

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7.2.3 $\Delta = -68$

Again, we are now able to compute the coordinates of the other rational CM points for $\mathcal{X}_{10}^*$ listed in Table 4. Moreover, we are also capable of calculating the norms of irrational CM points. As an example, we compute the norm of the irrational CM point with discriminant -68.

Take $z = 7\ell_1 - 13\ell_2 + \ell_3$ so that $Q(z) = 17$. Then the negative plane is spanned by $u_1 = -35\ell_2 + 11\ell_3$, $u_2 = 534\ell_1 - 1067\ell_2 + 137\ell_3$, and $Q(Xu_1 + Yu_2) = -2670(X^2 + 17Y^2)$. A basis of L_- is given by $\ell_1^- = -35\ell_2 + 11\ell_3$, $\ell_2^- = \ell_1 + 2\ell_2 - \ell_3$. The group $L/(L_- + L_+)$ is cyclic of order 17 and is generated by $\lambda = \ell_3 + (L_- + L_+)$. This has the decomposition

$$\lambda_+ = \frac{-35}{17} z + L_+, \quad \lambda_- = \frac{27}{17}\ell_1^- + \frac{245}{17}\ell_2^- + L_-.$$

Then computations as before yield

$$\kappa_0(1) = -6\log(2) - 6\log(5),$$
$$\kappa_0(3) = -8\log(2) - \frac{14}{3}\log(5).$$

This time the CM point with discriminant -68 is irrational, and Theorem 4.3 gives its norm (after renormalization) as

$$\prod_{z \in Z(-68)} \|\Psi(z, F_{f_{10}})\|^2 = 2^2 \cdot 5.$$

8 Tables

8.1 $D = 6$

8.1.1 Coordinates of Rational CM Points on $\mathcal{X}_6^*$

Table 2 gives the values of t_6 (as defined by (7.2)) at the rational CM points of $\mathcal{X}_6^*$. These values verify the values in [7, Table 2]. Denote $t_6(P_{CM}) = (r : s)$.

Table 2: Coordinates of Rational CM Points on $\mathcal{X}_6^*$.

Δ	r	s	Proved in [7]
-3	1	0	Y
-4	0	1	Y
-24	1	1	Y
-40	-3^7	5^3	Y
-52	$2^2 3^7$	5^6	Y
-19	-3^7	2^{10}	Y
-84	$2^2 7^2$	3^3	Y
-88	$3^7 7^4$	$5^6 11^3$	Y
-100	$-2^4 3^7 7^4 5$	11^6	Y

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Table 2: (continued)

Δ	r	s	Proved in [7]
-120	-7^4	$3^3 5^3$	Y
-132	$2^4 11^2$	5^6	Y
-148	$2^2 3^7 7^4 11^4$	$5^6 17^6$	N
-168	$7^2 11^4$	5^6	Y
-43	$-3^7 7^4$	$2^{10} 5^6$	Y
-51	7^4	2^{10}	Y
-228	$-2^6 7^4 19^2$	$3^6 5^6$	N
-232	$-3^7 7^4 11^4 19^4$	$5^6 23^6 29^3$	N
-67	$-3^7 7^4 11^4$	$2^{16} 5^6$	N
-75	-11^4	$2^{10} 3^3 5$	Y
-312	$7^4 23^4$	$5^6 11^6$	Y
-372	$2^2 7^4 19^4 31^2$	$3^3 5^6 11^6$	N
-408	$7^4 11^4 31^4$	$3^6 5^6 17^3$	N
-123	$7^4 19^4$	$2^{10} 5^6$	N
-147	$11^4 23^4$	$2^{10} 3^3 5^6 7$	Y
-163	$-3^{11} 7^4 19^4 23^4$	$2^{10} 5^6 11^6 17^6$	N
-708	$2^8 7^4 11^4 47^4 59^2$	$5^6 17^6 29^6$	N
-267	$7^4 31^4 43^4$	$2^{16} 5^6 11^6$	N

8.1.2 Norms of CM Points on $\mathcal{X}_6^*$ for $0 < -d \leq 250$

Table 3 gives the norms for all CM points of fundamental discriminant $\Delta = d$ or $4d$ for $0 < -d \leq 250$. This cut-off is arbitrary. It is also only for implementation reasons that we only compute for fundamental discriminants (*i.e.*, d squarefree).

Table 3: Norms of CM Points on $\mathcal{X}_6^*$ for $0 < -d \leq 250$.

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert(1-t_6)(\mathcal{P}_\Delta)\rvert$
-40	$\frac{3^7}{5^3}$	$\frac{2^3 17^2}{5^3}$
-52	$\frac{2^2 3^7}{5^6}$	$\frac{13^1 23^2}{5^6}$
-19	$\frac{3^7}{2^{10}}$	$\frac{13^2 19^1}{2^{10}}$
-84	$\frac{2^2 7^2}{3^3}$	$\frac{13^2}{3^3}$
-88	$\frac{3^7 7^4}{5^6 11^3}$	$\frac{2^5 17^2 41^2}{5^6 11^3}$
-120	$\frac{7^4}{3^3 5^3}$	$\frac{2^4 19^2}{3^3 5^3}$
-132	$\frac{2^4 11^2}{5^6}$	$\frac{3^4 13^2}{5^6}$
-136	$\frac{3^{14}}{11^2 17^3}$	$\frac{2^6 13^4 41^2}{11^6 17^2}$
-148	$\frac{2^2 3^7 7^4 11^4}{5^6 17^6}$	$\frac{13^2 37^1 47^2 71^2}{5^6 17^6}$
-168	$\frac{7^2 11^4}{5^6}$	$\frac{2^3 3^5 19^2}{5^6}$

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Table 3: (continued)

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert(1-t_6)(\mathcal{P}_\Delta)\rvert$
-43	$\frac{3^7 7^4}{2^{10} 5^6}$	$\frac{19^2 37^2 43^1}{2^{10} 5^6}$
-184	$\frac{3^{14} 7^8}{17^6 23^3}$	$\frac{2^8 13^4 89^2}{17^4 23^2}$
-51	$\frac{7^4}{2^{10}}$	$\frac{3^4 17^1}{2^{10}}$
-228	$\frac{2^6 7^4 19^2}{3^6 5^6}$	$\frac{13^2 17^2 37^2}{3^6 5^6}$
-232	$\frac{3^7 7^4 11^4 19^4}{5^6 23^6 29^3}$	$\frac{2^3 13^2 17^2 41^2 89^2 113^2}{5^6 23^6 29^3}$
-244	$\frac{2^6 3^{21} 19^4}{17^6 29^6}$	$\frac{19^4 37^2 47^2 61^1}{17^2 29^6}$
-264	$\frac{19^4}{3^9 11^1}$	$\frac{2^6 19^2 43^2}{3^9 11^3}$
-67	$\frac{3^7 7^4 11^4}{2^{16} 5^6}$	$\frac{13^2 43^2 61^2 67^1}{2^{16} 5^6}$
-276	$\frac{2^4 23^2}{11^2}$	$\frac{3^8 23^1 37^2}{11^6}$
-280	$\frac{3^{14} 7^4 23^4}{5^6 11^2 29^6}$	$\frac{2^{12} 13^4 23^2 113^2 137^2}{5^6 11^6 29^6}$
-292	$\frac{2^{10} 3^{14} 19^4}{5^{12} 23^2}$	$\frac{13^4 17^4 19^2 67^2 71^2}{5^{12} 23^6}$
-312	$\frac{7^4 23^4}{5^6 11^6}$	$\frac{2^4 3^5 13^1 17^2 43^2}{5^6 11^6}$
-328	$\frac{3^{18} 11^8 19^4}{5^{12} 17^6 41^3}$	$\frac{2^6 19^2 23^4 89^2 137^2}{5^{12} 17^4 41^2}$
-340	$\frac{2^4 3^{18} 7^8 23^4}{5^6 29^6 41^6}$	$\frac{13^4 17^2 19^4 23^2 61^2 167^2}{5^6 29^6 41^6}$
-91	$\frac{3^{14} 7^4}{2^{26} 11^2}$	$\frac{13^2 17^4 37^2 67^2}{2^{26} 11^6}$
-372	$\frac{2^2 7^4 19^4 31^2}{3^3 5^6 11^6}$	$\frac{13^2 23^2 37^2 61^2}{3^3 5^6 11^6}$
-376	$\frac{3^{28} 31^4}{23^2 41^6 47^3}$	$\frac{2^{16} 37^4 113^2}{23^2 41^4 47^2}$
-388	$\frac{2^{14} 3^{18} 31^4}{5^{12} 11^2 47^6}$	$\frac{13^4 17^4 43^2 167^2 191^2}{5^{12} 11^6 47^4}$
-408	$\frac{7^4 11^4 31^4}{3^6 5^6 17^3}$	$\frac{2^6 13^2 19^2 43^2 67^2}{3^6 5^6 17^3}$
-420	$\frac{2^{12} 7^4 23^4}{5^6 17^6}$	$\frac{3^8 23^2 61^2}{5^6 17^4}$
-424	$\frac{3^{25} 7^{12} 19^4}{29^6 47^6 53^3}$	$\frac{2^9 13^6 19^4 37^4 41^2 137^2}{29^6 47^6 53^3}$
-436	$\frac{2^6 3^{21} 7^{12} 31^4}{17^6 41^6 53^6}$	$\frac{13^6 43^4 71^2 109^1 191^2}{17^2 41^6 53^6}$
-456	$\frac{7^8 19^2}{11^2 17^6}$	$\frac{2^6 3^9 19^1 67^2}{11^6 17^4}$
-115	$\frac{3^{14} 19^4}{2^{20} 5^6 11^2}$	$\frac{13^4 19^2 23^2 61^2 109^2}{2^{20} 5^6 11^6}$
-472	$\frac{3^{21} 19^4 23^4 31^4}{5^{18} 53^6 59^3}$	$\frac{2^{17} 19^4 23^4 47^2 89^2 233^2}{5^{18} 53^6 59^3}$
-123	$\frac{7^4 19^4}{2^{10} 5^6}$	$\frac{3^4 13^2 23^2 41^1}{2^{10} 5^6}$
-516	$\frac{2^{14} 31^4 43^2}{3^{12} 17^6}$	$\frac{37^2 41^2 61^2}{3^{12} 17^2}$
-520	$\frac{3^{18} 7^8 19^4 43^4}{5^6 11^2 41^6 59^6}$	$\frac{2^6 13^2 17^4 19^2 37^4 113^2 233^2 257^2}{5^6 11^6 41^6 59^6}$
-532	$\frac{2^4 3^{14} 7^4 11^8 23^4 43^4}{5^{12} 29^6 53^6}$	$\frac{17^4 19^2 23^2 37^2 109^2 191^2 239^2 263^2}{5^{12} 29^6 53^6}$
-552	$\frac{19^4 43^4}{3^6 5^{12} 23^1}$	$\frac{2^6 13^4 19^2 43^2 67^2}{3^6 5^{12} 23^3}$

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Table 3: (continued)

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert(1-t_6)(\mathcal{P}_\Delta)\rvert$
-139	$\frac{3^{21}19^{4}23^{4}}{2^{36}17^{6}}$	$\frac{19^{4}23^{4}43^{2}139^{1}}{2^{36}17^{2}}$
-564	$\frac{2^{4}7^{8}47^{2}}{11^{2}23^{6}}$	$\frac{3^{8}13^{4}17^{4}47^{1}}{11^{6}23^{4}}$
-568	$\frac{3^{14}7^{8}23^{4}31^{4}}{5^{12}17^{6}47^{2}71^{3}}$	$\frac{2^{8}19^{4}23^{2}41^{2}137^{2}257^{2}281^{2}}{5^{12}17^{4}47^{6}71^{2}}$
-580	$\frac{2^{20}3^{32}43^{4}47^{4}}{5^{12}59^{6}71^{6}}$	$\frac{13^{8}41^{4}43^{4}47^{2}139^{2}263^{2}}{5^{12}59^{6}71^{6}}$
-616	$\frac{3^{28}7^{8}43^{4}}{11^{2}23^{2}53^{6}71^{6}}$	$\frac{2^{12}13^{8}37^{2}61^{4}233^{2}281^{2}}{11^{6}23^{2}53^{6}71^{6}}$
-628	$\frac{2^{6}3^{21}19^{4}31^{4}47^{4}}{5^{18}11^{8}41^{6}}$	$\frac{19^{4}61^{2}71^{2}157^{1}167^{2}239^{2}311^{2}}{5^{18}11^{12}41^{2}}$
-163	$\frac{3^{11}7^{4}19^{4}23^{4}}{2^{10}5^{6}11^{6}17^{6}}$	$\frac{13^{2}67^{2}109^{2}139^{2}157^{2}163^{1}}{2^{10}5^{6}11^{6}17^{6}}$
-660	$\frac{2^{4}7^{8}11^{4}43^{4}}{3^{12}5^{6}23^{6}}$	$\frac{17^{4}47^{2}61^{2}109^{2}}{3^{12}5^{6}23^{4}}$
-664	$\frac{3^{39}47^{4}}{29^{6}59^{6}83^{3}}$	$\frac{2^{27}37^{4}47^{4}61^{4}71^{2}89^{2}257^{2}}{11^{12}29^{6}59^{6}83^{3}}$
-696	$\frac{11^{2}23^{4}31^{4}}{3^{15}29^{3}}$	$\frac{2^{12}13^{6}23^{4}67^{2}}{3^{15}11^{6}29^{3}}$
-708	$\frac{2^{8}7^{4}11^{4}47^{4}59^{2}}{5^{6}17^{6}29^{6}}$	$\frac{3^{4}13^{2}19^{2}23^{2}37^{2}41^{2}109^{2}}{5^{6}17^{6}29^{6}}$
-712	$\frac{3^{32}19^{8}43^{4}59^{4}}{5^{24}83^{6}89^{3}}$	$\frac{2^{12}19^{4}41^{2}43^{2}47^{4}113^{2}281^{2}353^{2}}{5^{24}83^{6}89^{2}}$
-724	$\frac{2^{10}3^{39}59^{4}}{17^{6}53^{6}89^{6}}$	$\frac{17^{2}41^{4}43^{4}67^{4}157^{2}181^{1}359^{2}}{11^{12}53^{6}89^{6}}$
-744	$\frac{7^{12}31^{2}59^{4}}{23^{6}29^{6}}$	$\frac{2^{9}3^{13}41^{2}43^{2}}{23^{2}29^{6}}$
-187	$\frac{3^{18}11^{4}31^{4}}{2^{20}5^{12}23^{2}}$	$\frac{13^{4}17^{2}19^{4}37^{2}163^{2}181^{2}}{2^{20}5^{12}23^{6}}$
-760	$\frac{3^{14}7^{8}11^{8}23^{4}31^{4}47^{4}}{5^{6}41^{6}71^{6}89^{6}}$	$\frac{2^{8}13^{4}17^{4}19^{2}23^{2}47^{2}61^{4}137^{2}233^{2}353^{2}}{5^{6}41^{6}71^{6}89^{6}}$
-772	$\frac{2^{14}3^{14}7^{8}31^{4}43^{4}}{5^{12}23^{6}59^{2}83^{6}}$	$\frac{13^{4}17^{4}43^{2}139^{2}239^{2}311^{2}359^{2}383^{2}}{5^{12}23^{4}59^{6}83^{6}}$
-195	$\frac{19^{4}31^{4}}{2^{26}5^{6}}$	$\frac{3^{8}13^{2}19^{2}47^{2}}{2^{26}5^{6}}$
-804	$\frac{2^{18}11^{2}19^{4}67^{2}}{3^{12}29^{6}}$	$\frac{13^{6}17^{6}19^{4}109^{2}}{3^{12}11^{6}29^{6}}$
-808	$\frac{3^{25}23^{4}31^{4}59^{4}67^{4}}{5^{18}11^{8}47^{6}101^{3}}$	$\frac{2^{9}13^{6}23^{4}37^{2}41^{2}89^{2}257^{2}401^{2}}{5^{18}11^{12}47^{2}101^{3}}$
-820	$\frac{2^{8}3^{28}7^{16}47^{4}67^{4}}{5^{12}29^{6}89^{6}101^{6}}$	$\frac{37^{4}41^{2}47^{2}67^{4}109^{2}167^{2}181^{2}263^{2}383^{2}}{5^{12}29^{6}89^{6}101^{6}}$
-840	$\frac{7^{4}43^{4}67^{4}}{3^{12}5^{6}11^{2}17^{6}}$	$\frac{2^{6}13^{4}19^{4}23^{4}139^{2}}{3^{12}5^{6}11^{6}17^{4}}$
-211	$\frac{3^{25}7^{12}31^{4}}{2^{36}17^{6}23^{6}}$	$\frac{41^{4}61^{2}157^{2}211^{1}}{2^{36}17^{2}23^{2}}$
-852	$\frac{2^{4}59^{4}71^{2}}{5^{12}11^{2}23^{2}}$	$\frac{3^{8}13^{4}19^{4}47^{2}61^{2}71^{1}}{5^{12}11^{6}23^{6}}$
-856	$\frac{3^{21}7^{12}11^{2}19^{4}31^{4}71^{4}}{53^{6}83^{6}101^{6}107^{3}}$	$\frac{2^{19}13^{6}17^{6}19^{4}37^{4}281^{2}353^{2}401^{2}}{11^{6}53^{6}83^{6}101^{6}107^{3}}$
-868	$\frac{2^{28}3^{28}7^{8}67^{4}}{5^{24}71^{2}107^{6}}$	$\frac{37^{4}41^{4}67^{2}163^{2}191^{2}211^{2}359^{2}431^{2}}{5^{24}71^{4}107^{6}}$
-219	$\frac{7^{8}23^{4}}{2^{26}3^{9}}$	$\frac{13^{4}23^{2}41^{2}71^{2}}{2^{26}3^{9}}$
-888	$\frac{31^{4}47^{4}71^{4}}{5^{18}29^{6}}$	$\frac{2^{12}3^{13}37^{1}41^{2}67^{2}139^{2}}{5^{18}29^{6}}$
-904	$\frac{3^{32}7^{16}19^{4}67^{4}}{17^{6}59^{2}89^{6}107^{6}113^{3}}$	$\frac{2^{12}13^{8}19^{6}43^{2}61^{4}449^{2}}{59^{6}89^{4}107^{6}113^{2}}$
-916	$\frac{2^{10}3^{35}19^{4}43^{4}71^{4}}{41^{6}101^{6}113^{6}}$	$\frac{13^{10}19^{8}43^{4}229^{1}311^{2}383^{2}431^{2}}{11^{12}41^{2}101^{6}113^{6}}$

Table 3: (continued)

Δ	$\lvert t_6(\mathcal{P}_\Delta)\rvert$	$\lvert(1-t_6)(\mathcal{P}_\Delta)\rvert$
-235	$\frac{3^{14}7^{8}19^{4}31^{4}}{2^{20}5^{6}11^{2}29^{6}}$	$\frac{17^{4}19^{2}47^{2}139^{2}181^{2}211^{2}229^{2}}{2^{20}5^{6}11^{6}29^{6}}$
-948	$\frac{2^{6}19^{4}31^{4}67^{4}79^{2}}{3^{15}5^{18}}$	$\frac{19^{4}37^{2}47^{2}71^{2}109^{2}157^{2}}{3^{15}5^{18}}$
-952	$\frac{3^{28}7^{8}23^{2}71^{4}79^{4}}{5^{24}17^{6}113^{6}}$	$\frac{2^{16}43^{4}47^{4}71^{2}233^{2}401^{2}449^{2}}{5^{24}17^{4}23^{4}113^{4}}$
-964	$\frac{2^{34}3^{42}59^{4}79^{4}}{17^{12}47^{2}83^{6}107^{6}}$	$\frac{13^{12}37^{4}67^{4}239^{2}479^{2}}{17^{4}47^{2}83^{6}107^{6}}$
-984	$\frac{7^{12}11^{2}79^{4}}{3^{12}23^{6}41^{3}}$	$\frac{2^{16}37^{2}43^{2}139^{2}163^{2}}{3^{12}11^{6}23^{2}41^{3}}$
-996	$\frac{2^{16}7^{12}71^{4}83^{2}}{17^{6}29^{6}41^{6}}$	$\frac{3^{14}13^{6}47^{2}157^{2}}{17^{2}29^{6}41^{6}}$

8.2 $D = 10$

8.2.1 Coordinates of Rational CM Points on $\mathcal{X}_{10}^*$

Table 4 gives the values of t_{10} (as defined by (7.19)) at the rational CM points of $\mathcal{X}_{10}^*$. These values verify the values in [7, Table 4]. Again denote $t_{10}(P_{CM}) = (r : s)$.

Table 4: Coordinates of Rational CM Points on $\mathcal{X}_{10}^*$.

Δ	r	s	Proved in [7]
-3	0	1	Y
-8	1	0	Y
-20	2	1	Y
-40	3^3	1	Y
-52	$-2^1 3^3$	5^2	N
-72	5^3	$3^1 7^2$	Y
-120	-3^3	7^2	Y
-88	$3^3 5^3$	$2^1 7^2$	N
-27	$-2^6 3$	5^2	Y
-35	2^6	7	Y
-148	$2^1 3^3 11^3$	$5^2 7^2 13^2$	N
-43	$2^6 3^3$	$5^2 7^2$	N
-180	$-2^1 11^3$	13^2	Y
-232	$3^3 11^3 17^3$	$2^2 5^2 7^2 23^2$	N
-67	$-2^6 3^3 5^3$	$7^2 13^2$	N
-280	$3^3 11^3$	$2^1 7^1 23^2$	N
-340	$2^1 3^3 23^3$	$7^2 29^2$	N

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Table 4: (continued)

Δ	r	s	Proved in [7]
-115	$2^9 3^3$	$13^2 23$	N
-520	$3^3 29^3$	$2^3 7^2 13^1 47^2$	N
-163	$-2^9 3^3 5^3 11^3$	$7^2 13^2 29^2 31^2$	N
-760	$3^3 17^3 47^3$	$7^2 31^2 71^2$	N
-235	$2^6 3^3 17^3$	$7^2 37^2 47$	N

8.2.2 Norms of CM Points on $\mathcal{X}_{10}^*$ for $0 < -d \le 250$

Table 5: Norms of CM Points on $\mathcal{X}_{10}^*$ for $0 < -d \le 250$.

Δ	$\lvert t_{10}(\mathcal{P}_\Delta)\rvert$	$\lvert(2 - t_{10})(\mathcal{P}_\Delta)\rvert$
-40	$\frac{3^3}{1}$	$\frac{5^2}{1}$
-52	$\frac{2^1 3^3}{5^2}$	$\frac{2^3 13^1}{5^2}$
-68	$\frac{2^2 5^1}{1}$	$\frac{2^4 17^1}{5^2}$
-88	$\frac{3^3 5^3}{2^1 7^2}$	$\frac{11^1 17^2}{2^1 7^2}$
-120	$\frac{3^3}{7^2}$	$\frac{5^3}{7^2}$
-132	$\frac{2^2 3^6 5^1}{13^2}$	$\frac{2^4 11^2}{5^2}$
-35	$\frac{2^6}{7^1}$	$\frac{2^1 5^2}{7^1}$
-148	$\frac{2^1 3^3 11^3}{5^2 7^2 13^2}$	$\frac{2^5 17^2 37^1}{5^2 7^2 13^2}$
-152	$\frac{11^3}{2^1 5^1}$	$\frac{11^4 19^1}{2^1 5^4}$
-168	$\frac{3^6 11^3}{2^2 5^4 7^2}$	$\frac{11^2 37^2}{2^2 5^4 7^2}$
-43	$\frac{2^6 3^3}{5^2 7^2}$	$\frac{2^1 19^2}{5^2 7^2}$
-212	$\frac{2^3 5^4 11^3}{7^6}$	$\frac{2^{11} 11^4 53^1}{5^2 7^6}$
-228	$\frac{2^2 3^6 5^1 17^3}{7^4 13^2}$	$\frac{2^4 19^2 37^2}{5^2 7^4}$
-232	$\frac{3^3 11^3 17^3}{2^2 5^2 7^2 23^2}$	$\frac{13^2 19^2 53^2}{2^2 5^2 7^2 23^2}$
-248	$\frac{5^2 17^3}{2^2 23^2}$	$\frac{17^2 19^4 31^1}{2^2 5^4 23^2}$
-260	$\frac{2^2 17^3}{7^4 13^1}$	$\frac{2^4 5^4}{7^4}$
-67	$\frac{2^6 3^3 5^3}{7^2 13^2}$	$\frac{2^1 11^2 31^2}{7^2 13^2}$
-280	$\frac{3^3 11^3}{2^1 7^1 23^2}$	$\frac{5^3 13^2}{2^1 7^1 23^2}$
-292	$\frac{2^2 3^6 5^1 17^3}{13^4 29^2}$	$\frac{2^4 17^2 53^2 73^1}{5^2 13^4 29^2}$
-308	$\frac{2^4 5^2 11^3 23^3}{7^4 29^2}$	$\frac{2^{14} 11^2 19^4}{5^4 7^4 29^2}$

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Table 5: (continued)

Δ	$\lvert t_{10}(\mathcal{P}_\Delta)\rvert$	$\lvert(2-t_{10})(\mathcal{P}_\Delta)\rvert$
-312	$\frac{3^6 17^3 23^3}{2^2 5^4 7^4 31^2}$	$\frac{11^4 13^2 73^2}{2^2 5^4 7^4 31^2}$
-328	$\frac{3^6 5^1 23^1}{2^3 31^2}$	$\frac{11^4 37^2}{2^3 5^2 23^2}$
-83	$\frac{2^{18}}{5^1 13^2}$	$\frac{2^3 13^2 19^2}{5^4}$
-340	$\frac{2^1 3^3 23^3}{7^2 29^2}$	$\frac{2^3 5^2 13^2 17^1}{7^2 29^2}$
-372	$\frac{2^2 3^9 11^3 23^3}{5^4 7^4 13^2 37^2}$	$\frac{2^8 11^2 31^2 73^2}{5^4 7^4 37^2}$
-388	$\frac{2^2 3^6 17^3 29^1}{5^4 13^2 37^2}$	$\frac{2^4 11^4 17^2 97^1}{5^4 29^2 37^2}$
-408	$\frac{3^9 5^1 11^3 29^3}{7^4 13^4 31^2}$	$\frac{11^2 17^2 19^4 97^2}{5^2 7^4 13^4 31^2}$
-420	$\frac{2^2 3^6 29^3}{7^2 37^2}$	$\frac{2^4 5^6 17^2}{7^2 37^2}$
-107	$\frac{2^{21}}{5^1 7^6}$	$\frac{2^3 17^4 31^2}{5^4 7^6}$
-440	$\frac{11^3 23^3}{2^2 13^4}$	$\frac{5^6 11^1}{2^2 13^2}$
-452	$\frac{2^4 11^6 17^3 29^1}{5^3 7^8}$	$\frac{2^8 11^4 17^2 31^4 113^1}{5^6 7^8 29^2}$
-115	$\frac{2^9 3^3}{13^2 23^1}$	$\frac{2^1 5^2 11^2}{13^2 23^1}$
-472	$\frac{3^9 5^4 29^3}{2^1 23^2 31^2 47^2}$	$\frac{19^2 59^1 73^2 113^2}{2^1 5^2 23^2 31^2 47^2}$
-488	$\frac{11^9 17^3}{2^4 13^4 47^2}$	$\frac{11^4 13^2 17^4}{2^4 5^6 47^2}$
-123	$\frac{2^{15} 3^6 5^1}{7^4 23^2}$	$\frac{2^2 13^4 59^2}{5^2 7^4 23^2}$
-520	$\frac{3^3 29^3}{2^3 7^2 13^1 47^2}$	$\frac{5^4 11^2 17^2}{2^3 7^2 13^1 47^2}$
-532	$\frac{2^2 3^6 5^6 11^3 23^1}{7^2 29^2 37^2 53^2}$	$\frac{2^{10} 11^2 19^2 113^2}{7^2 23^2 29^2 37^2}$
-548	$\frac{2^4 11^6 29^3 41^3}{5^3 7^8 13^2 53^2}$	$\frac{2^8 11^4 13^4 19^4 137^1}{5^6 7^8 53^2}$
-552	$\frac{3^{15} 5^2 41^3}{2^6 13^6 31^2}$	$\frac{19^4 31^2 59^2}{2^6 5^4 13^4}$
-568	$\frac{3^6 17^3 23^1 41^3}{5^4 7^4 47^2}$	$\frac{17^2 31^2 71^1 97^2 137^2}{5^4 7^4 23^2 47^2}$
-580	$\frac{2^2 3^6 41^3}{13^2 29^1 53^2}$	$\frac{2^4 5^7}{29^1 53^2}$
-155	$\frac{2^{12} 11^3}{7^4 31^1}$	$\frac{2^2 5^6 11^2}{7^4 31^1}$
-628	$\frac{2^3 3^9 5^4 11^3 47^3}{29^2 31^2 53^2 61^2}$	$\frac{2^{13} 11^4 19^2 137^2 157^1}{5^2 29^2 31^2 53^2 61^2}$
-632	$\frac{5^2 11^6 41^3 47^1}{2^2 7^8 13^4}$	$\frac{11^4 17^6 79^1 113^2}{2^2 5^4 7^8 47^2}$
-163	$\frac{2^9 3^3 5^3 11^3}{7^2 13^2 29^2 31^2}$	$\frac{2^1 19^2 59^2 79^2}{7^2 13^2 29^2 31^2}$
-660	$\frac{2^2 3^9 47^3}{7^4 23^2 61^2}$	$\frac{2^8 5^4 11^2 17^2}{7^4 23^2 61^2}$
-680	$\frac{11^3 17^3 41^3}{2^4 7^6 23^2}$	$\frac{5^{10} 11^4}{2^4 7^6 23^2}$
-692	$\frac{2^7 17^6 47^3}{5^4 23^1 31^2 53^2}$	$\frac{2^{27} 17^6 31^2 173^1}{5^{10} 23^4 53^2}$
-708	$\frac{2^2 3^6 5^6 17^3 41^3 53^3}{7^4 13^2 23^4 29^2 37^2 61^2}$	$\frac{2^4 11^4 59^2 97^2 157^2}{7^4 23^4 29^2 37^2 61^2}$
-712	$\frac{3^{12} 47^1 53^3}{2^7 5^3 71^2}$	$\frac{19^4 53^2 79^2 173^2}{2^7 5^6 47^2 71^2}$

Table 5: (continued)

Δ	$\lvert t_{10}(\mathcal{P}_\Delta)\rvert$	$\lvert(2-t_{10})(\mathcal{P}_\Delta)\rvert$
-728	$\frac{17^6 29^3 53^3}{2^4 5^2 7^6 13^4 71^2}$	$\frac{17^4 19^4 59^4 137^2}{2^4 5^8 7^6 13^2 71^2}$
-740	$\frac{2^4 11^6 53^3}{23^4 29^2 37^1}$	$\frac{2^8 5^8 11^4}{23^4 29^2}$
-187	$\frac{2^{12} 3^6 5^1 11^3}{23^2 31^2 37^2}$	$\frac{2^2 13^4 71^2}{5^2 23^2 37^2}$
-760	$\frac{3^3 17^3 47^3}{7^2 31^2 71^2}$	$\frac{5^2 11^2 13^2 19^1 37^2}{7^2 31^2 71^2}$
-772	$\frac{2^2 3^6 5^1 29^3 41^3 53^1}{7^4 13^2 37^2 61^2}$	$\frac{2^4 17^2 113^2 173^2 193^1}{5^2 7^4 53^2 61^2}$
-195	$\frac{2^{12} 3^6}{13^2 29^2}$	$\frac{2^2 5^6 19^2}{13^2 29^2}$
-788	$\frac{2^5 17^6 47^3 59^3}{7^{10} 13^4 23^1}$	$\frac{2^{21} 13^2 17^2 19^2 59^4 197^1}{5^6 7^{10} 23^4}$
-808	$\frac{3^9 5^4 41^3 59^3}{2^2 13^2 23^2 47^2 71^2 79^2}$	$\frac{11^6 13^2 157^2 197^2}{2^2 5^2 23^2 47^2 71^2 79^2}$
-203	$\frac{2^{27} 5^2 11^3}{7^4 13^4 37^2}$	$\frac{2^4 11^6 79^2}{5^4 7^4 37^2}$
-820	$\frac{2^2 3^6 59^3}{7^4 31^2 37^2}$	$\frac{2^8 5^7}{7^4 31^2}$
-840	$\frac{3^6 23^3 53^3}{2^4 7^2 13^4 79^2}$	$\frac{5^4 11^4 17^2 19^2}{2^4 7^2 13^4 79^2}$
-852	$\frac{2^4 3^{12} 5^2 11^3 47^3 59^3}{13^6 23^6 61^2}$	$\frac{2^{12} 11^6 19^4 71^2 193^2}{5^4 13^4 23^6 61^2}$
-868	$\frac{2^4 3^{12} 5^2 53^1}{7^4 29^1 37^2}$	$\frac{2^8 31^2 137^2 197^2}{5^4 7^4 29^4}$
-872	$\frac{11^6 17^9 59^3}{2^4 7^{10} 29^4 71^2}$	$\frac{11^6 19^2 31^4 71^2 173^2}{2^4 5^6 7^{10} 29^4}$
-888	$\frac{3^{18} 41^3 47^3}{2^4 5^2 29^4 31^2 79^2}$	$\frac{31^2 37^2 59^4 71^2 97^2}{2^4 5^8 29^4 79^2}$
-227	$\frac{2^{33} 17^3}{13^6 31^2}$	$\frac{2^5 17^4 37^4}{5^6 13^2 31^2}$
-920	$\frac{23^2 59^3}{2^3 29^1 47^2}$	$\frac{5^{15} 37^2}{2^3 23^1 29^4 47^2}$
-932	$\frac{2^6 17^3 23^2 41^3 53^3}{5^2 7^{12} 31^4}$	$\frac{2^{12} 17^6 19^4 53^2 71^4 233^1}{5^8 7^{12} 23^4 31^4}$
-235	$\frac{2^6 3^3 17^3}{7^2 37^2 47^1}$	$\frac{2^1 5^2 11^2 19^2}{7^2 37^2 47^1}$
-948	$\frac{2^6 3^{21} 59^3 71^3}{5^2 31^4 37^2 47^2 61^2}$	$\frac{2^{26} 19^4 79^2 157^2}{5^8 37^2 47^2 61^2}$
-952	$\frac{3^{12} 5^2 71^1}{2^2 7^4 23^1 79^2}$	$\frac{17^2 113^2 193^2 233^2}{2^2 5^4 7^4 23^4 71^2}$

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